

# TWISTED COHOMOLOGY

## AND DIRAC GEOMETRY

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① Courant brackets. Dirac structures.

Twisting.

② Clifford algebras. Dirac structures  
on Lie groups.

③ Dynamical r-matrices. Higher  
twistings.

Based on joint works with

D. Xu, P. Severa, E. Meinrenken, H. Bursztyn

$M$  = manifold       $\{x^i\}$  = local  
coordinates

$$u = \sum_i u_i(x) \frac{\partial}{\partial x_i}$$

$\mathcal{X}(M) = \Gamma(M, TM)$

$$\sigma = \sum_j \sigma_j(x) \frac{\partial}{\partial x^j}$$

vector fields  
on  $M$

• LIE BRACKET :

$$[u, \sigma] = \sum_i (u_i \frac{\partial \sigma_j}{\partial x^i} - \sigma_i \frac{\partial u_j}{\partial x^i}) \frac{\partial}{\partial x^j}$$

QUESTION : extend this bracket to

$$\Gamma(M, TM \oplus T^*M)$$

E

# DERIVED BRACKETS

(Y. KOSMANN - Schwarzbach)

$\Omega(M)$  = differential forms on  $M$

- contractions  $v \in \mathcal{X}(M)$

$$z(v) \omega = \omega(v, \dots)$$

- LIE DERIVATIVES  $L(v) \in \underline{\text{DER}}(\Omega(M))$

- de Rham differential  $d_{DR}$   $\mathbb{Z}_2$ -graded

$$\alpha(\delta \wedge \omega) = \alpha(\delta) \wedge \omega + (-1)^{|a||\delta|} \delta \wedge \alpha(\omega)$$

- CARTAN MAGIC FORMULA

$$[d_{DR}, z(v)] = L(v)$$

||

$$d_{DR} z(v) \underset{\equiv}{=} z(v) d_{DR}$$

||

$$- (-1)^{|d||z|}$$

$$\kappa: \Gamma(M, TM \oplus T^*M) \longrightarrow \text{End}(\Omega(M))$$

$$(u, \alpha) \longmapsto \iota(u) + \alpha \wedge$$

exterior multiplication

$$\kappa([u, \alpha], [v, \beta]) =$$

$$= [[d_{DR}, \kappa(u, \alpha)], \kappa(v, \beta)] =$$

$$= [[d_{DR}, \iota(u) + \alpha \wedge], \iota(v) + \beta \wedge] =$$

$$= [L_u + d\alpha \wedge, \iota(v) + \beta \wedge] =$$

$$= \iota([u, v]) + \underset{\text{LIE}}{(L_u \beta) \wedge} - (uv) d\alpha \wedge \\ \qquad \qquad \qquad \parallel \\ \qquad \qquad \qquad - L_v \alpha + d \langle \alpha, v \rangle$$

Courant bracket:

$$[(u, \alpha), (\nu, \beta)] =$$

$$= ([u, \nu]_{\text{LIE}}, L_u \beta - L_\nu \alpha + d_{\text{DR}} \langle \alpha, \nu \rangle)$$

- Anchor map  $\rho: E = TM \oplus T^*M \longrightarrow TM$

$$(u, \alpha) \longmapsto u$$

$$\rho([e_1, e_2]) = [\rho(e_1), \rho(e_2)]_{\text{LIE}}$$

- SCALAR PRODUCT  $e = (u, \alpha)$

$$(e, e) = \langle \alpha, u \rangle$$

$$[e, e] = d(e, e)$$

- LEIBNIZ IDENTITY

$$[e, [e_1, e_2]] = [[e, e_1], e_2] + [e_1, [e, e_2]]$$

• LEIBNIZ IDENTITY , PROOF :

