

Algebroids and Sigma Models

Srni 2007

Part I: basic definitions,
facts, philosophy

Part II: Current Algebras,
(Poisson &) Dirac sigma models

Part III: General higher gauge ths,
(Lie) Algebroid YM theories

1) Algebroids

Def.: 1) An algebroid $(E, \rho, [,])$ is



$$\text{with } [,]: \Gamma(E) \otimes \Gamma(E) \rightarrow \Gamma(E)$$

s.t. $[\psi, f\psi'] = f[\psi, \psi'] + (\rho(\psi)f)\psi'$
(i) (Leibniz)

2) A Loday algebroid if also

$$[\psi_1, [\psi_2, \psi_3]] = [[\psi_1, \psi_2], \psi_3] + [\psi_2, [\psi_1, \psi_3]]$$

3) A Lie algebroid if besides (ii)

the bracket is antisymmetric.

Examples:

1) $E = TM$, $\rho = \text{id}$, $[,]$ Lie bracket of vector fields
the standard Lie algebroid

2) $M = \text{pt.}$, $E \cong g$ Lie algebra

3) M Poisson, $E = T^*M$, $\rho = \pi^*$,
 $\{f, g\} = \langle \pi, df \wedge dg \rangle$ $[df, dg] = d\{f, g\}$

4) $E = TM \oplus T^*M$, $\rho = \pi_1$,
 $[,]$... Courant bracket
(Dorfmann)

5) a Dirac structure

} cf. previous talk

1) ÷ 5) Lotay algebroids

1), 2), 3), 5) even Lie algebroids

Philosophy: Generalize geometry as defined
on TM to $E \leftarrow$ vector bundle
with algebroid structure

e.g.: tensors: $t \in \Gamma(T_q^P(M))$

$$\downarrow \qquad \qquad \qquad \uparrow T^{*P} \otimes T^*M^{\otimes q}$$

E -tensors $\stackrel{E}{t} \in \Gamma(E^{\otimes P} \otimes E^{*\otimes q})$.

The E -Lie derivative $\stackrel{E}{\mathcal{L}}$ then permits to
differentiate E -tensors along sections $\psi \in \Gamma(E)$:

$$\stackrel{E}{\mathcal{L}}_{\psi}(f) \stackrel{\text{def}}{=} p(\psi)f, \quad \stackrel{E}{\mathcal{L}}_{\psi}(\psi') \stackrel{\text{def}}{=} [\psi, \psi']$$

extended by an ordinary Leibniz rule
to all E -tensors

$$\stackrel{E}{\mathcal{L}}_{\psi}(\psi_1 \otimes \psi_2) = \stackrel{E}{\mathcal{L}}_{\psi}\psi_1 \otimes \psi_2 + \psi_1 \otimes \stackrel{E}{\mathcal{L}}_{\psi}\psi_2$$

$$\stackrel{E}{\mathcal{L}}_{\psi}(\langle \alpha, \psi \rangle) = \langle \stackrel{E}{\mathcal{L}}_{\psi}(\alpha), \psi \rangle + \langle \alpha, \stackrel{E}{\mathcal{L}}_{\psi}(\psi) \rangle$$

($\alpha \in \Gamma(E^*) \equiv \stackrel{E}{\Omega}^1(M)$... E -1-form) etc.

Prop.: E Loday algebroid, then

(1) $\rho: \Gamma(E) \rightarrow \Gamma(TM)$ a morphism of brackets.

$$(2) [\overset{E}{\mathcal{L}}_{\psi_1}, \overset{E}{\mathcal{L}}_{\psi_2}] = \overset{E}{\mathcal{L}}_{[\psi_1, \psi_2]}$$

(The E -Lie derivative provides a repr. of the bracket on E -tensors)

Proof: ad (1) :

$$\rho(\psi_i)f \cdot \psi_3 + f[\psi_i, \psi_3]$$

$$(i) \Rightarrow \underbrace{[[\psi_1, \psi_2], f\psi_3]}_{(i)} = \underbrace{[\psi_1, [\psi_2, f\psi_3]]}_{(i)} - (\psi_1 \leftrightarrow \psi_2)$$

$$\rho([\psi_1, \psi_2])f \cdot \psi_3 + f[[\psi_1, \psi_2], \psi_3] \quad (i)^2$$

$$\rho(\psi_1)\rho(\psi_2)f \cdot \psi_3 + f[\psi_1, [\psi_2, \psi_3]] +$$

$$+ \underbrace{\rho(\psi_2)f \cdot [\psi_2, \psi_3] + \rho(\psi_3)f \cdot [\psi_1, \psi_2]}_{\text{symmetric in } (\psi_1 \leftrightarrow \psi_2)}$$

$$\Rightarrow \underline{\rho([\psi_1, \psi_2])f \cdot \psi_3} = [\rho(\psi_1), \rho(\psi_2)]f \cdot \psi_3 \quad \#$$

ad (2) : from (1), (ii) & ext. by Leibniz. $\#$ $\forall \psi_i \in \Gamma(E), f \in C^\infty(M)$

For Lie algebroids can go further:

$$\omega \in {}^E\Omega^p(M) = \Gamma(\wedge^p E^*) \quad (E\text{-differential forms})$$

$$\begin{aligned} {}^E\mathrm{d}\omega(\psi_1, \dots, \psi_{p+1}) := & \sum_{i=1}^{p+1} (-1)^{i+1} \rho(\psi_i) \cdot \omega(\psi_1, \dots, \hat{\psi}_i, \dots, \psi_{p+1}) \\ & + \sum_{i < j} (-1)^{ij} \omega([\psi_i, \psi_j], \dots, \hat{\psi}_i, \dots, \hat{\psi}_j, \dots, \psi_{p+1}) \end{aligned}$$

defines $\overset{E}{d}: {}^E\Omega^{\bullet} \rightarrow {}^E\Omega^{\bullet+1}$ with $\overset{E}{d} \circ \overset{E}{d} = 0$.

generalizes the
de Rham differential

(E-exterior derivative)

Prop.: Lie algebroid str. $\xleftarrow{1:1}$ differential complex on
on a vector bundle. \Leftrightarrow $E \rightarrow M$ $({}^E\Omega^{\bullet}(M), \wedge)$

Generalizing quadratic Lie algebras

Prop.: E Lie algebroid. It admits an invariant fiber metric $\overset{E}{g}$ only if $p \in O$ ($\Rightarrow E \cong$ bundle of Lie algebras).

$$\text{Proof: } \underbrace{\overset{E}{\mathcal{L}_f} \overset{E}{g}}_O = \underbrace{f \overset{E}{\mathcal{L}_{\psi}} \overset{E}{g}}_O + 2 \overset{E}{\langle df, \psi \rangle} \overset{E}{g} \quad \begin{matrix} f \in C^{\infty}(n) \\ \psi \in \Gamma(E) \end{matrix}$$

$$\text{but } \langle \overset{E}{df}, \overset{E}{\psi} \rangle = p(\psi) f \quad \#$$

Option 1: Courant algebroids

Def.: A Courant algebroid is a Loday algebroid $(E, p, [\cdot, \cdot])$ with an invariant E -metric $\overset{E}{g}$ s.t.:

$$\overset{E}{g}([\psi, \psi], \psi') = \frac{1}{2} p(\psi') (\overset{E}{g}(\psi, \psi)) . \quad (*)$$

Remarks:

- For $M = \{ \cdot \}$ indeed this reproduces a quadratic Lie algebra.

