Analysis and geometry of shape spaces I - V

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Content:

Introduction: What are shape spaces and what are they good for. A little bit of history.

Shape spaces of plane curves:

The topology of shape space.

Hamiltonian background and conserved momenta.

A bunch of metrics, their geodesics and curvatures:

The L^2-metric and its vanishing of geodesic distance.

Almost local metrics.

Immersion Sobolev metrics.

The scale invariant Sobolov H^1-metric and its relation to the Grassmannian of 2-planes in an infinite dimensional space, and Neretin geodesics.

A covariant formula for curvature and its relation to O'Neill's curvature formulas.

Shape spaces as quotients of diffeomorphism groups:

Right invariant metrics on diffeomorphism groups, their geodesics and curvatures.

Landmark space, geodesics and curvatures.

Spaces of submanifolds (plane curves and higher dimensional ones)

High dimensional shape space $\operatorname{Imm}(M,N)/\operatorname{Diff}(M)$

Vanishing geodesic distance on diffeomorphism groups. Burgers equation corresponds to this phenomenon. The Camassa-Holm equation has to positive geodesic distance. For the Korteweg-de Vries equation we do not know.

The universal Teichmueller space with the Weil-Peterssen metric as shape space. (not done)

Based on:

P.M. and D. Mumford. Riemannian geometries on spaces of plane curves. *J. Eur. Math. Soc. (JEMS)* 8 (2006), 1-48, arXiv:math.DG/0312384.

H. Kodama, P.M. The homotopy type of the space of degree 0 immersed curves. *Revista Matemática Complutense* 19 (2006), 227-234. arXiv:math/0509694.

P.M. and D. Mumford. An overview of the Riemannian metrics on spaces of curves using the Hamiltonian approach. *Appl. Numerical Harmonic Analysis* 23 (2007), 74-113.
arXiv:math.DG/0605009

P.M., David Mumford, Jayant Shah, Laurent Younes: A Metric on Shape Space with Explicit Geodesics. *Rend. Lincei Mat. Appl.* 9 (2008) 25-57. arXiv:0706.4299

Mario Micheli, P.M., David Mumford: Landmarks. In preparation.

David Mumford: Lectures at the Chennai Mathematical Institute.

P.M., David Mumford: Vanishing geodesic distance on spaces of submanifolds and diffeomorphisms, Documenta Math. 10 (2005), 217–245. arXiv:math.DG/0409303

V.Cervera, F.Mascaro, P.M.: The action of the diffeomorphism group on the space of immersions. Diff. Geom. Appl. 1 (1991), 391–401

For background material: Peter W. Michor: Some Geometric Evolution Equations Arising as Geodesic Equations on Groups of Diffeomorphism, Including the Hamiltonian Approach. IN: Phase space analysis of Partial Differential Equations. Birkhauser Verlag 2006. Pages 133-215. arXiv:math/0609077

Introduction:

What are shapes, why are they interesting, and how are they arranged in shape spaces.

Albrecht Dürer was the first to look at the effect of diffeomorphisms on shape

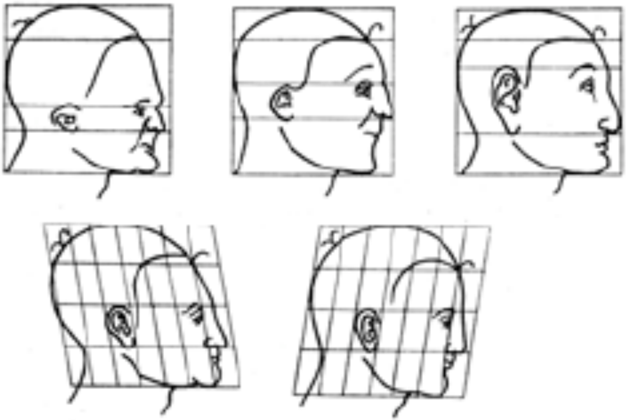
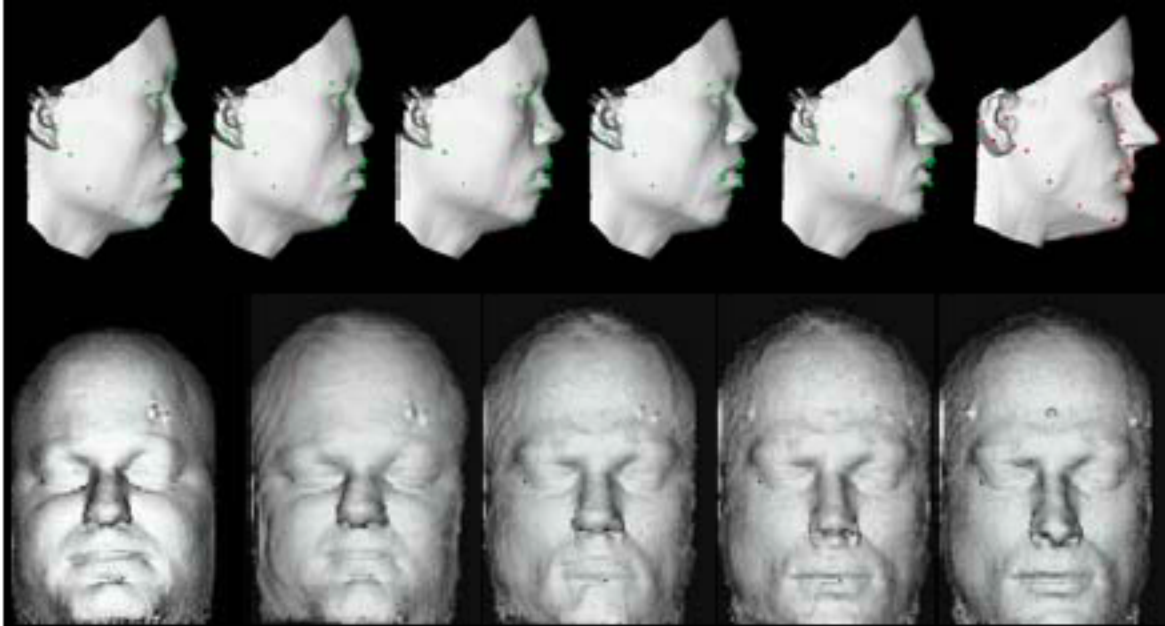


Fig. 139. (After Albrecht Dürer.)