$$[d, e] \equiv de$$

$$[e, [e_1, e_2]] = [de, [de_1, e_2]]$$

$$[[e, e_1], e_2] = [d [de, e_1], e_2]$$

→ //

$$[[de, de_1], e_2]$$

$$d^2 = 0$$

$$[e_1, [e, e_2]] = [de_1, [de, e_2]]$$

∴ Jacobi identity

for  $[\cdot, \cdot]$

## Properties of generating operator:

①

$$[[d, e_1], e_2] \in \Gamma(E)$$

②

$$[d, [d, e]] = \underbrace{\frac{1}{2}}_{\text{In our case}} [d^2, e] = 0$$

$$\text{In our case } d^2 = 0$$

③  $d$  is a 1st order operator

$$[d, f] = df - \Gamma(E)$$

$$f \in C^\infty(M)$$

$$[[d, f], g] = 0$$

## TWISTING (Several)

$$\eta \in \Omega^3(M), \quad d\eta = 0$$

• generating operator

$$D = d_{DR} + \eta \wedge \quad D^2 = 0$$

• twisted bracket

$$[(u, \alpha), (v, \beta)]^\eta =$$

$$= ([u, v]_{\text{Lie}}, L_u \beta - L_v \alpha + d \langle \alpha, v \rangle + \underline{\underline{\eta(u, v, \cdot)}})$$

Remark  $\tilde{D} = D + \kappa(e)$

defines the same bracket

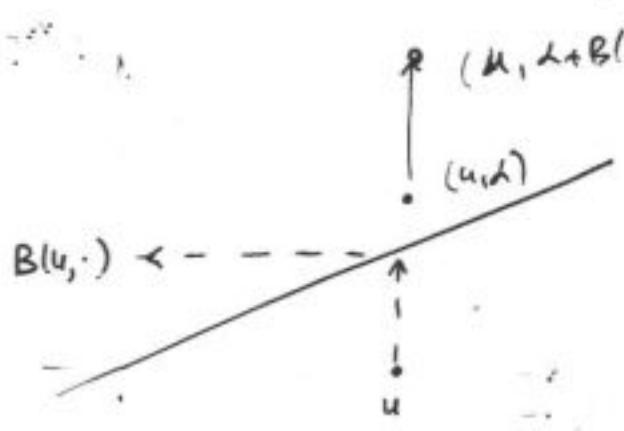
## "Gauge transformations"

$$B \in \Omega^2(M) , \quad e^B = \sum_{n=0}^{\infty} \frac{B^n}{n!}$$

$$e^B \in \text{End } (\Omega(M))$$

$$\bullet \quad e^{-B} (d + \eta) e^B = d + (\eta + dB)$$

$$\bullet \quad e^{-B} (z(u) + \alpha) e^B = \\ = z(u) + \underbrace{(B(u, \cdot) \wedge + \alpha \wedge)}_{\xrightarrow{(u, \lambda + B(u, \cdot))}}$$



Conclusion:  $[ ]^\eta$  only depends on  
 $[\eta] \in H^3(M, \mathbb{R})$

## Dirac structures

Definition:  $D \subset E = TM \oplus T^*M$  is  
a Dirac structure if

(1)  $D$  is a maximal isotropic  
subbundle of  $E$

$$(1a) \quad (e, e) = 0 \quad \forall e \in D$$

$$(1b) \quad \dim D = \dim M$$

(2)  $D$  is integrable

$$[e_1, e_2] \in \Gamma(M, D)$$

$$\forall e_1, e_2$$

$\rho(D) \subset TM$  is an  
integrable distribution

"Pure spinors" :

$$\omega \in \Omega(M) , \quad \omega_m \neq 0 \quad \forall m \in M$$

$E_m$

$$D_\omega^m = \{ e \in E_m \mid \kappa(e) \omega_m = 0 \}$$

Assume  $\dim D_\omega^m = \dim M \quad \forall m \in M$

$\Rightarrow \omega$  is a pure spinor

$D_\omega$  is maximal isotropic :

$$0 = [K(e_+), K(e_-)] \omega_m = 2(e_+, e_-) \omega_m$$

Proposition (Guatieri) :