- For any Courant algebroid E

$$0 \rightarrow T^*M \xrightarrow{\rho^\sharp} E \xrightarrow[\rho]{\lrcorner} TM \rightarrow 0 \quad (+)$$

and $\rho^\sharp(T^*M) \subseteq \text{Ker } \rho$. If (+) exact, E is called an exact Courant algebroid,

- $E \cong T^*M \oplus TM$, determined up to iso by $[H] \in H^3_{dR}(M)$.

Choose isotropic splitting \lrcorner , then

$$\Gamma(E) \ni \psi = \alpha \oplus u \in \Omega^1(M) \oplus \mathcal{X}(M)$$

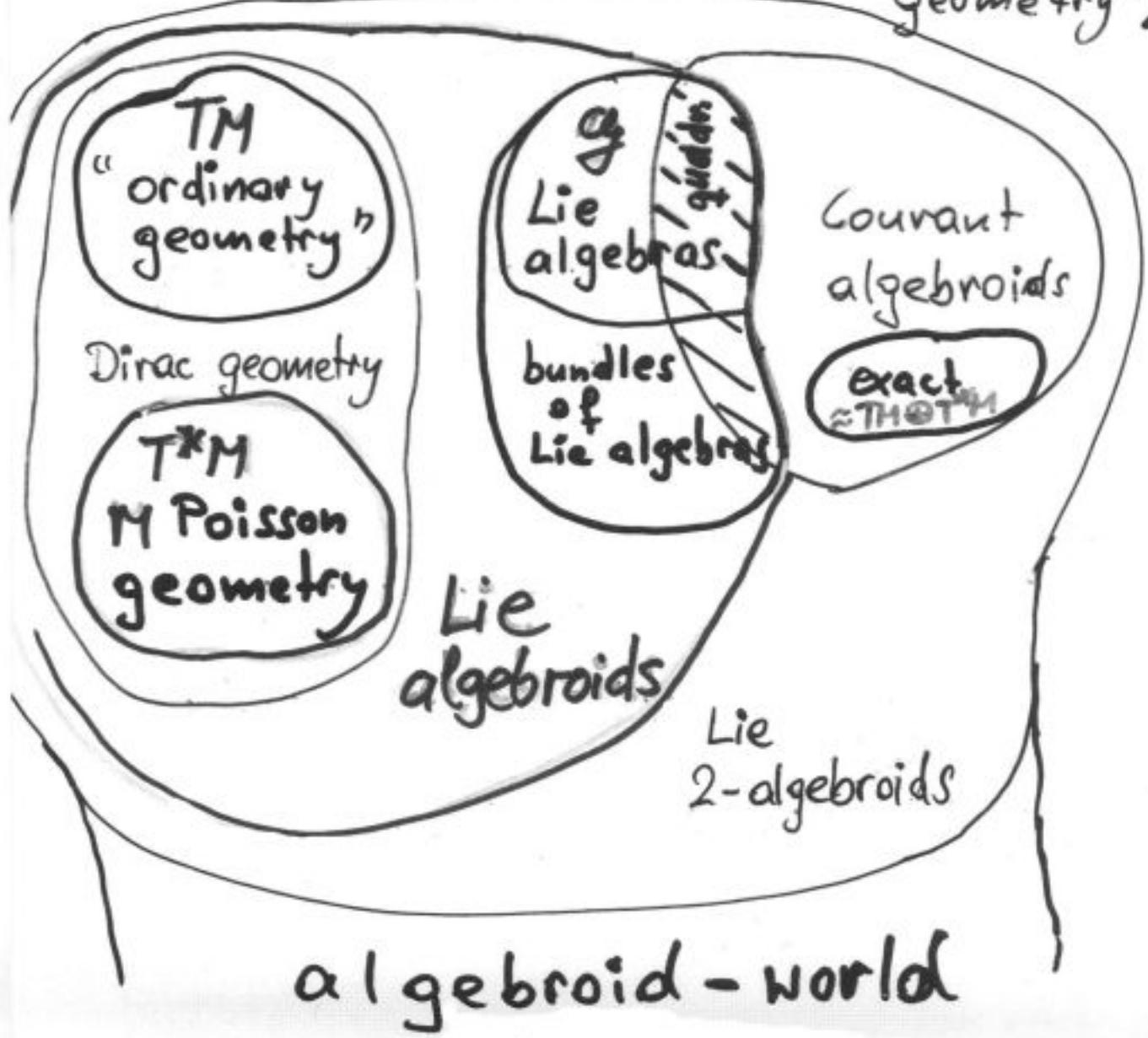
$$\tilde{\text{g}}_\psi(\alpha \oplus u, \beta \oplus v) = \langle \alpha, v \rangle + \langle \beta, u \rangle$$

$$[\alpha \oplus u, \beta \oplus v] = [u, v] \oplus \mathcal{L}_u \beta - i_v(d\alpha + i_H)$$

and $\lrcorner \rightsquigarrow \lrcorner'$ gives $H \rightsquigarrow H + dB$.

$$\lrcorner' B \in \Omega^2(M)$$

- Any involutive, isotropic DCE carries canonically a Lie algebroid structure. If E exact and D maximal, this is called a Dirac structure.
- Resulting Landscape ("generalizing geometry")



Q: \exists generalization of quadr. Lie algs.
within Lie algebroids with $\rho \neq 0$?

YES. For example (option 2)

Def.: E Loday algebroid, \tilde{g} fiber metric.

$(E, \rho, [,], \tilde{g})$ is called homogeneous

E - (pseudo) Riemannian. if

\exists (possibly overcomplete) basis of sections

$\psi_x \in \Gamma(E)$, $\langle \psi_x(x) \rangle = E_x \quad \forall x \in M$,

s.t. $\sum \frac{\partial}{\partial x^i} \psi_j = 0$.

- Remarks:
- If E a Lie algebroid and M is a point this is a quadr. Lie alg.
 - If E is the standard Lie algebroid this is a homog. (pseudo) Riemannian manifold.
 - \exists further options (related to E -connections)

2) Physics Theories, Sigma Models

$\Sigma \dots d$ -dim. spacetime manifold

$h \dots$ Lorentzian or Euclidean metric on Σ

$* : \Omega^p(\Sigma) \rightarrow \Omega^{d-p}(\Sigma)$ Hodge operator

"scalar fields": $\phi \in C^\infty(\Sigma)$

$$S[\phi] := \int_{\Sigma} d\phi_1 * d\phi, \quad (\text{e.g. } \Sigma \text{ comp., } \partial\Sigma = 0)$$

action functional

several scalar fields: $\phi^a \in C^\infty(\Sigma)$, $a=1, \dots, n$

$$\boxed{\varphi: \Sigma \rightarrow \mathbb{R}^n} \quad (1a)$$

$$S[\varphi] \equiv S[\phi^1, \dots, \phi^n] = \frac{1}{2} \int_{\Sigma} d\phi^a * d\phi^a \quad (1b)$$

Philosophy for obtaining a sigma model:

Replace \mathbb{R}^n in (1a) by n -dim manifold M

equipped with geometrical structures

entering a generalization of (1b).