Treatise on Proportion, 1528

A modern view: geodesics between faces

Shortest paths in the space of diffeos carrying one face to the other (Vaillant, Trouve, Younes)



D’Arcy Thompson: *Growth and Form*, 1917, was the first to systematically study the forms of homologous biological shapes

“The study of form may be descriptive merely or it may become analytical. We begin by describing the shape of an object in the simple words of common speech: we end by defining it in the precise language of mathematics. … The mathematical description of a ‘form’ has a quality of precision that is quite lacking in our earlier stage of mere description … We are brought in touch with Galileo’s aphorism that ‘the Book of Nature is written in the characters of Geometry’.”, p. 269.

"In a very large part of morphology, our essential task lies in the comparison of related forms rather than in the precise definition of each; and the *deformation* of a complicated figure may be a phenomenon easy of comprehension, though the figure itself have to be left unanalyzed and undefined. … This method is the <u>Theory of Transformations</u>." p.271

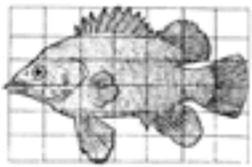


Fig. 150. *Polyprion.*

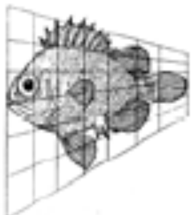


Fig. 151. *Pseudopriacanthus altus.*

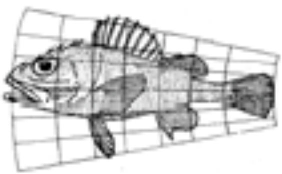


Fig. 152. *Scorpaena sp.*

Fig. 153. *Antigonia capros.*

Fig. 154.

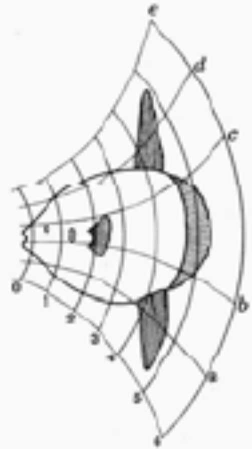


Fig. 155. *Orthagoriscus.*

Primate skulls are of particular interest

To right: named 'landmark points' on skulls

Below: D'Arcy Thompson's skulls

Below right: Bookstein's deformations

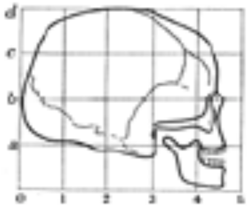
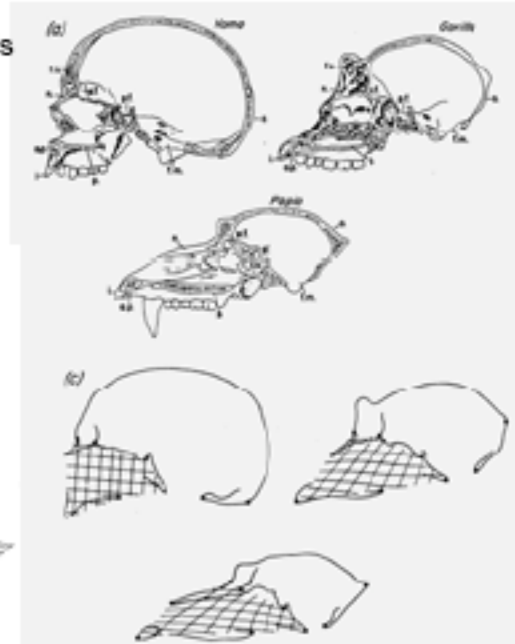


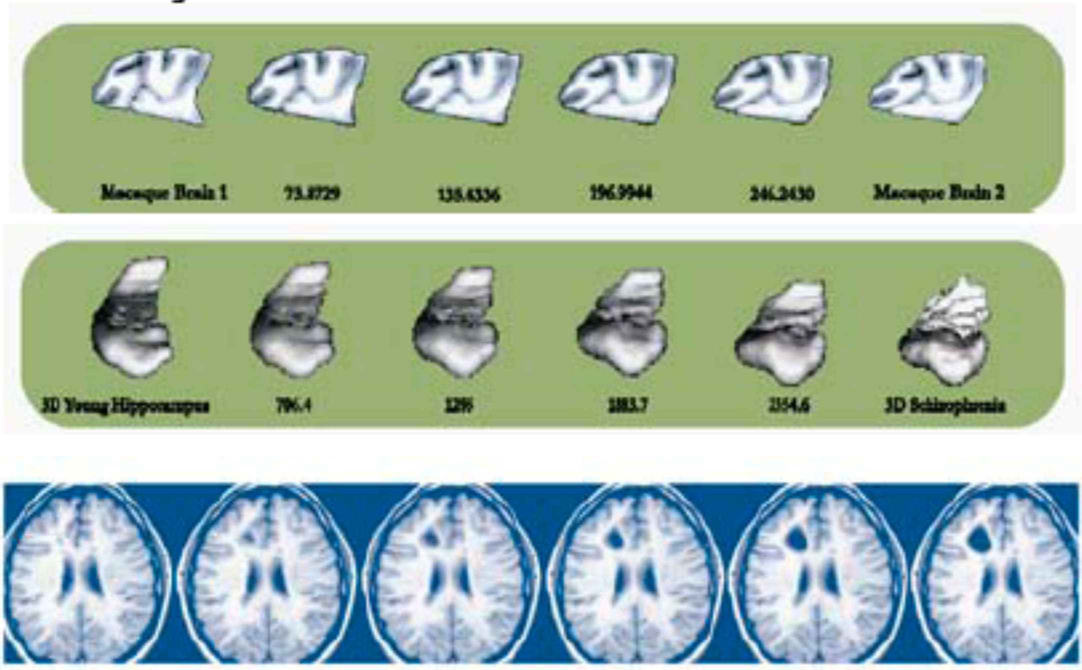
Fig. 177. Human skull.

Fig. 180. Skull of baboon.

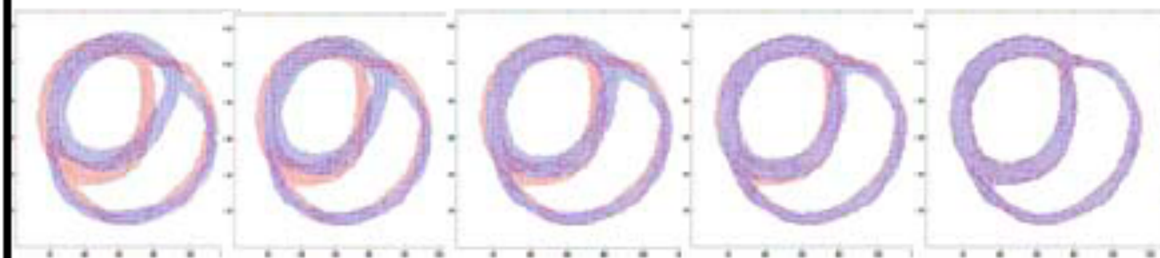
Fig. 181. Skull of dog, compared with the human skull.