$D_\omega$  is a Dirac structure iff

$\exists e \in \Gamma(E)$  s.t.

$$\underline{(d + \gamma) \omega = \kappa(e) \omega}$$

$$[2(u) + \alpha, 2(u) + \alpha] = 2 \cdot 2(u) \alpha$$

PROOF

$$\Leftrightarrow D = d + \eta^{\wedge}$$

$$\tilde{D} = d + \eta^{\wedge} - \kappa(e)$$

$$[e_1, e_2]^\eta \omega =$$

$$= [[D, \kappa(e_1)], \kappa(e_2)] \omega =$$

$$= [[\tilde{D}, \kappa(e_1)], \kappa(e_2)] \omega = 0$$

$$[e_1, e_2]^\eta : \Gamma(D)$$

## Examples

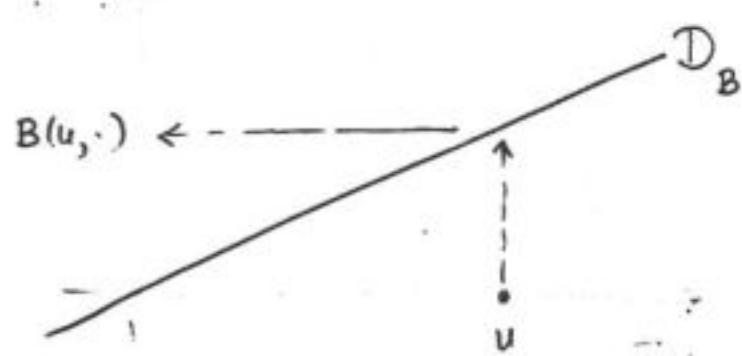
$$\textcircled{1} \quad B \in \Omega^2(M) , \quad dB = 0$$

$$\omega_B = e^{-B} , \quad \eta = 0.$$

$$\mathcal{D}_B = \{ (u, \alpha) \mid \alpha = B(u, \cdot) \}$$

$$\textcircled{1'} \quad \eta \neq 0 . \quad dB = \eta$$

$$(d + \eta) e^{-B} = (\eta - dB) e^{-B} = 0$$



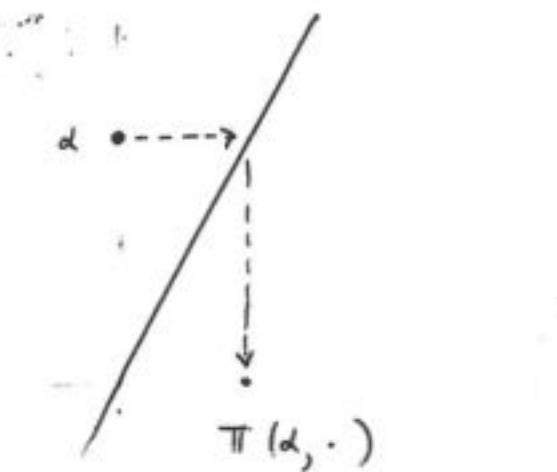
## Examples

②  $V \in \Omega(M)^{\text{top}}$  volume form

$\pi \in \Gamma(M, \Lambda^2 TM)$  bivector

$$\omega_\pi = \exp(-\varphi(\pi)) V$$

- $D_\pi = \{(u, \alpha) \mid u = \pi(\alpha, \cdot)\}$



$$e^{\varphi(\pi)} d_{DR} e^{-\varphi(\pi)} = d_{DR} - [d, \varphi(\pi)] + \frac{1}{2} [[d, \varphi(\pi)], \varphi(\pi)]$$

$$[d, \varphi(\pi)] V = K(e) V$$

for some  $e \neq 0$

$$\varphi(\pi, \pi)$$

~~Schouten  
bracket~~

## CLIFFORD ALGEBRAS :

$W = \text{vector space } / \mathbb{R} \quad \dim W < +\infty$

$Q : W \times W \rightarrow \mathbb{R} \quad \text{scalar product}$

- $\text{Cl}(W, Q) = TW / \langle x \otimes y + y \otimes x - Q(x, y) \rangle$   
 $\forall x, y \in W$