e.g. (M, g, B) ← d-form on M
 \uparrow
 (pseudo) Riemannian metric

fields: $x \in C^{\infty}(\Sigma, M)$ (2a)

$$\begin{array}{ccc} T\Sigma & \xrightarrow{x_*} & TM \\ \downarrow & & \downarrow \\ x: \Sigma & \longrightarrow & M \end{array} \Leftrightarrow dx \in \Omega^1(\Sigma, X^*TM)$$

$$g \circ x \in \Gamma(X^*T^*M \otimes X^*T^*M)$$

action functional (standard sigma model)

$$S[x] = \frac{1}{2} \int_{\Sigma} g \circ x (dx, \gamma^* dx) + \int_{\Sigma} x^* B \quad (2b)$$

reduces to (1b) for $(\mathbb{R}^n, g_{st}, 0)$

Notation: $\alpha \in \Omega^p(\Sigma, X^*TM)$

$$\|\alpha\|^2 := g \circ x (\alpha, \gamma^* \alpha) \in \Omega^q(\Sigma)$$

then $S[x] = \int_{\Sigma} \frac{1}{2} \|dx\|^2 + x^* B \quad (2b')$

Rewriting (2b) using (local) coords. x^i on M :



"spacetime"
"world sheet"
"base mfd"

$$\dim \Sigma = d$$



"target space"

$$\dim M = n$$

$$x^i := x^*(x^i) \in C^\infty(\Sigma) \quad \text{or} \quad C^\infty(\Sigma_U)$$

$$\Rightarrow dx^i \in \Omega^1(\Sigma), \quad i=1, \dots, n \quad (\text{or } \Omega^1(\Sigma_U))$$

$$S[x] = \int_{\Sigma} \frac{1}{2} g_{ij}(x) dx^i \wedge dx^j + \frac{1}{d!} B_{i_1 \dots i_d}(x) dx^{i_1} \wedge \dots \wedge dx^{i_d}$$

(2b'')

Remarks:

- $d=2$ String theory

- Integration over Σ : change charts on M when necessary ((2b'') covar.).
- (1b) results from $M = \mathbb{R}^n$, $g_{ij}(x) = \delta_{ij}$, $B=0$, with $(X^i)_{i=1}^n \leftrightarrow (\phi^a)_{a=1}^n$.

Wess-Zumino terms: ($\partial\Sigma = \emptyset$ case)

classical theory { Euler-Lagrange eqs.
Hamiltonian formul.

B enters always only via dB

Replace dB by $H \in \Omega_{closed}^{d+1}(M)$ in

Indicate this by writing (symbolic.):

$$S = \frac{1}{2} \sum \|dx\|^2 + "S_H" \quad (2b)$$

quantum theory: $[\frac{\hbar}{2\pi} H] \in H_{deform}^{d+1}(M, \mathbb{Z})$

needed in addition

Important example in $d=2$:

$M := G$... semisimple Lie group

g ... Killing metric, H ... Cartan 3-form

then (2b) defines the WZW-model.

"vector fields": $A \in \Omega^1(\Sigma)$

$$S[A] := \int_{\Sigma} dA \wedge *dA$$

$$\rightarrow d * dA = 0$$

"electromagnetism"

with $F = dA := E dt + B$
vacuum Maxwell eqs,

gauge invariance: $S[A+d\lambda] = S[A]$

↳ a "gauge theory" $\# \lambda \in C^\infty(\Sigma)$

more general: Yang-Mills theories

$$A \in \Omega^1(\Sigma, g) \quad (3a)$$

(trivial bundles)

$$(A = (\bar{A}^a) e_a, \langle e_a \rangle = g)$$

g ... quadr. Lie alg.

several vector fields

$$(\mathfrak{g}, K)$$

$$F = dA + \frac{1}{2} [A, \wedge A]$$

$$S[A] = -\frac{1}{2} \int_{\Sigma} K(F, \wedge *F) \quad (3b)$$

gauge invariance: $S[A^g] = S[A] \quad (3c)$

$$(A^g = Ad_g(A) + g^{-1} dg)$$

$$\# g \in C^\infty(\Sigma, g)$$

(non-trivial bundles ... connections ...)

Q: Can we - and then in what sense - make a sigma model generalizing (3a) - (3c) ?

before: $\varphi: \Sigma \rightarrow \mathbb{R}^n$ (with flat metric)



M with geometr. str.

now: idea: view $A \in \Omega^1(\Sigma, g)$

as a vector bundle morphism

$a: T\Sigma \rightarrow g$ (with e.g. Killing metric)



E an algebroid

$$\begin{array}{ccc} a: T\Sigma \rightarrow E & \xleftrightarrow{\quad} & X \in C^\infty(\Sigma, M) \\ \downarrow & & \downarrow \\ x: \Sigma \rightarrow M & & A \in \Omega^1(\Sigma, X^*E) \end{array}$$

(4a)

With this philosophy,
what can we choose for E , (4b), (4c)?

easier warmup question,

do this for Chern-Simons:

$$S_{CS}[A] = \frac{1}{2} \int_{\Sigma} k(A_j dA + \frac{1}{3} [A_j, A])$$

\Downarrow $\dim \Sigma = 3$

S_{CSM}, E... Courant algebroid

Courant sigma model, N. Ikeda 2001

still easier, top. sigma model in $d=2$

Poisson sigma model:

(M, π) ... Poisson P. Schaller + T.S '94
N. Ikeda '94

$$\alpha: T\Sigma \rightarrow T^*M \Leftrightarrow \chi \in C^\infty(\Sigma, M)$$

$$A \in \Omega^1(\Sigma, \chi^* T^* M)$$

$$S_{PSH}[\alpha] = \int_{\Sigma} \langle A_j d\chi \rangle + \frac{1}{2} \langle \pi \circ \chi, A \wedge A \rangle$$

$$= \int_{\Sigma} A_i \wedge d\chi^i + \frac{1}{2} \pi^{ij}(\chi) A_{ij} \wedge A_j$$

1 II) Current algebras & diff. geometry

A.Alekseev + T.S. 2004

1) 2d sigma models

{ current algebras
 ↓ Hamiltonian formulation

Dirac geometry,
exact Courant algebroids

$\Sigma := S^1 \times \mathbb{R}$ ("closed strings")

phase space: (T^*LM, ω)

$LM = \{x: S^1 \rightarrow M\}, \quad TLM \simeq \{S^1 \rightarrow TM\}$

$T^*LM := \{p: TS^1 \rightarrow T^*M\}$

$= \{(x, p) \mid x \in LM, p \in T_x^*T^*M\}$

canonical sympl.

form,

$$x^i(\sigma), \quad R(\sigma) = \dot{p}_i(\sigma) d\sigma$$

$$\omega = \oint_{S^1} \delta x^i(\sigma) \wedge \delta p_i(\sigma), \quad \{x^i(\sigma), p_j(\bar{\sigma})\} = \delta^i_j \delta(\sigma - \bar{\sigma})$$