Medical Scans require shape analysis to detect defects – cortex



Medical scans II – heart defects



- All vertebrates with their internal structures are 3D diffeomorphic (more or less); all healthy male (e.g. without tumors) and all healthy female humans are really clearly diffeomorphic with only moderate distortion.

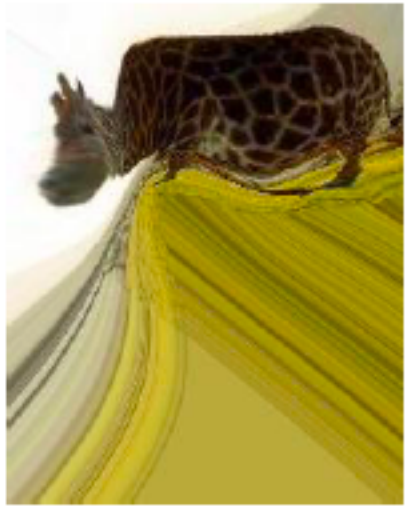
- Can you, then, form an ideal 3D computer model of a male human and female human including all organs/bones/vessels etc.?

- THEN: for each MRI or other scan of each patient, find an *optimal* diffeomorphism of the scanned region with the ideal model, revealing individual differences.

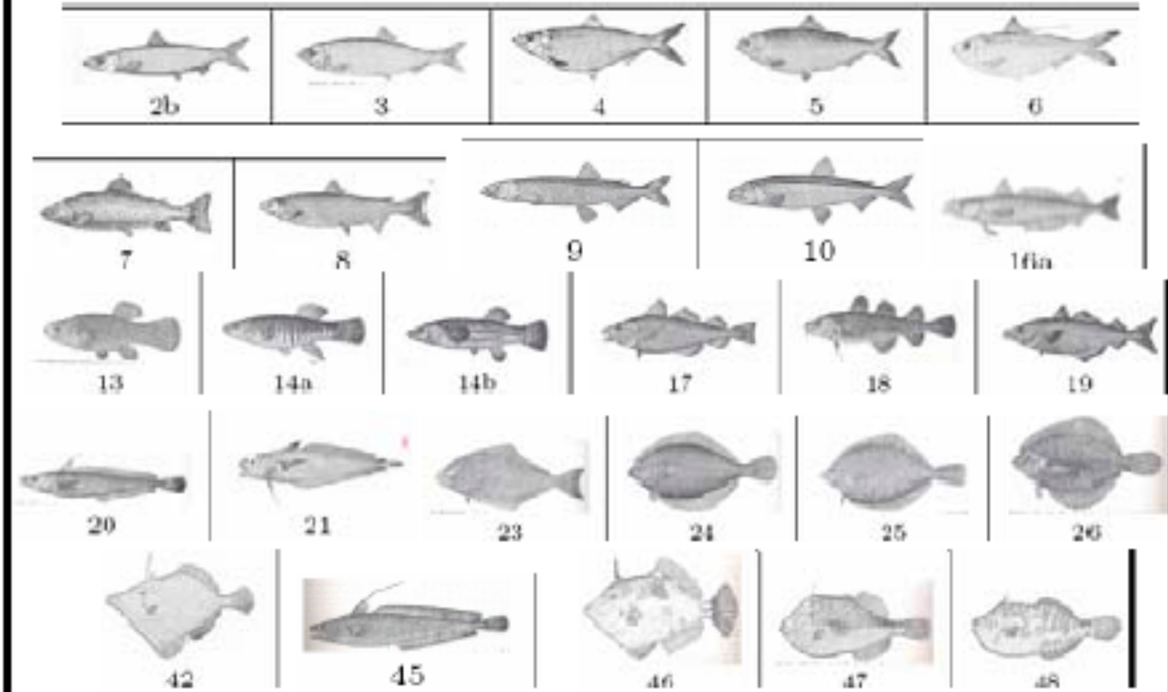
A hippopotamus and a giraffe are indeed diffeomorphic! (2D matching of outlines with surface markings carried over)

A hippopotaffe and a girotamus

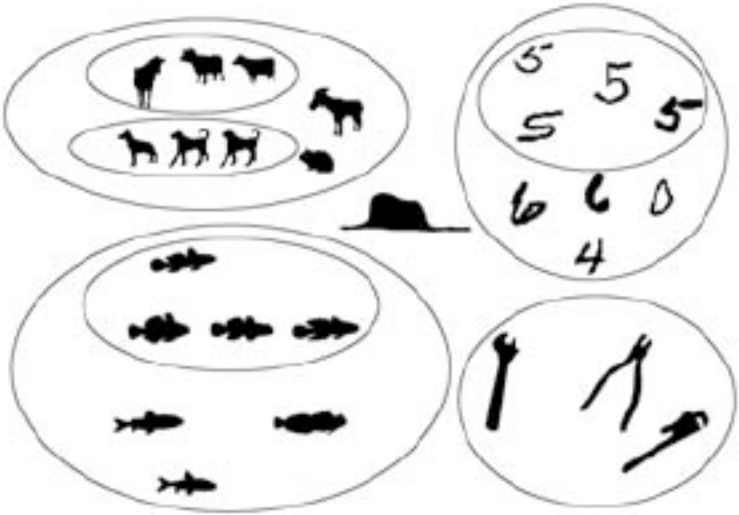
Matching by landmark points: J. Glaunes



Fish – can we classify them by their shape or by diffeomorphism to a prototype?



Clustering of shapes (and a well-know Boa Constrictor)

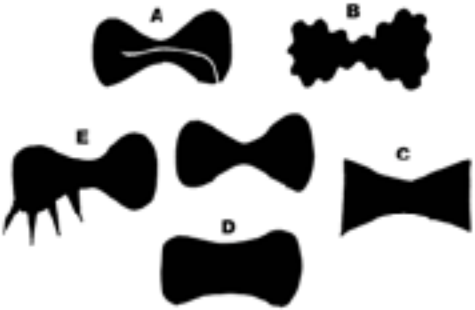


What mathematics can we bring to this?

- People find it natural to judge whether two shapes are 'similar'. We should seek a *metric* on the set of shapes to describe this
- It is natural to compare two shapes by *warping* one to the other. We should look for *geodesics*, shortest paths between shapes
- People will *cluster* shapes into different categories. We need to study datasets of shapes with various statistical algorithms.
- People will say that shape looked *more* like a dog than a cat. We need to put *probability measures* on shape space and take their ratios.

There are many possible metrics!

The central shape is similar in various respects to all 5 of the shapes around it – but in different metrics!