### Properties :

- $\mathbb{Z}_2$ -grading       $1, xy, yx \in \text{Cl}^{\text{even}}$   
 $x, y \in \text{Cl}^{\text{odd}}$

- Chevalley Isomorphism :

$\text{Asym} : \Lambda W \longrightarrow \text{Cl}(W, Q)$

$$1 \longmapsto 1$$

$$x \longmapsto x \quad x \in W$$

$$x \wedge y \longmapsto \frac{1}{2} (xy - yx)$$

$$x_1 \wedge \dots \wedge x_n \longmapsto \frac{1}{n!} \sum_{\sigma \in S_n} (-1)^{| \sigma |} x_{\sigma(1)} \cdot \dots \cdot x_{\sigma(n)}$$

Example :

$\mathfrak{g}$  = LIE algebra

$Q : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{R}$  invariant scalar product

$$Q([x, y], z) + Q(y, [x, z]) = 0$$

e.g.  $\mathfrak{g}$  = semisimple

$Q$  = Killing form

- Cartan 3-form

$$C_Q \in (\Lambda^3 \mathfrak{g}^*)^{\mathfrak{g}}$$

$$C_Q(x, y, z) = Q([x, y], z)$$

coadjoint  
action

- $\Lambda^3 \mathfrak{g}^* \cong \Lambda^3 \mathfrak{g} \longrightarrow Cl(\mathfrak{g})$

$$\underline{q = \frac{1}{6} Asym(C_Q)}$$

"canonical cubic element"

Prop  $q^2 = \text{const}$



$d_{Cl} = [q, \cdot]$  is a differential :

$$d_{Cl}^2 = [q, [q, \cdot]] = [q^2, \cdot] = 0$$

Prop  $\underbrace{[[q, x], y]}_{\text{commutators}} = [x, y]_g$

in  $Cl(g)$

Prop  $\tau : x \rightarrow [q, x] \in Cl(g)$

is a Lie algebra homomorphism

Proof :

$$[[q, x], [q, y]] = [q, [x, [q, y]]]$$

$$- [q, [[q, y], x]]$$

$$- [q, [y, x]]_g = \tau([x, y])$$

$$d_{Cl}^2 = 0$$

Aside :

$W$  = vector space,  $Q : W \times W \rightarrow \mathbb{R}$   
scalar product

$$C \in \Lambda^3 W^* \cong \Lambda^3 W \longrightarrow Cl(W, Q) \ni q$$

$$\cdot \quad \tau : x \mapsto [q, x] \in Cl(W)$$

Theorem (Rohr) :

Assume (1)  $Q$  is positive definite

(2)  $\text{im}(\tau) \subset Cl(W)$  is  
a Lie subalgebra ( $= \mathfrak{g}$ )

$\Rightarrow [x, y] = [[q, x], y]$  is a  
Lie bracket on  $W$  and

$$(W, [ , ]) \cong (\mathfrak{g}, [ , ]_{\mathfrak{g}}) \oplus (\ker \tau, 0)$$

Aside: noncommutative Weil algebra

$$\mathcal{W}_{\mathfrak{g}} = U_{\mathfrak{g}} \otimes Cl(\mathfrak{g})$$

U

$$\mathfrak{g} \otimes \mathfrak{g} \ni c = \sum_i e_i \otimes e^i$$

canonical element  
corresponding to Q

$\mathcal{D} = c + 1 \otimes q$

"Dirac operator"