2d sigma models

$$S_{\pi}[\alpha] = \int \sum A_\mu dx - \frac{1}{2} \pi^*(A)$$

constraints
"CURRENTS"
symmetry generators

$$\begin{aligned} J^i(\sigma) &= \partial X^i(\sigma) + \pi^{ij}(X(\sigma)) \tilde{\pi}_j(\sigma) \\ (J(\sigma) &= (dx - \pi^*(p))_i) \end{aligned}$$

$$S_{\pi, H}[\alpha] := S_{\pi}[\alpha] + \int H \quad \text{the same}$$

$$S_{g, H}[x] = \int \sum \frac{1}{2} \|dx\|^2 + \int H$$

with Killing vectors v_a
and $\iota_{v_a} H = d\alpha_a$

$$\begin{aligned} J_a(\sigma) &= v_a^i(X(\sigma)) \tilde{\pi}_i(\sigma) + \\ &+ (\alpha_a)_i(X(\sigma)) \partial X^i(\sigma) \end{aligned}$$

$$S_{WZW}[x] \quad \left\{ \begin{array}{l} M = G \text{ Lie gp.} \\ g \dots \text{inv. Metric} \\ H \dots \text{Cartan 3-f.} \end{array} \right\} \quad J_L^f(\sigma), J_R^f(\sigma) \quad (f \in \text{Lie } G)$$

$$S_{GWZW}[x, A]$$

$$J^f(\sigma) = J_L^f(\sigma) - J_R^f(\sigma)$$

all of the form

$$\alpha_i \partial X^i + V^i \tilde{p}_i$$

for some pairs (α_i, V^i)

current-
(Poisson) algebra

geometric structures

$$\left\{ \begin{array}{l} \text{involutive} \\ \uparrow \\ J^i(\alpha) = 0 \text{ a} \\ \text{coisotropic sub-} \\ \text{manifold of } T^*M \end{array} \right\} \Leftrightarrow \begin{array}{l} \pi \text{ Poisson} \\ \\ \Leftrightarrow_2 [\pi, \pi] = \langle H, \pi^{\otimes 3} \rangle \\ H\text{-twisted Poisson} \end{array}$$

$$\begin{array}{c} \text{involutivity} \\ \text{conditions} \end{array} \leftrightarrow \begin{array}{c} \text{equivariant} \\ \text{cohomology} \end{array}$$

Kac-Moody algebra
("anomaly")

→ Cartan-Dirac-str.
on G

Hamiltonian formulation for S_{π} :

(Dirac's procedure needed)

$$S_{\pi} [a] = \int \sum A_i \wedge dx^i + \frac{1}{2} \pi^{ij} A_i \wedge A_j =$$

$$\Sigma = \overset{\text{"space"} }{S^M} \times \overset{\text{"time"} }{\mathbb{R}_t}, \quad A_i = \tilde{p}_i d\sigma + \lambda_i dt$$

$$= \int_R dt \oint_{S^M} \phi d\sigma \left[\tilde{p}_i \dot{x}^i - \lambda_i \underbrace{\left(\partial x^i + \pi^{ij}(\sigma) \tilde{p}_j \right)}_{=: J^i} \right]$$

Compare with

$$S_{\text{Ham}}[\tilde{p}_i, q^i] = \int dt \left[\tilde{p}_i \dot{q}^i - H(q, p) - \lambda_a J^a(q, p) \right]$$

$$\omega_{sympl} = dq^i \wedge dp_i, \quad H \dots \text{Hamiltonian}$$

$J^a(q, p) = 0 \dots \text{(primary) constraints}$

⇒ $\omega = \oint_{S^M} d\sigma \delta x^i (\sigma) \wedge \delta \tilde{p}_i (\sigma), \quad H \equiv 0$

$$J^i(\sigma) =: J^i(x(\sigma), \tilde{p}_i(\sigma)) = \partial x^i(\sigma) + \pi^{ij}(x(\sigma)) \tilde{p}_j(\sigma) \stackrel{!}{=} 0$$

the constraints of S_{π}

How to introduce a Wess-Zumino-term?

Compare with S_{Ham} : (point particles)

$$S_{Ham}[q, p] = \int_{\mathbb{R}} dt \left(p_i \dot{q}^i - H(q, p) + \underbrace{A_i(q) \dot{q}^i}_{\text{new contribution}} \right)$$

changes $p_i dq^i$ into $p_i dq^i + A_i(q) dq^i$

and $\omega = dq \wedge dp$ into $\omega = dq \wedge dp + dA$

this we can also replace by
 a closed 2-form (magn. field)
 (coming from the base of T^*M)

2d sigma models: Now T^*LM

closed 3-form H on $M \rightarrow$

closed 2-form
on LM

ev: $S^1 \times LM \rightarrow M$
 $(\xi, x) \mapsto x(\xi)$

$\int_{S^1} ev^* H$

Given $v \oplus \alpha \in \mathcal{X}(M) \oplus \Omega^1(M) = \Gamma(TM \oplus T^*M)$,

want a function on T^*LM .

- $\chi^* \alpha \in \Omega^1(S^1)$, test function $\varphi \in C^\infty(S^1)$

then $\int_{S^1} \varphi \cdot \chi^* \alpha \in C^\infty(LM) \subset C^\infty(T^*LM)$

- $v \in \Gamma(TM) \subset C^\infty(T^*M)$

$T^*LM = \left\{ \begin{array}{l} p: TS^1 \rightarrow T^*M \\ \downarrow \\ x: S^1 \rightarrow M \end{array} \right\}$

$p^* v \in C^\infty(TS^1)$
even $\in \Omega^1(S^1)$

Thus $J_\psi[\varphi] \equiv J_{v \oplus \alpha}[\varphi] = \int_{S^1} \varphi \cdot (p^* v + \chi^* \alpha)$

is a function on phase space T^*LM .

- Let $f \in C^\infty(M)$ and $\mu \in \Omega^1(S^1)$ test 1-form

then likewise

$$F[f] := \int \chi^* f \cdot \mu$$

a function on
 LM and T^*LM

6

Compute Poisson brackets on T^*LM , with
the canonical sympl. form twisted by $H \in \Omega_{\omega}^3(M)$

Result: $\varphi, \bar{\psi} \in \Gamma(E)$, $E = TM \oplus T^*M$

$$\{F_\varphi[\mu], F_\varphi[\bar{\mu}]\} = 0$$

$$\{J_\varphi[\varphi], F_\varphi[\mu]\} = F_{\rho \circ J_\varphi}[\varphi \mu] \quad \text{"anomaly"}$$

$$\{J_\varphi[\varphi], J_{\bar{\psi}}[\bar{\varphi}]\} = J_{[\varphi, \bar{\psi}]}[\varphi \bar{\varphi}] + F_{(J_\varphi, \bar{\psi})}[\varphi d\bar{\varphi}]$$

(*)

where (for $\psi = v \oplus \alpha$, $\bar{\psi} = \bar{v} \oplus \bar{\alpha}$):

$$\rho(\psi) = \rho_v(v \oplus \alpha) = v \quad \text{anchor}$$

$$(\psi, \bar{\psi}) = \langle \bar{\alpha}, v \rangle + \langle \alpha, \bar{v} \rangle \quad \text{inner product}$$

$$[\psi, \bar{\psi}] = [v, \bar{v}] \oplus \alpha \bar{\alpha} - \iota_{\bar{v}} d\alpha + \iota_v d\bar{\alpha} H$$