We can adapt function theory ideas –

L^p-norms on k derivatives

a) In L^1, distances are:
   A < B,C < D,E
b) In L^∞, distances are:
   B < C,D <A,E
c) In L^∞ with 1-jets:
   D < B,C < A,E
d) In L^1 with 2-jets:
   D < A,B < C,E
e) To make E close, need 'robust' non-convex metrics that discard outliers.
d) To make D far, qualitative ideas of 'parts' are needed – as it doesn't break into 2 parts.

Advantages of Riemannian metrics

- Have gradients of functionals, gradient flow
- Can expect, at least locally, to have unique geodesics, hence optimal paths from one shape to another
- Can analyze departure from flatness via Riemann curvature tensor
- Can carry over classical statistical data analysis via the exponential map
- Can expect to have diffusion, Brownian motion, hence base probability measures

Let me go into some detail here.

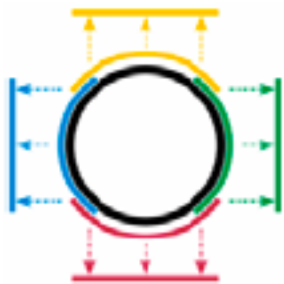
First we need to make the set of 'shapes' into a manifold so we can do differential geometry on it

Riemann introduced the idea of manifolds in his Habilitation Lecture in 1854. He also imagined the infinite dimensional version:

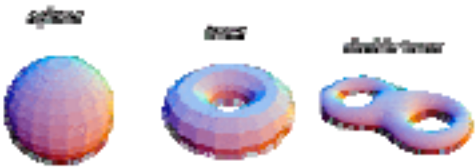
"There are however manifolds in which the fixing of position requires not a finite number but either an infinite series or a continuous manifold of determinations of quantity. Such manifolds are constituted for example by … the possible shapes of a figure in space, etc."

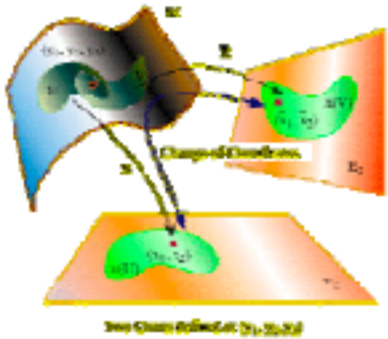
The idea of an "atlas" – some illustrations off the web!

The abstract idea: many pieces, on each have coordinates $x_1, \ldots, x_n$

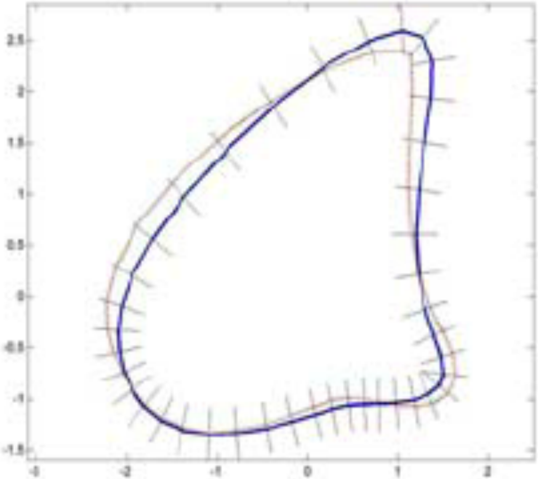


In dimension two, there are tori, pretzels, surfaces with handles. Can (with some pain) make an atlas for each.





The set $\mathcal{S}$ of all smooth plane curves forms a manifold!



Start with a fixed curve $C \in \mathcal{S}$

parametrized by $s \mapsto \phi(s)$

Define a local chart near ϕ:

$\psi_a(s) = \phi(s) + a(s)\vec{n}(s)$,

$\vec{n}(s) =$ unit normal to C,

$C_a =$ image of ψ_a

$U_\phi = \{a \mid \psi_a \text{ smooth}\}$

$\quad \subset (\text{v.sp.of fcns. } a)$

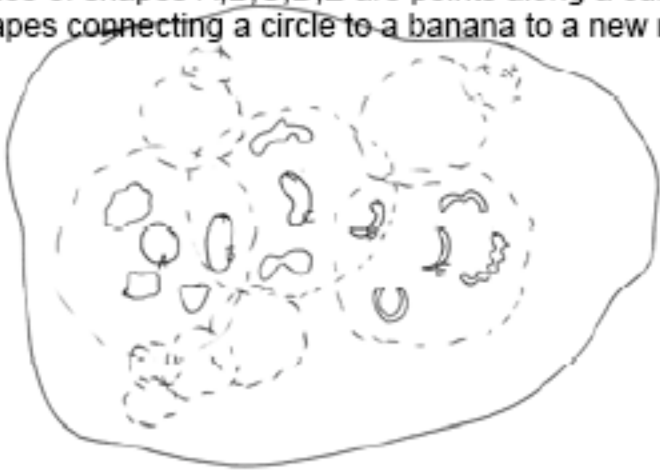
$a \mapsto C_a$ is the chart,

$a(s)$ the local linear coord.

$\mathcal{S} = \bigcup_\phi U_\phi$, gives the atlas

An abstract view of what we are doing

- This whole blob represents the space of all plane curves
- Each curve represents a single point in the space
- The dotted lines represents parts which can be represented as deformations of the central shape – forming a coordinate chart
- The sequence of shapes A,B,C,D,E are points along a curve in the space of shapes connecting a circle to a banana to a new moon.



SIX ingredients of differential geometry

1. Charts/local coordinates at every point $P \in M$:

$$P \overset{\approx}{\longmapsto} \left(x_1(P), x_2(P), \cdots, x_n(P)\right)$$

2. A tangent space $T_P M$ to M, which in coordinates is the vector space of infinitesimal changes $\left(dx_1, dx_2, \cdots, dx_n\right)$. We can associate to every curve $\gamma : [0,1] \to M$ its tangents
$\dot{\gamma}(t) = \left(\cdots, d/dt\,(x_i(\gamma(t))), \cdots\right) \in T_{\gamma(t)} M.$

3. A way of measuring size in $T_P M$, a 'Riemannian metric':

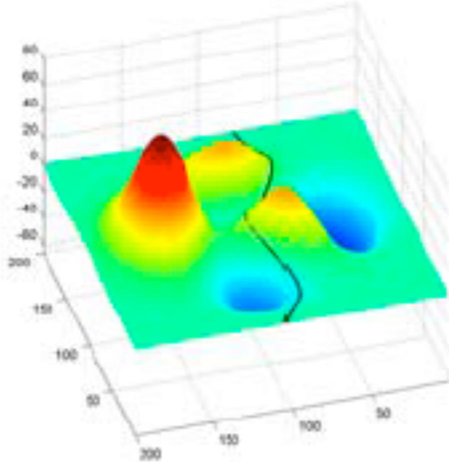
$$\left\|\left(dx_1, \cdots, dx_n\right)\right\| = \sqrt{\sum_{i,j=1}^{n} g_{i,j}(P)\, dx_i dx_j}$$

4. Integrating this, we get the length of paths: $\ell(\gamma) = \int_0^1 \|\dot{\gamma}(t)\|\, dt$

<u>All this will carry over to infinite dimensional manifolds</u>

Ingredient 5. Geodesics

Navigating the earth, a shortest path is seldom a straight line: you must weave to avoid hills and valleys.