Prop  $\mathcal{D}^2 \in Z(U_{\mathfrak{g}}) \otimes 1$



$d_{\mathcal{W}} = [\mathcal{D}, \cdot]$  is a differential

Prop  $x, y \in \mathfrak{g}$

$$\underbrace{[[\mathcal{D}, 1 \otimes x], 1 \otimes y]}_{\text{commutator in } \mathcal{W}_{\mathfrak{g}}} = 1 \otimes [x, y]_{\mathfrak{g}}$$

commutator in  $\mathcal{W}_{\mathfrak{g}}$

of simple

2.4

$$(\Lambda g^*)^\# \cong \Lambda [c_{2m_j+1}, j = 1, \dots, r]$$

$m_1 = 1$ ,  $c_3$  = Cartan form

Theorem (Kostant) :

$$Cl(g) = \tau(u_g) \otimes Cl(g)^\#$$

$$\text{End}(V_\rho) \quad Cl[q_{2m_j+1}] = \text{Asym}(c_{2m_j+1}) \quad j = 1, \dots, r$$



$$H(Cl(g), d_C = [q, \cdot]) = 0$$

Prop  $H(Cl(g))^\#$ ,  $\tilde{\mathcal{L}} = [q+x, \cdot]$ ,  $x \in \{$

$$\begin{cases} Cl(g), & x \in W(\rho) \\ 0 & \text{otherwise} \end{cases}$$

$$\mathfrak{g} = n_+ \oplus \mathfrak{f} \oplus n_- \quad \underline{x} = \rho$$

$$s \subset \mathfrak{f}$$

. Cocycles :

$$\textcircled{1} \quad \alpha = 1$$

$$\textcircled{2} \quad \alpha_s = \text{Vol}_{n_-} \cdot \text{Vol}_s \cdot \text{Vol}_{n_+}$$

$$\text{Cl}(g) = \Lambda_{n_+} \cdot \text{Cl}(f) \cdot \Lambda_{n_-} = \Lambda_{n_-} \cdot \text{Cl}(f) \cdot \Lambda_{n_+}$$

$$\text{hc} \downarrow \quad \begin{matrix} \text{"Clifford Harish-Chandra} \\ \text{projection"} \end{matrix}$$

$$\text{Cl}(f)$$

$$\underline{\text{Prop}} \quad \text{hc} \circ [g + \rho, \cdot] = 0$$

$$\underline{\text{Prop}} \quad \text{hc}(\alpha_s) = \text{Vol}_s$$

Kostant's conjecture :

Find  $\text{hc}(g_{2m_j+1})$  ??

Example :  $\underline{\text{hc}}(g_3) = -\rho$

$$\underline{W} = V \oplus V^*$$

$$Q((u, \alpha), (v, \beta)) = \langle \alpha, v \rangle + \langle \beta, u \rangle$$

signature  $(n, n)$  scalar product

$\exists!$  irreducible representation of  $Cl(W, Q)$ :

- $S_{V^*} = \Lambda V^*$

$$(u, \alpha) : \omega \longmapsto (z(u) + \alpha \wedge) \omega$$

- $S_V = \Lambda V$

$$(u, \alpha) : p \longmapsto (u \wedge + z(\alpha)) p$$

- Choose  $\vartheta \in \Lambda^{\text{top}} V^*$

isomorphism  $S_V \xrightarrow{\cong} S_{V^*}$

$$p \longmapsto z(p) \vartheta$$

Example :  $\mathfrak{g}$  = quadratic Lie algebra  
 $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}$   
 $Q_{\mathfrak{d}} = Q_g^1 - Q_g^2 \Rightarrow$  signature  $(n, n)$