(twisted) Courant bracket

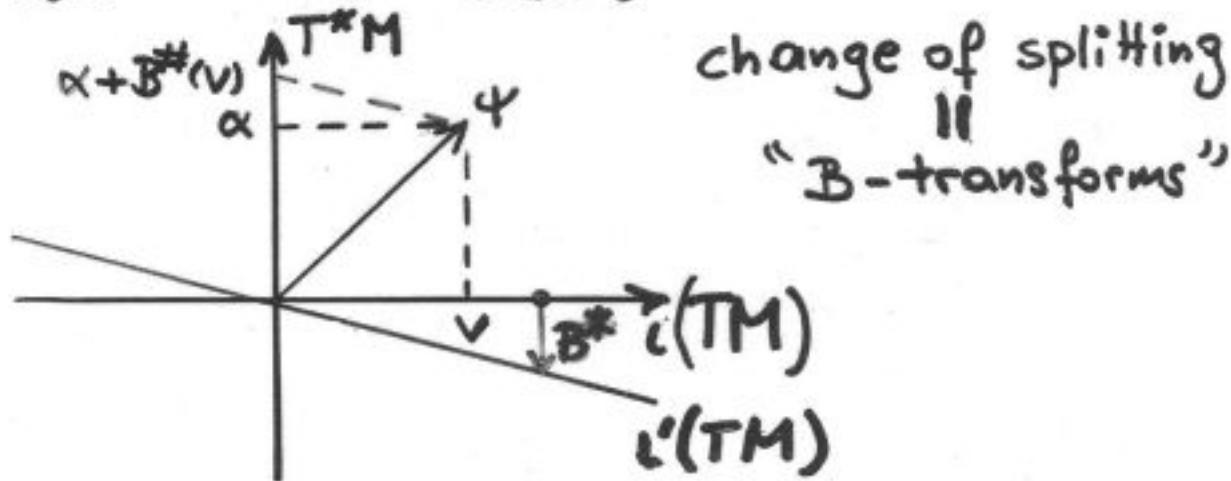
of the exact Courant algebroid-

with splitting: $E = TM \oplus T^*M$ & $H \in \Omega_{\omega}^3(M)$

Discussion

- change WZ-term $H \mapsto H + dB$ can be compensated by $X^i(\sigma) \mapsto X^i(\sigma)$, $P_i(\sigma) \mapsto P_i(\sigma) + B_{ij}(X(\sigma)) dX^j(\sigma)$. Induces

$J_{V \oplus \alpha} \mapsto J_{V \oplus \alpha + B^*(V)}$, i.e.



change of splitting
"B-transforms"

so also in this setting:

up to change of splitting, the structures depend only on

$$[H] \in H^3_{dR}(M)$$

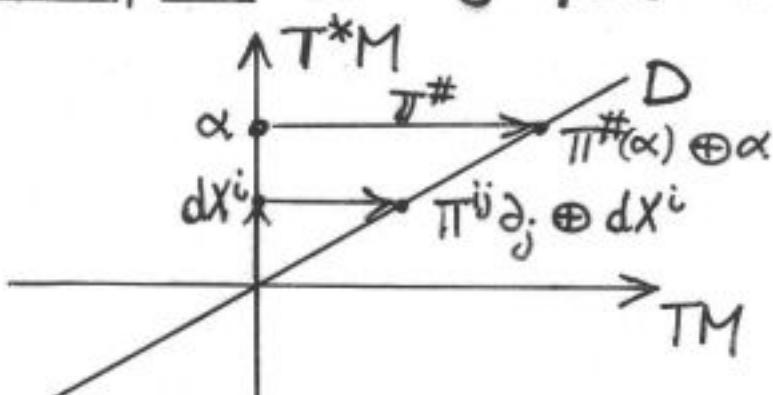
Several cases

- maximal involutive subalgebras generated by J_ψ for $\psi \in \Gamma(D)$, $D \subset E$

$\uparrow \downarrow 1:1$

Dirac structures $D \subset E$

- example: $D = \text{graph}(\pi^\#)$



$$J_{\pi^i_j \partial_j + dx^i} [\varphi] = \oint_{S^1} \varphi \underbrace{[dx^i(\sigma) + \pi^i_j(\chi(\sigma)) p_j(\sigma)]}_{J^i(\sigma)}$$

the constraint funs. of $S_{\pi}[\alpha]$

- G simple Lie group, $E = TG \oplus T^*G \subset TG \oplus TG$
embed $\text{Lie}(G) = g$ diagonally into $\Gamma(E)^G$
or ${}^G\Gamma(E)$, $l(g) \in \Gamma(E)^G$, $r(g) \in {}^G\Gamma(E)$,
then $J_{l(g)} = J_L^g$, $J_{r(g)} = J_R^g$ Kac Moody

- In another notation, $E_F[\psi] = \oint_S E_F(\sigma) d\sigma$, $J_F[\psi] = \oint_S J_F(\sigma) \psi(\sigma) d\sigma$, the current algebra reads:

$$\boxed{\begin{aligned} \{E_F(\sigma), E_F(\tau)\} &= 0 \\ \{J_F(\sigma), E_F(\tau)\} &= \delta(\sigma-\tau) F_{F,F}(0) \\ \{J_F(\sigma), J_F(\tau)\} &= \delta(\sigma-\tau) J_{F,F}(0) - F_{F,F}(0) \delta'(\sigma-\tau) \end{aligned}} \quad (*)$$

- also obviously

$$\boxed{\begin{aligned} F_{f_1, f_2}(\sigma) &= E_F(\sigma) \cdot E_F(\sigma) \\ J_{f_1, f_2}(\sigma) &= E_F(\sigma) \cdot J_F(\sigma) \end{aligned}} \quad (**) \quad \text{(*)}$$

- the F - and J -currents are linearly dependent:

$$\partial E_F(\sigma) = J_{df}(\sigma) \Leftrightarrow \boxed{E_F[d\varphi] = -J_{df}[\varphi]} \quad (***)$$

2) Generalizations

(i) General (degenerate) Courant algebroids
as particular current algebras

simple fact: vector space V

let $j: V \hookrightarrow P$ be an embedding

into a Poisson algebra P s.t.

$\forall u, v \in V \exists w \in V$ s.t. $\{ju, jv\} = jw$

then $w = [u, v]$, $(V, [\cdot, \cdot])$ a Lie algebra

$$(\{ju, jv\} = j[u, v])$$

OR: linear Poisson structures $\xrightleftharpoons[1:1]{\quad}$ Lie algebras
 $(V \cong g^*)$ (g)

Theorem: Let E be a vector bundle,

$\varphi: E \rightarrow TM$, $[\cdot, \cdot]$ and (\cdot, \cdot) product and
(fiberwise) bilin. form, resp., $\tau: T^*M \rightarrow E$

and $J: \Gamma(E) \rightarrow \mathcal{C}$, $F: C^\infty(M) \rightarrow \mathcal{C}$
 $\otimes C^\infty(S^1)$ $\otimes \Omega^1(S^1)$

$$\text{s.t. } J_\psi[\varphi] = 0 \quad \forall \varphi \Rightarrow \psi = 0$$

$$E_\varphi[\mu] = 0 \quad \forall \mu \Rightarrow f = 0$$

s.t. $(*)$ and $(**)$ hold true, as well as

$$F_f[d\varphi] = -J_{\tau(d\varphi)}[\varphi] \quad (*)$$

then E is a "degenerate Courant alg."
(if (\cdot, \cdot) non deg., a Courant algebroid)

Remark: For E exact, split,
the previous formulas provide
a sympl. realization (T^*LM) of \mathcal{C} .

one step in the proof :

$$0 = \{ J_\psi[\varphi], J_\psi[\varphi] \} \stackrel{(*)}{=} J_{[\psi, \psi]}[\varphi^2] +$$

↑

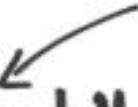
antisymmetry of
the Poisson bracket
on \mathcal{C}

$$F_{(\psi, \psi)}[\varphi d\varphi] = \frac{1}{2} F_{(\psi, \psi)}[d(\varphi^2)] \stackrel{(*)}{=} \\ \stackrel{(*)}{=} - J_{\frac{1}{2}\tau(d(\psi, \psi))}[\varphi^2]$$

$$\Rightarrow [\psi, \psi] = \frac{1}{2} \tau(d(\psi, \psi))$$

mostly open \rightarrow Exercises!