Start with the variational principle:

$$\delta\left(\text{path length } = \int_0^1 \left\|\frac{dx}{dt}\right\| dt\right) = 0$$

On a manifold with coordinates $x^1, \ldots, x^n$, get:

$$\frac{d^2x^i}{dt^2}(t) = \sum_{j,k}\Gamma^i_{jk}(x)\cdot\frac{dx^j}{dt}(t)\cdot\frac{dx^k}{dt}(t)$$

<u>This too will work in infinite dimensional manifolds</u>

Exploratory data analysis can be done geodesics

Start with a dataset of points $\{P_i\}$ in $\mathbb{R}^n$

1. Form their mean $\bar{P} = \left(\sum_i P_i\right) \Big/ N$

2. Form their covariance matrix $C = \left(\sum_i (P_i - \bar{P})^t \otimes (P_i - \bar{P})\right) \Big/ N$

3. Take its eigenvectors with large eigenvalues:
   principal components of the dataset

4. In other cases, seek first to break the dataset into clusters
   a. *k*-means
   b. nearest neighbor clustering

   with distinct means and principal components

These are the standard work horses for data in linear spaces. On a manifold, we use geodesics.

Data analysis via geodesics

- Given a dataset $\{ P_i \}$ on a manifold M, its Karcher mean is a point Q minimizing

$$\sum_i \left(\text{length of geodesics } P_i \text{ to } Q\right)^2$$

- Once you have the mean, take the shortest geodesics from each P_i to Q and let $t_i \in T_Q M$ be the tangent vector to this geodesic at Q.
- Then take the principal components via the linear theory on $\{ t_i \}$.
- k-means can also be done via Karcher means.

This approach has been applied, e.g. to the shape of the hippocampus and the diagnosis of schizophrenia and Alzeimer's; to the shape of the heart in various conditions; to the shape of the prostate; etc.

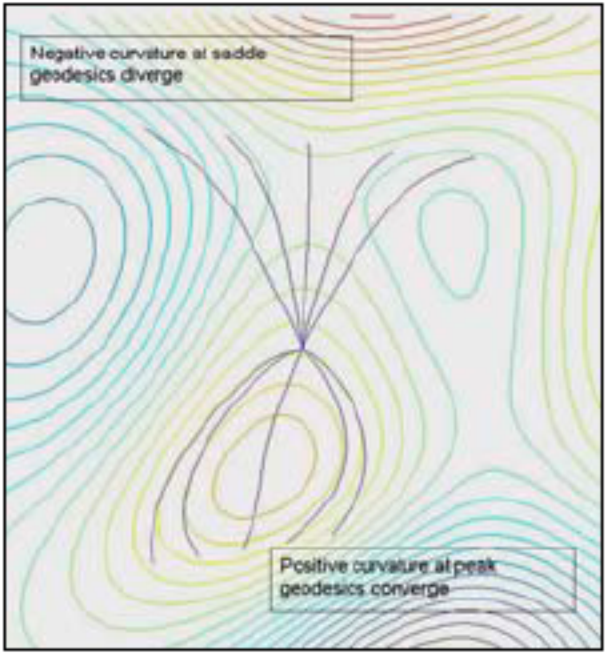
Ingredient 6. Geodesics are not always act like straight lines: **curvature**

The idea of curvature:

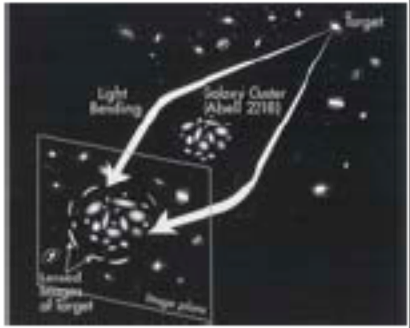
Euclid: parallel lines stay the same distance apart; ZERO CURVATURE

Non-Euclidean geometry (Bolyai, Gauss): geodesics diverge exponentially, e.g. at mountain passes; NEGATIVE CURVATURE

Spherical geometry: great circles come together at antipodes; similar thing at mountain peaks or valleys. POSITIVE CURVATURE



Gravitational lensing: positive curvature in our space-time. What you *see* is not what is *out there*!



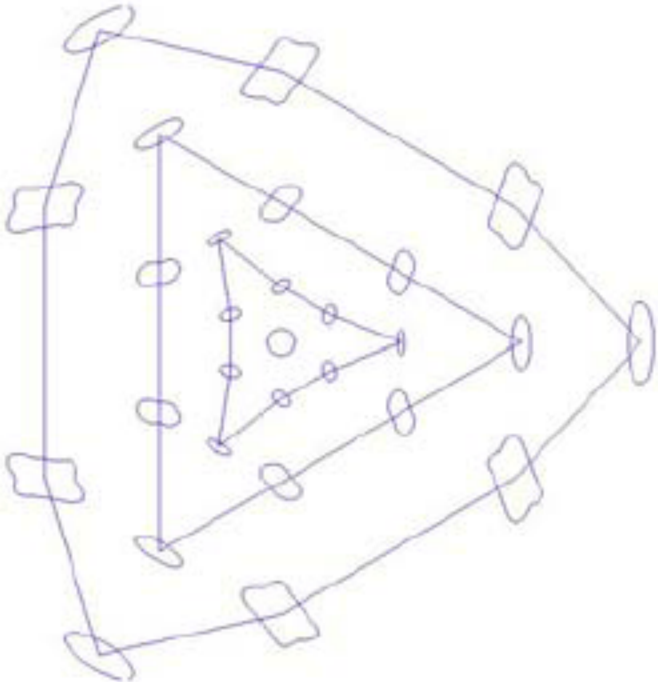
Curvature also carries over to infinite dimensional spaces

Geodesic triangles in the space of plane curves (Michor-M metric):

There is more than one way to rotate an ellipse!

For small shapes, curvature is negative and the path nearly goes back to the circle (= the 'origin'). Angle sum = 102 degrees.