- generating operator

$$q_{\mathfrak{d}} = q_g^1 - q_g^2$$

- derived bracket

$$[x, y] = [[q_{\mathfrak{d}}, x], y] = [x, y]_{\mathfrak{d}}$$

- irreducible representation / Clifford module  $C(\mathfrak{d})$

$$S \cong C(\mathfrak{g}) \quad (x, y) : \omega \mapsto x\omega - \omega y$$

- differential :  $q_{\mathfrak{d}}(\omega) = [q_g, \omega]$

- "Dirac structures" = subspaces, <sup>maximal</sup> isotropic and closed under  $[,]_{\mathfrak{d}}$

||  
maximal    isotropic  
Lie    subalgebras  
of     $\mathfrak{d}$

Examples:

$$\textcircled{1} \quad D = \{ (x, x) \in d \mid x \in \mathbb{R} \}$$

$$\underline{\omega_D} = 1 : \quad (x, x) (\underline{\omega_D}) = x \cdot 1 - 1 \cdot x = 0$$

$$\textcircled{2} \quad D_s = (n_+, 0) \oplus (0, n_-) \oplus$$

$$\oplus \{ (x, -x) \mid x \in s \} \oplus \{ (x, x) \mid x \in s, x \in s^\perp \}$$

$$\underline{\omega_{D_s}} = \underline{\alpha_s} :$$

$$(x_+, 0) \alpha_s = x_+ \alpha_s = \underbrace{x_+}_{\parallel} \text{vol}_{n_+} \cdot \text{vol}_s \cdot \text{vol}_{n_-}$$

$$(0, x_-) \alpha_s = \pm \alpha_s x_- = \pm \underbrace{\text{vol}_{n_+}}_0 \cdot \text{vol}_s \cdot \underbrace{\text{vol}_{n_-}}_0 x_-$$

 $\parallel$  $0$

$G = \underline{\text{simple}}$ , 1-connected Lie group

$$\Omega(G)^{G \times G} \cong (\Lambda g^*)^{\mathfrak{g}}$$

↓                  ↓  
     $\gamma$        $\longleftrightarrow$      $C_G$     Cartan 3-form

$$S := \underline{\Omega(G) \otimes Cl(\mathfrak{g})}$$

$x^L, x^R$  left- and right-invariant  
 $x \in \mathfrak{g}$  vector fields on  $G$

$\theta^L, \theta^R \in \Omega(\mathfrak{g}, \mathfrak{g})$  left- and right-invariant  
Maurer-Cartan forms on  $G$

$\tau : G \rightarrow Cl(\mathfrak{g})$  lift of the Lie  
homomorphism  $\tau : \mathfrak{g} \rightarrow Cl(\mathfrak{g})$

$\Lambda^L \in \Omega(\mathfrak{g})^{G^L} \otimes Cl(\mathfrak{g})$   
    llz                  llz                  canonical  
 $\Lambda g^*$                $\Lambda g$               elements

$\Lambda^R \in \Omega(\mathfrak{g})^{G^R} \otimes Cl(\mathfrak{g})$

$$\bullet \quad S = \Omega(G) \otimes Cl(g)$$

↑

Clifford action

$$\Gamma(G, TG \oplus T^*G) \oplus \mathbb{D}$$

$$\bullet \quad D = d_{DR} + \eta^\wedge + [q_g, \cdot]$$

generator of derived bracket

Prop  $D$  spanned by

- $\tau(x^L) - \frac{1}{2} Q(x, \theta^L) + (0, x) = a(x)$
- $\tau(x^R) + \frac{1}{2} Q(x, \theta^R) + (x, 0) = b(x)$

is a Dirac structure defined by

$$\omega_D = \Lambda^L \tau(g) = \tau(g) \Lambda^R$$

Proof.