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(ii)  What are current algebras
with higher derivatives on
the r.h.s. ? (s, s', s'', \dots)

- Relation to Vertex Poisson algebras
- Colombeau theory
- higer dimensions.

e.g. 3d sigma models



twist of a Courant algebroid

by a closed 4-form $\lambda \in \Omega^4_{cl}(M)$

- include fermionic fields, susy
 \rightarrow c.f. M. Zabzine last year

• • •

III.

1. Applications of Poisson sigma models
2. Dirac sigma models
3. Action functionals for
Courant sigma models
and Lie Algebroid YM-theories

unfinished due
to lack of electricity
in the night... :)



Applications of the PSM:

1) 2-dim Gravity - Yang-Mills

$\dim \Sigma = 2$, metric h now part of fields

fields: $\phi \in C^\infty(\Sigma)$, $A_{YM} \in \Omega^1(\Sigma, g)$, h

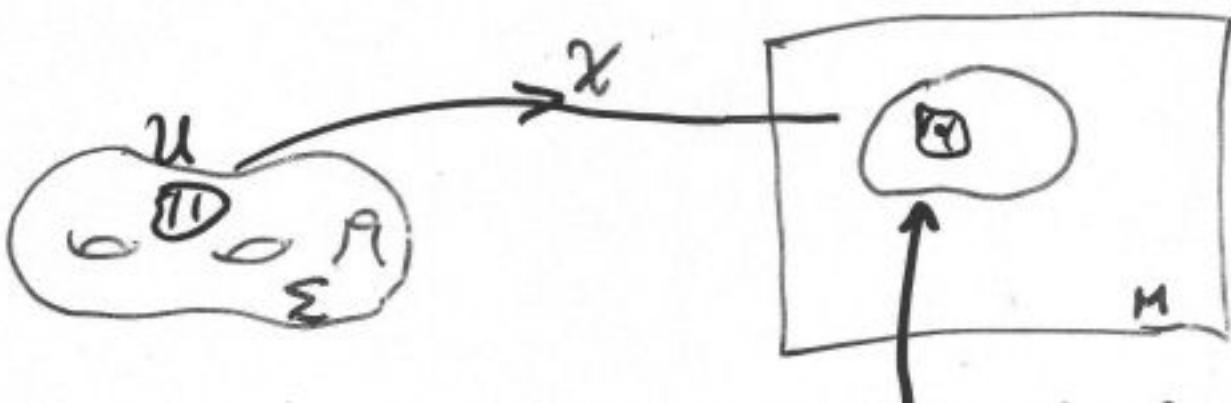
action functional:

$$S[\phi, A_{YM}, h] = \int_{\Sigma} \frac{1}{2} \|d\phi\|^2 + U(\phi) \wedge (F_{YM} \wedge \star F_{YM}) + (W(\phi, R_0) + V(\phi)) \text{vol}_{\Sigma} \quad (*)$$

↑ ↑ ↑ ↓
 scalar field Yang-Mills connection metric curvature of h

Theorem (T.Klösch, T.S.'95): Upon an appropriate choice of (M, π) and map of fields ($h = (\Lambda \circ \chi, A \otimes A), \dots$) are the Euler-Lagrange eqs. modulo gauge symmetries of $(*)$ equiv. to those of $S_{\pi}[a]$

one consequence: permits local trivialization of E.L. eqs. using Weinstein's splitting thm. from Poisson geom



$$S_{\pi}[\alpha] = \int_{\Sigma} A_i \wedge dx^i + \sum + \frac{1}{2} \underbrace{\pi^{ij}(x)}_{\text{rank } (\pi) = \text{const.}} A_i \wedge A_j$$

neighborhood of
a point where
 $\text{rank } (\pi) = \text{const.}$

and $\pi = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

for $U \subset \Sigma$ this is just
(in appropriate target coords.)



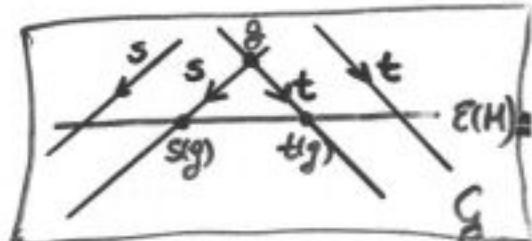
then S_{π} quadratic and E.L.eqs. Linear!

(and in the case of (*) the form of
 $\Delta = \frac{1}{2} \Lambda^{ij} \partial_i \otimes \partial_j$ etc. could be computed)

2) Integration of Lie algebroids

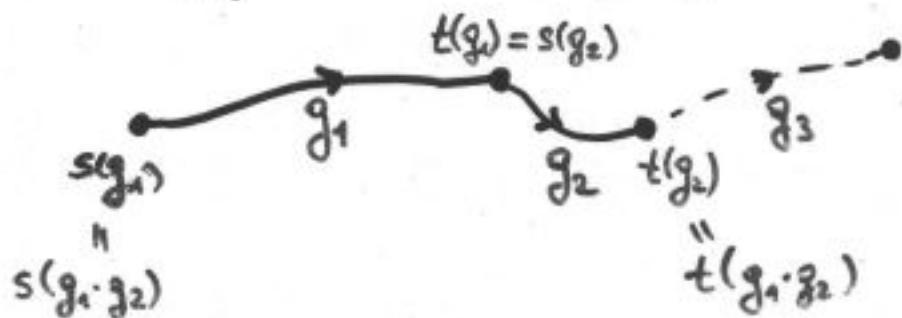
Lie groupoids:

$$\begin{array}{ccc} G & & \\ \downarrow s & \downarrow t & \\ M & & \end{array}$$



example: fundamental groupoid $\mathcal{F}(M)$

homotopy classes of paths on a mfd. M



associated Lie algebroid:

- $M = \text{pt.}, G = G$ Lie group, $\mathfrak{g} \cong T_e G \cong \mathfrak{X}(G)$
 $(\mathfrak{g}, [\cdot, \cdot])$ left invariant vector fields on G
- $\text{Lie}(G) \cong$ vectors at $E(M)$ tangent to $-$ -fibers
 \cong Left invariant vector fields on G tangent to \sim
- e.g. $\text{Lie}(\mathcal{F}(M)) = TM$





Q: Can one vice versa find a Lie groupoid for any Lie algebroid E , or, if not, under which conditions?

Theorem (A.Cattaneo, G.Felder '00):

The reduced phase space of the PSM for $\Sigma \cong I \times \mathbb{R}$ carries a groupoid structure.