For large shapes, curvature is positive, 2 protrusions grow while 2 shrink. Angle sum = 207 degrees.



Geometry behind curvature

- Get curvature at each point in each 2-plane: Riemann's *sectional curvature* and curvature tensor R_{ijkl}, Ricci and scalar curvatures.
- When positive, beyond *cut locus*, geodesics are not unique. Datasets may not have means.
- When negative, easy to get lost, space is big – but datasets do have means, geodesics are unique.

What will we look at in this course?

- There are *three* very different mathematical approaches to putting Riemannian metrics on the space of shapes
  a. Metrics on the group of diffeomorphisms and its quotients, e.g. *N*-tuples of pts.
  b. Local metrics on the nonlinear Grassmannian of submfds $\{N \subset M \mid M \text{ fixed}\}$
  c. Conformal approaches for plane domains and their boundaries $\Omega \subset \mathbb{C}$, $C = \partial\Omega$ and, esp.:

$$\mathrm{Diff}(S^1)/SL_2 \simeq \{\Omega\}/\text{transl.,scalings}$$

Shape spaces of plane curves:

Some spaces:

$\text{Diff}(S^1)$ a regular Lie group, $= \text{Diff}^+(S^1) \sqcup \text{Diff}^-(S^1)$.

$\text{Emb} = \text{Emb}(S^1, \mathbb{R}^2)$, the manifold of all smooth embeddings $S^1 \to \mathbb{R}^2$.
$T\,\text{Emb}(S^1, \mathbb{R}^2) = \text{Emb}(S^1, \mathbb{R}^2) \times C^\infty(S^1, \mathbb{R}^2)$.

$\text{Imm} = \text{Imm}(S^1, \mathbb{R}^2)$, the manifold of all smooth immersions $S^1 \to \mathbb{R}^2$.
$T\,\text{Imm}(S^1, \mathbb{R}^2) = \text{Imm}(S^1, \mathbb{R}^2) \times C^\infty(S^1, \mathbb{R}^2)$.

$\text{Imm}_{free} = \text{Imm}_{\text{free}}(S^1, \mathbb{R}^2)$, the manifold of all free smooth immersions $S^1 \to \mathbb{R}^2$, i.e., those with trivial isotropy group for the right action of $\text{Diff}(S^1)$ on $\text{Imm}(S^1, \mathbb{R}^2)$.

$B_e = B_e(S^1, \mathbb{R}^2) = \mathrm{Emb}(S^1, \mathbb{R}^2)/\operatorname{Diff}(S^1)$, the manifold of 1-dimensional connected submanifolds of $\mathbb{R}^2$,

$B_i = B_i(S^1, \mathbb{R}^2) = \mathrm{Imm}(S^1, \mathbb{R}^2)/\operatorname{Diff}(S^1)$, an infinite dimensional 'orbifold'

$B_{i,\mathrm{free}} = \mathrm{Imm}_{\mathrm{free}}(S^1, \mathbb{R}^2)/\operatorname{Diff}(S^1)$, a manifold, the base of a principal fiber bundle,

Notation. We work mostly with arclength ds, arclength derivative D_s and the unit tangent vector v to the curve:

$$ds = |c_\theta| d\theta$$
$$D_s = \partial_\theta / |c_\theta|$$
$$v = c_\theta / |c_\theta|$$

Attention: Given a family of curves $c(\theta, t)$, then ∂_θ and ∂_t commute but D_s and ∂_t don't. Rotation through 90 degrees (complex multiplication by $\sqrt{-1}$) will be denoted by:

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The unit normal vector to the image curve is thus

$$n = Jv.$$

Curvature and length on $\mathrm{Imm}(S^1,\mathbb{R}^2)$

$$\kappa : \mathrm{Imm}(S^1,\mathbb{R}^2) \to C^\infty(S^1,\mathbb{R}),$$

$$\kappa(c) = \frac{\det(c_\theta, c_{\theta\theta})}{|c_\theta|^3} = \langle n, D_s v\rangle$$

$$d\kappa(c)(h) = \frac{\langle Jh_\theta, c_{\theta\theta}\rangle}{|c_\theta|^3} + \frac{\langle Jc_\theta, h_{\theta\theta}\rangle}{|c_\theta|^3} - 3\kappa(c)\frac{\langle h_\theta, c_\theta\rangle}{|c_\theta|^2}.$$

$$= \langle D_s^2(h), n\rangle - 2\kappa\langle D_s(h), v\rangle$$

The length function

$$\ell : \mathrm{Imm}(S^1,\mathbb{R}^2) \to \mathbb{R}, \qquad \ell(c) = \int_{S^1} |c_\theta|\, d\theta$$

$$d\ell_c(h) = \int_{S^1} \frac{\langle h_\theta, c_\theta\rangle}{|c_\theta|} d\theta = \int_{S^1} \langle D_s(h), v\rangle ds$$

$$= -\int_{S^1} \langle h, D_s(v)\rangle ds = -\int_{S^1} \kappa(c)\langle h, n\rangle ds$$

The degree of immersions. The degree or rotation degree of an immersion $c : S^1 \to \mathbb{R}^2$ is the winding number around 0 of the tangent $c' : S^1 \to \mathbb{R}^2$. $\mathrm{Imm}(S^1, \mathbb{R}^2)$ decomposes into the disjoint union of the open submanifolds $\mathrm{Imm}^k(S^1, \mathbb{R}^2)$ for $k \in \mathbb{Z}$ according to the degree k. These are connected according to a theorem of Whitney and Graustein (1931-32)

Theorem. *The manifold* $\operatorname{Imm}^k(S^1, \mathbb{R}^2)$ *of immersed curves of degree* k *contains* S^1 *as a strong smooth deformation retract.*
For $k \neq 0$ *the manifold*
$B_i^k(S^1, \mathbb{R}^2) := \operatorname{Imm}^k(S^1, \mathbb{R}^2)/\operatorname{Diff}^+(S^1)$
is contractible.
For $k = 0$ *we have (surprise, Kodama-M.)*

$$\begin{aligned}
&\pi_1(B^0(S^1, \mathbb{R}^2)) = \mathbb{Z}, \\
&\pi_2(B^0(S^1, \mathbb{R}^2)) = \mathbb{Z}, \\
&\pi_k(B^0(S^1, \mathbb{R}^2)) = 0 \qquad \text{for } k > 2.
\end{aligned}$$

The tangent bundle is
$T\,\mathrm{Imm}(S^1,\mathbb{R}^2) = \mathrm{Imm}(S^1,\mathbb{R}^2) \times C^\infty(S^1,\mathbb{R}^2)$, the cotangent bundle is
$T^*\,\mathrm{Imm}(S^1,\mathbb{R}^2) = \mathrm{Imm}(S^1,\mathbb{R}^2) \times \mathcal{D}(S^1)^2$
where the second factor consists of periodic distributions.