$$(1) \quad a(x) \omega_D = 0$$

$$\uparrow\downarrow \\ \wedge^L = \text{canonical element} = \wedge^R$$

$$\downarrow \\ b(x) \omega_D = 0$$

(2)

$$[\mathbb{D}, a(x)] = L(x^L) - (0, \tau(x)) = \mathcal{L}^L(x)$$

$$[\mathbb{D}, b(x)] = L(x^R) + (\tau(x), 0) = \mathcal{L}^R(x)$$

$$\mathcal{L}^L(x) \omega_D = 0$$

$\wedge^L$  invariant,  $\tau$  = representation,  $\wedge^R$  invariant

$$\downarrow \\ \mathcal{L}^R(x) \omega_D = 0$$

$$(3) \quad a(x) \mathbb{D} \omega_D = [\mathbb{D}, a(x)] \omega_D = 0$$

$$b(x) \mathbb{D} \omega_D = [\mathbb{D}, b(x)] \omega_D = 0$$

$$\Rightarrow \mathbb{D} \omega_D = f \omega_D, f \in C^\infty(\mathbb{B})$$

But  $\omega_D$  and  $\mathbb{D} \omega_D$  have different  $\mathbb{Z}_2$ -grading

## TWISTED COHOMOLOGY

$G$  = compact, simple, 1-connected  
LIE GROUP

- $H(G, \mathbb{R}) = H(\Omega(\mathfrak{g}), d_{DR}) \cong (\wedge \mathfrak{g}^*)^{\frac{m}{2}}$   
 $\wedge [C_{2m_j+1}, j=1, \dots, r]$
- $\eta = C_3$  Cartan 3-form
- $H^7(G) = H(\Omega(\mathfrak{g}), d_{DR} + \eta \wedge) = 0$
- $\eta = \sum_j \lambda_j C_{2m_j+1}$  primitive element
- $H^9(G) := H(\Omega(\mathfrak{g}), d_{DR} + \eta \wedge) = 0$

Recall: equivariant cohomology

Theorem (H. Cartan):

$G \subset M$  = manifold

"  
compact connected  
Lie group

$$H_G(M, \mathbb{R}) := H(M \times_G EG, \mathbb{R})$$

$$H\left(\Omega(M) \otimes S\mathfrak{g}^*\right)^G, \underbrace{d_{DR} \otimes 1 - 2(e_i)_* \otimes i}_{{d_G}}$$

- where
- $\{e_i\}$  = basis of  $\mathfrak{g}$
  - $(e_i)_*$  = fundamental vector fields on  $M$
  - $\{\zeta^i\}$  = dual basis in  $\mathfrak{g}^*$

- $T \subset G$  maximal torus  
 $\uparrow$   
action by conjugation  $t : g \mapsto tgt^{-1}$
- $H_F(G, \mathbb{R}) \cong \underbrace{H(G, \mathbb{R})}_{\downarrow} \otimes St^*$
- $G$  is "equivariantly formal"
- $\wedge [C_{2m_j+1}^G, j=1, \dots, r]$

Example :

$$C_3^G = C_3 + \frac{1}{2} Q(\xi, \underline{\theta^L + \theta^R})$$

$\overset{\text{deg} = 2}{\nearrow}$        $\downarrow$        $\overset{\text{deg} = 1}{\searrow}$   
 $\text{deg} = 3$

$$\eta_i = C_3(\xi) = C_3 + \frac{1}{2} Q(\xi, \theta^L + \theta^R), \quad \xi \in t$$

Theorem (Severa, AA) :

$$H_\xi^\gamma(G) := H(\Omega(G)^T, d_{\text{de}} - z(\xi_m) + \eta_i) \approx$$

$$\approx \begin{cases} H(\nu) & \text{if } \xi \in \Lambda_{\text{reg}}^* \text{ regular weight} \\ 0 & \text{otherwise} \end{cases}$$

## Cocycles:

$$\omega_D = \wedge^L \tau(g) = \tau(g) \wedge^R$$

†

$$\Omega(G) \otimes Cl(\mathfrak{g})$$

- $(d_{DR} + \eta + [q, \cdot]) \omega_D = 0$
- $\left. \begin{array}{l} (-z(x^L) + \frac{1}{2} Q(x, \theta^L) - (0, x)) \omega_D = 0 \\ (z(x^R) + \frac{1}{2} Q(x, \theta^R) + (x, 0)) \omega_D = 0 \end{array} \right\}$
- $[\omega_D] \in H(\Omega(G), d_{DR} + \eta) \otimes H(Cl, [q, \cdot])$   
 $\parallel$   
0       $\parallel$   
D