If smooth, it is the symplectic ^{Lie}groupoid integrating $T^*M \rightarrow \text{dann } \underline{\text{Thm Crainic-Fernandes}}$

Recall: constraints $J = dX - \pi^*(p) = 0$

RPS = $(T^*PM|_{J=0})/\sim$ generated by J
(gauge symms)

e.g. $M = \mathfrak{g}^* \Rightarrow \pi$ lin. in X . $p \in \Omega^1(I, \mathfrak{g})$

$$T^*M \cong \mathfrak{g} \times \mathfrak{g}^*$$

integrating Lie gpds.:

$$J = \nabla_p X \quad \text{YM connection}$$

$$(J=0)/\sim$$

$$\mathfrak{g} \times \mathfrak{g}^* \cong \begin{matrix} T^*G \\ s \downarrow t \end{matrix} \mathfrak{g}^*$$

$s = id$
 $t = \text{Ad}_g$

moment
maps of
left/right
action

$$X(0) \in \mathfrak{g}^*$$

$$P \exp \int_P \in G$$

3) The Kontsevich formula

• o o

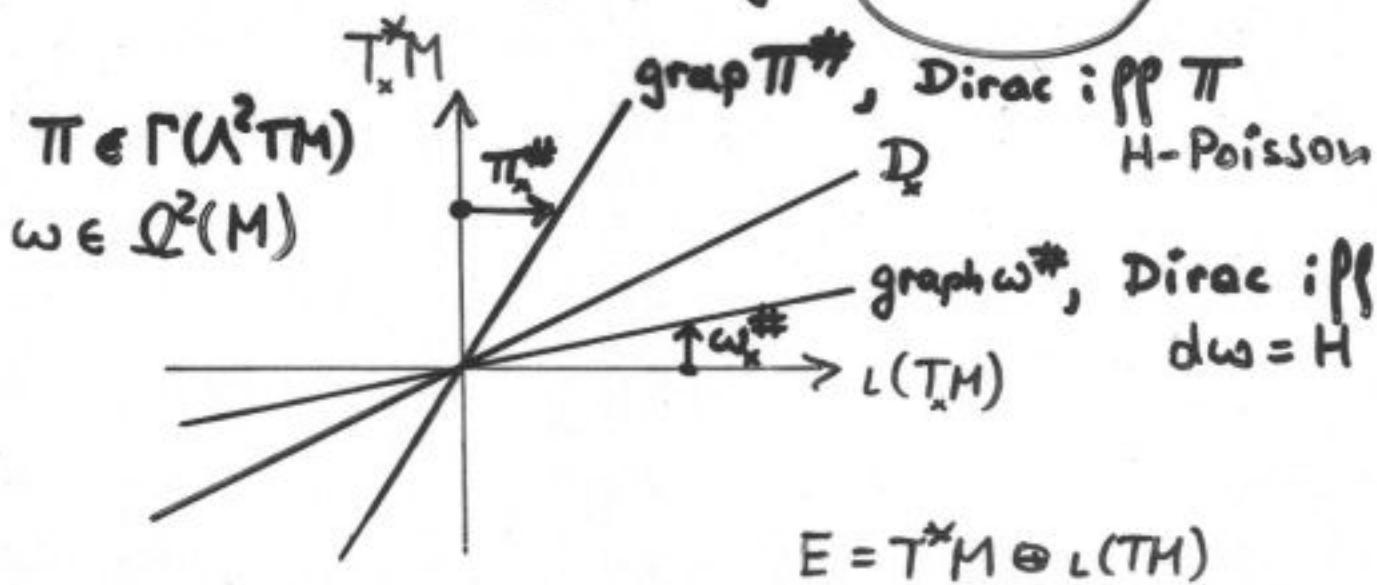
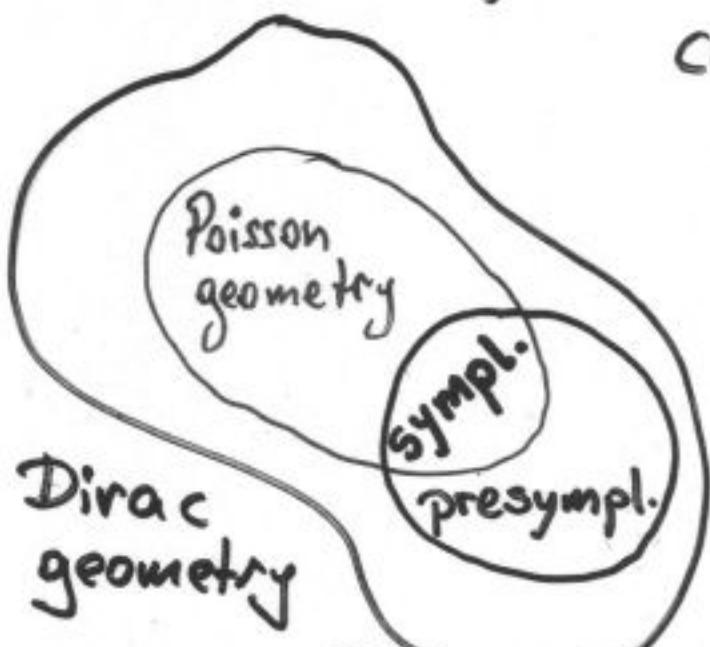
Dirac Sigma Models

A.Kotov, P.Schaller, T.S.

CMP05

Recall:

or likewise
twisted by
 $H \in \Omega_{cl}^3(M)$



$$0 \rightarrow T^*M \rightarrow E \xleftarrow{L} TM \rightarrow 0, H \in \Omega_{cl}^3(M)$$

D Dirac str. \Leftrightarrow • max. isotropic

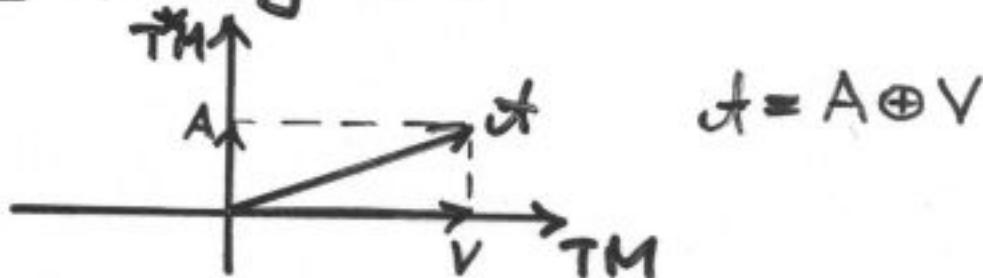
$$\bullet [\Gamma(D), \Gamma(D)] \subset \Gamma(D)$$

Let $D \subset E$ max. isotropic.

fields:

$$\begin{array}{ll} \text{vector bundle} & a: T\Sigma \rightarrow D \\ \text{morphisms} & \chi: \Sigma \rightarrow M \end{array} \quad \left\{ \begin{array}{l} \chi \in C^{\infty}(M, \Sigma) \\ A \in \Omega^1(\Sigma, X^*D) \end{array} \right.$$

choose splitting $\iota: TM \rightarrow E$



choose also further auxiliary structures

g ... metric on M ,

h ... metric on Σ , $\underline{\alpha \in R \setminus 0}$

action functional:

$$S_D[a] = \int_{\Sigma} \frac{\alpha}{2} \|dX - V\|^2 + \int_{\Sigma} \langle A, dX - \frac{1}{2}V \rangle + \int_H$$

for $\beta \in \Omega^p(\Sigma, X^*TM)$: $\|\beta\|^2 = g \circ \chi(\beta \wedge \star \beta)$

here \hbar enters

Theorem:

Hamiltonian formulation of S_D :

T^*LM (or T^*PM) with constraints

$$\boxed{J_\psi = 0} \quad \forall \psi \in \Gamma(D)$$

↑
(as in last lecture)

Corollary:

- 1) The (reduced) phase space does
not depend on auxiliary structures.
- 2) The model is topological[⊕] iff
 D is a Dirac structure.