We consider smooth Riemannian metrics on $\mathrm{Imm}(S^1,\mathbb{R}^2)$, i.e., smooth mappings

$$G : \mathrm{Imm}(S^1,\mathbb{R}^2) \times C^\infty(S^1,\mathbb{R}^2) \times C^\infty(S^1,\mathbb{R}^2) \to \mathbb{R}$$
$$(c,h,k) \mapsto G_c(h,k), \qquad \text{bilinear in } h,k$$
$$G_c(h,h) > 0 \qquad \text{for } h \neq 0.$$

Each such metric is *weak* in the sense that G_c, viewed as bounded linear mapping

$$G_c : T_c\,\mathrm{Imm}(S^1,\mathbb{R}^2) = C^\infty(S^1,\mathbb{R}^2) \to$$
$$\to T_c^*\,\mathrm{Imm}(S^1,\mathbb{R}^2) = \mathcal{D}(S^1)^2$$
$$G : T\,\mathrm{Imm}(S^1,\mathbb{R}^2) \to T^*\,\mathrm{Imm}(S^1,\mathbb{R}^2)$$
$$G(c,h) = (c, G_c(h, \quad))$$

is injective, but can never be surjective.

In the sequel we shall further assume that that *the weak Riemannian metric G itself admits G-gradients with respect to the variable c in the following sense:*

$$\boxed{dG_c(m)(h,k) = G_c(m, H_c(h,k)) = G_c(K_c(m,h),k)}$$

$$H, K : \text{Imm} \times C^\infty \times C^\infty \to C^\infty$$

$$(c, h, k) \mapsto H_c(h,k), K_c(h,k)$$

smooth and bilinear in h, k.

We will check and compute these gradients for several concrete metrics below.

The fundamental symplectic form on $T\,\mathrm{Imm}(S^1,\mathbb{R}^2)$ pulled back from the canonical symplectic form on the contangent bundle via the mapping $G : T\,\mathrm{Imm}(S^1,\mathbb{R}^2) \to T^*\,\mathrm{Imm}(S^1,\mathbb{R}^2)$ is then:

$$\begin{aligned}
&\omega_{(c,h)}((k_1,\ell_1),(k_2,\ell_2)) = \\
&= -dG_c(k_1)(h,k_2) - G_c(\ell_1,k_2) \\
&\quad + dG_c(k_2)(h,k_1) + G_c(\ell_2,k_1) \\
&= G_c(k_2, H_c(h,k_1) - K_c(k_1,h)) \\
&\quad + G_c(\ell_2,k_1) - G_c(\ell_1,k_2)
\end{aligned}$$

The geodesic equation. The Hamiltonian vector field of the Riemann energy function

$$E(c,h) = \frac{1}{2} G_c(h,h), \qquad E : T\,\mathrm{Imm}(S^1, \mathbb{R}^2) \to \mathbb{R}$$

is the geodesic vector field:

$$\begin{aligned} \mathrm{grad}^{\omega}_1(E)(c,h) &= h \\ \mathrm{grad}^{\omega}_2(E)(c,h) &= \tfrac{1}{2} H_c(h,h) - K_c(h,h) \end{aligned}$$

and the geodesic equation becomes:

$$\begin{cases} c_t &= h \\ h_t &= \tfrac{1}{2} H_c(h,h) - K_c(h,h) \end{cases}$$

$$\boxed{c_{tt} = \tfrac{1}{2} H_c(c_t,c_t) - K_c(c_t,c_t)}$$

The momentum mapping for a G-isometric group action. Consider a (possibly infinite dimensional regular) Lie group with Lie algebra $\mathfrak{g}$ with a right action $g \mapsto r^g$ by isometries on $\mathrm{Imm}(S^1, \mathbb{R}^2)$. Fundamental vector field mapping $\zeta : \mathfrak{g} \to \mathfrak{X}(\mathrm{Imm}(S^1, \mathbb{R}^2))$, a bounded Lie algebra homomorphism, given by

$$\zeta_X(c) = \partial_t|_0 r^{\exp(tX)}(c).$$

momentum map $j : \mathfrak{g} \to C^\infty_G(T\,\mathrm{Imm}(S^1, \mathbb{R}^2), \mathbb{R})$:

$$\boxed{j_X(c,h) = G_c(\zeta_X(c), h).}$$

$$\mathcal{J} : T\,\mathrm{Imm}(S^1, \mathbb{R}^2) \to \mathfrak{g}', \quad \langle \mathcal{J}(c,h), X\rangle = j_X(c,h).$$

It fits into the following commmutative diagram and is a homomorphism of Lie algebras:

$$\begin{tikzcd}
0 \arrow[r] & H^0 \arrow[r, "i"] & C^\infty_G \arrow[r, "\mathrm{grad}^\omega"] & \mathfrak{X}_\omega \arrow[r] & H^1 \arrow[r] & 0 \\
& & & \mathfrak{g} \arrow[u, "\zeta^{T\,\mathrm{Imm}}"'] \arrow[ul, "j"] & &
\end{tikzcd}$$

$\mathcal{J}$ is equivariant for the group action. Along any geodesic $t \mapsto c(t, \quad)$ this momentum mapping is constant, thus for any $X \in \mathfrak{g}$

$$\boxed{\langle \mathcal{J}(c, c_t), X \rangle = j_X(c, c_t) = G_c(\zeta_X(c), c_t)}$$

is constant in t.

We can apply this construction to the following group actions on $\mathrm{Imm}(S^1, \mathbb{R}^2)$.

• The smooth right action of the group $\mathrm{Diff}(S^1)$ on $\mathrm{Imm}(S^1, \mathbb{R}^2)$, given by composition from the right: $c \mapsto c \circ \varphi$ for $\varphi \in \mathrm{Diff}(S^1)$. For $X \in \mathfrak{X}(S^1)$ the fundamental vector field is then given by

$$\zeta_X^{\mathrm{Diff}}(c) = \zeta_X(c) = \partial_t|_0(c \circ \mathrm{Fl}_t^X) = c_\theta.X.$$

The *reparametrization momentum*, for any vector field X on S^1 is thus:

$$j_X(c, h) = G_c(c_\theta.X, h).$$

Assuming the metric is reparametrization invariant, it follows that on any geodesic $c(\theta, t)$, the expression $G_c(c_\theta.X, c_t)$ is constant for all X.