•  $x = \rho$   
 $(d_{DR} - z_\rho + \eta_\rho + [q+\rho, \cdot]) \omega_D = 0$

$$[\omega_D] \in H(\Omega(G)^T, d_{DR} - z_\rho + \eta_\rho) \otimes H(Cl^T, [q+\rho, \cdot])$$

$\parallel_2$                      $\parallel_2$   
 $H(G)$                      $\otimes$                      $Cl(\mathfrak{g})$

duality

Trace = Berezin integral :

$$\text{Tr} : \mathcal{Cl}(g) \longrightarrow \mathbb{R}$$
$$\Downarrow$$
$$\wedge g \longrightarrow \wedge^{\text{top}} g \quad \begin{matrix} \uparrow \\ \text{by choice of} \\ \text{volume element} \end{matrix}$$

$\alpha \in \mathcal{Cl}(g)$  pure spinor representing  
a Dirac structure



$$\text{Tr}(\omega_D \cdot \alpha) \in \Omega(G)^T$$

is a cocycle for

$$(d_{DR} - \tau_p + \eta_p) = D_p$$

and defines a Dirac structure

on  $G$

APPLICATION : group-valued moment maps

$$\Phi : M \longrightarrow G$$

$\begin{matrix} G \\ \downarrow \\ G \end{matrix}$        $\begin{matrix} G \\ \downarrow \\ G \end{matrix}$  conjugations

Assume  $[\Phi^* c_3^\zeta] = 0$

$$\Phi^* c_3^\zeta = d_\zeta \zeta$$

•  $[\omega] \in H_{\mathfrak{g}}^{c_3^\zeta}(G)$

$\Rightarrow$   $e^{-\zeta}; \Phi^* \omega$

a cocycle for

$$d_3^M = d_{DR}^M - \iota(\zeta_M)$$

## Example

$\Sigma$  = oriented 2-manifold of genus  $g$  with  $\partial\Sigma = S^1$

$$\begin{array}{c} \text{Hom}(\pi_1(\Sigma), G) \cong G^{2g} \xrightarrow{\phi} G \\ \text{U} \\ \text{closed surface} \end{array}$$

$\phi = [a_1, b_1] \dots [a_g, b_g]$

$$\begin{array}{ccc} \text{Hom}(\pi_1(\bar{\Sigma}), G) & \longrightarrow & M_g(G) \\ \nearrow & & \parallel \\ \text{Hom}(\pi_1(\bar{\Sigma}), G)/G & & \text{symplectic space} \\ & & (\text{Atiyah-Bott}) \end{array}$$

$$\text{Vol}(M_g(G)) =$$

$$= C \sum (\dim V_\lambda)^{2-2g}$$

$$\lambda \in P_+$$

(Witten)

nonvanishing  $H_s^\eta(\mathfrak{h})$

## Higher twistings

$p \in (Sg^*)^g$  invariant polynomial

$\downarrow$  transgression

$\eta_p \in (\Lambda g^*)^g \cong \Omega(G)^{G \times G}$

primitive element

$\eta_p^G \in (\Omega(G) \otimes Sg^*)^G$  equivariant cocycle  
extending  $\eta_p$

## Theorem

$$H_{\eta_p^G}(G) = H(\Omega(G)^T, d_{\text{de}} - z(\xi) + \eta_p^G(\xi)) \approx$$

$$\approx \begin{cases} H(G) & \text{if } p'(\xi) \in \Lambda_{\text{reg}}^* \\ 0 & \text{otherwise} \end{cases}$$