* in the (classical) sense of:

finite dimensional RPS,

not depending on metric h on Σ .

Remarks:

- One may argue that the quotient E.L. eqs. is ~the same also for gauge symms.
 $\alpha = 0$. However, the α -dependent part serves as "regulator".
- two extreme examples of S_D (let $H=0$):
 - 1) $D = \text{graph}(\pi^\#)$. $\Rightarrow \mathcal{A} = A \oplus \pi^\#(A)$, with $A \in \Omega^1(\Sigma, X^* T^* M)$.

$$S_D[a] = \underbrace{\sum_{\Sigma} \frac{\alpha}{2} \| dX - \pi^\#(A) \|^2}_{\substack{\nearrow \\ \text{E.L. eqs.}}} + \underbrace{\sum_{\Sigma} \langle A_\lambda, dX - \frac{1}{2} \pi^\#(A) \rangle}_{\substack{\searrow \\ \text{unmodified} \\ \text{when } \alpha = 0!}} S_{\pi}[a]$$

$$2) D = TM \Rightarrow \mathcal{A} = V \in \Omega^1(\Sigma, X^* TM)$$

$$S_D[a] = \frac{\alpha}{2} \sum_{\Sigma} \| dX - V \|^2. \quad \begin{array}{l} \text{E.L. eqs.} \\ \text{modified} \\ \text{when } \alpha = 0 \end{array}$$

For $\alpha \neq 0$, E.L. eqs $dX^i = V^i$

and gauge symmetry $\delta_\varepsilon X^i = \varepsilon^i$
 $\delta_\varepsilon V^i = d\varepsilon^i$ $\forall \varepsilon \in \Gamma(\Sigma, X^*)$
invariance of S_D (infinitesimal version)

integrated: homotopy of zero dimens.
base map $\Sigma \rightarrow M$. quotient space

For $\alpha = 0$: No E.L. eqs (in this extreme case 2)

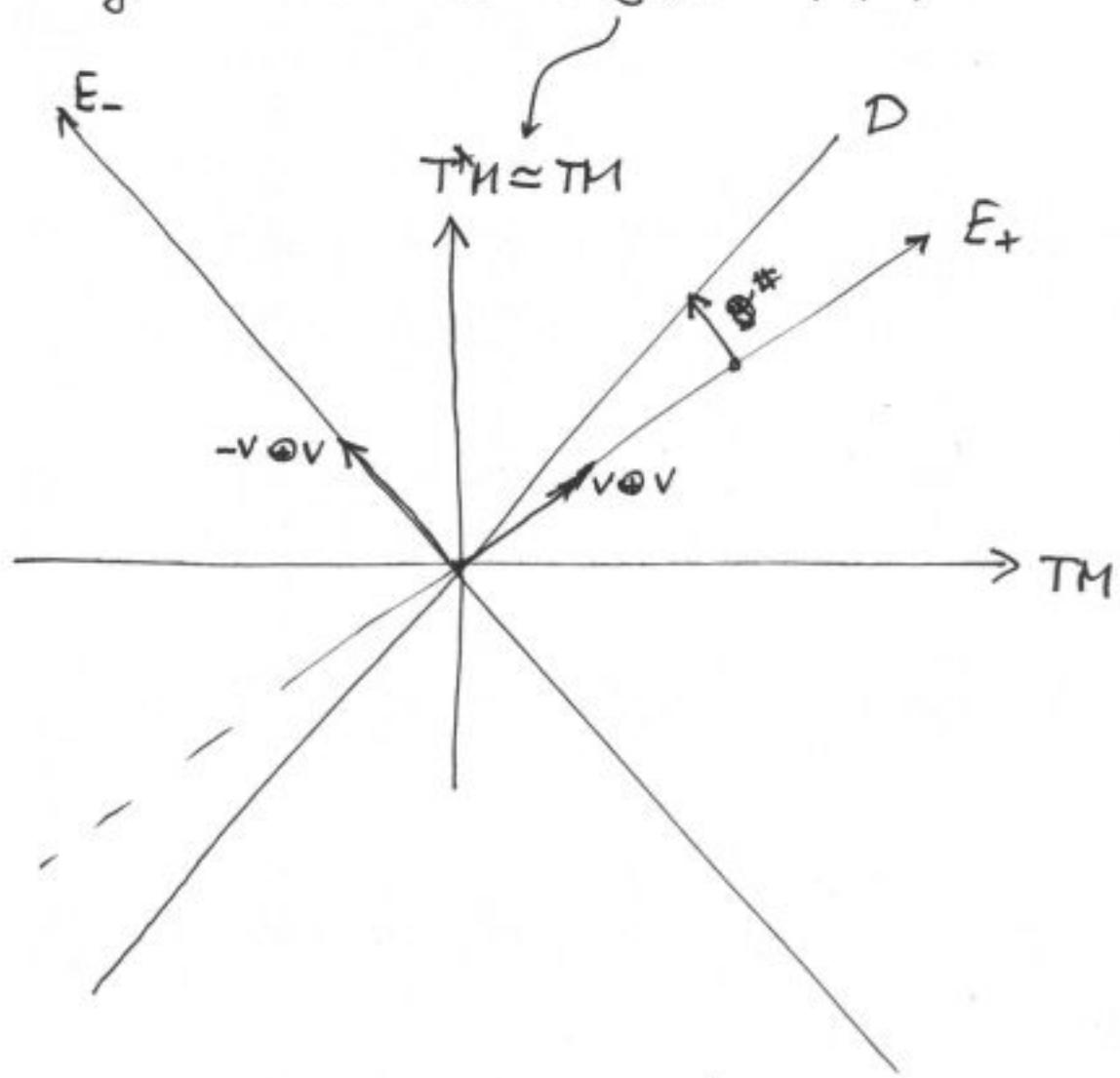
and gauge transfos: arbitrary changes of a .



quotient a point

- In the general case of $D \subset E$ and $\alpha = 0$ it is not always clear what are gauge transformations (qualitative changes on lower dim. submfds. of M, \dots).

Now parametrization of any Dirac structure
by means of an orthogonal op. on $T^*(TM)$
(using the aux. metric g): . . .



e.g. if D graph of bivector π

one finds $O = \frac{1 + \pi}{1 - \pi}$ Cayley

• • •

Theorem: Let \mathcal{D} Dirac structure, $\alpha \neq 0$.

The E.L. egs. are satisfied



$a: T\Sigma \rightarrow \mathcal{D}$ a Lie algebroid
morphism.

◦ ◦ ◦ (including def. and
remarks on Lie algebroid
morphisms).

CSM: $\epsilon \Omega^2(\Sigma, X^*TM)$

$$S_{CSM} [a, \lambda] = \sum_A \langle \lambda, dX - g(A) \rangle - \frac{1}{2} \langle A \wedge dA - \frac{1}{3}[A, A],$$

Σ

$a: T\Sigma \rightarrow E$ 3-dim Ikeda '02

↑
Covariant
algebroid

LAYM: $\epsilon \Omega^{d-1}(\Sigma, X^*TM)$

$$S_{LAYM} [a, \lambda] = \sum_A \langle \lambda, dX - g(A) \rangle - \frac{1}{4} \langle F_A \wedge *F_A \rangle$$

Σ

d -dim. $F_A = dA - \frac{1}{2}[A, A]$

$a: T\Sigma \rightarrow E$

↑
E Lie algebroid
with E-homogeneous
fiber metric $(,)$