• The left action of the Euclidean motion group $M(2) = \mathbb{R}^2 \rtimes SO(2)$ on $\mathrm{Imm}(S^1, \mathbb{R}^2)$ given by $c \mapsto e^{aJ}c + B$ for $(B, e^{aJ}) \in \mathbb{R}^2 \times SO(2)$. The fundamental vector field mapping is

$$\zeta_{(B,a)}(c) = aJc + B$$

The *linear momentum* is thus $G_c(B, h), B \in \mathbb{R}^2$ and if the metric is translation invariant, $G_c(B, c_t)$ will be constant along geodesics. The *angular momentum* is similarly $G_c(Jc, h)$ and if the metric is rotation invariant, then $G_c(Jc, c_t)$ will be constant along geodesics.

• The action of the scaling group of $\mathbb{R}$ given by $c \mapsto e^r c$, with fundamental vector field $\zeta_a(c) = a.c$. If the metric is scale invariant, then the *scaling momentum* $G_c(c, c_t)$ will also be invariant along geodesics.

If the Riemannian metric G on Imm is invariant under the action of Diff(S^1) it induces a metric on the quotient B_i as follows. For any $C_0, C_1 \in B_i$, consider all liftings $c_0, c_1 \in$ Imm such that $\pi(c_0) = C_0, \pi(c_1) = C_1$ and all smooth curves $t \mapsto (\theta \mapsto c(t,\theta))$ in Imm$(S^1, \mathbb{R}^2)$ with $c(0,\cdot) = c_0$ and $c(1,\cdot) = c_1$. Since the metric G is invariant under the action of Diff(S^1) the arc-length of the curve $t \mapsto \pi(c(t,\cdot))$ in $B_i(S^1, \mathbb{R}^2)$ is given by

$$\begin{aligned}
&L_G^{\text{hor}}(c) := L_G(\pi(c(t,\cdot))) \\
&= \int_0^1 \sqrt{G_{\pi(c)}(T_c\pi.c_t, T_c\pi.c_t)}\,dt \\
&= \int_0^1 \sqrt{G_c(c_t^\perp, c_t^\perp)}\,dt \\
&\text{dist}_G^{B_i(S^1,\mathbb{R}^2)}(C_1, C_2) = \inf_c L_G^{\text{hor}}(c).
\end{aligned}$$

The simplest (L^2-) metric.

$$G^0_c(h,k) = \int_{S^1} \langle h,k\rangle ds = \int_{S^1} \langle h,k\rangle |c_\theta|\, d\theta$$

We compute the G^0-gradients of $c \mapsto G^0_c(h,k)$:

$$dG^0(c)(m)(h,k) = G^0_c(K^0_c(m,h),k) = G^0_c\Big(m, H^0_c(h,k)\Big),$$

$$K^0_c(m,h) = \langle D_s(m), v\rangle h, \qquad D_s = \frac{\partial_\theta}{|c_\theta|}, \qquad v = \frac{c_\theta}{|c_\theta|}.$$

$$H^0_c(h,k) = -D_s\Big(\langle h,k\rangle v\Big)$$

Geodesic equation

$$c_{tt} = -\frac{1}{2|c_\theta|}\partial_\theta\Big(\frac{|c_t|^2\, c_\theta}{|c_\theta|}\Big) - \frac{1}{|c_\theta|^2}\langle c_{t\theta}, c_\theta\rangle c_t.$$

Horizontal Geodesics for G^0

$\langle c_t, c_\theta\rangle = 0$ and $c_t = an = aJ\frac{c_\theta}{|c_\theta|}$ for $a \in C^\infty(S^1,\mathbb{R})$. We use functions a, $s = |c_\theta|$, and κ, only holonomic derivatives:

$$s_t = -a\kappa s, \qquad a_t = \tfrac{1}{2}\kappa a^2,$$
$$\kappa_t = a\kappa^2 + \frac{1}{s}\left(\frac{a_\theta}{s}\right)_\theta = a\kappa^2 + \frac{a_{\theta\theta}}{s^2} - \frac{a_\theta s_\theta}{s^3}.$$

We may assume $s|_{t=0} \equiv 1$. Let $v(\theta) = a(0,\theta)$, the initial value for a. Then

$\frac{s_t}{s} = -a\kappa = -2\frac{a_t}{a}$, so $\log(sa^2)_t = 0$, thus

$s(t,\theta)a(t,\theta)^2 = s(0,\theta)a(0,\theta)^2 = v(\theta)^2$,

a conserved quantity along the geodesic. We substitute $s = \frac{v^2}{a^2}$ and $\kappa = 2\frac{a_t}{a^2}$ to get

$$
\begin{aligned}
&a_{tt} - 4\frac{a_t^2}{a} - \frac{a^6 a_{\theta\theta}}{2v^4} + \frac{a^6 a_\theta v_\theta}{v^5} - \frac{a^5 a_\theta^2}{v^4} = 0,\\
&a(0,\theta) = v(\theta),
\end{aligned}
$$

a nonlinear hyperbolic second order equation. Note that wherever $v = 0$ then also $a = 0$ for all t. So substitute $a = vb$. The outcome is

$$
\boxed{
\begin{aligned}
&(b^{-3})_{tt} = -\frac{v^2}{2}(b^3)_{\theta\theta} - 2vv_\theta(b^3)_\theta - \frac{3vv_{\theta\theta}}{2}b^3,\\
&b(0,\theta) = 1.
\end{aligned}
}
$$

This is the codimension 1 version where Burgers' equation is the codimension 0 version.

Now the big surprise for the L^2-metric:

Theorem. *For* $c_0, c_1 \in \text{Imm}(S^1, \mathbb{R}^2)$ *there exists always a variation through immersions* $t \mapsto c(t,\cdot)$ *with* $c(0,\cdot) = c_0$ *and* $\pi(c(1,\cdot)) = \pi(c_1)$ *for any given immersions* c_0 *and* c_1 *such that* $L_{G^0}^{hor}(c)$ *is arbitrarily small.*

Thus the distance $\text{dist}_{G^0}^{B_i}$ on $B_i(S^1, \mathbb{R}^2)$ vanishes. The simplest (L^2-) metric G^0 is useless on shape space.

The general almost local metric G^Φ.

$$G_c^\Phi(h,k) := \int_{S^1} \Phi(\ell_c, \kappa_c(\theta))\langle h(\theta), k(\theta)\rangle ds.$$

The metric G^Φ is invariant under the reparametization group Diff(S^1) and under the Euclidean motion group.

We compute the G^Φ-gradients of $c \mapsto G^\Phi_c(h,k)$:

$$
\begin{aligned}
dG^\Phi(c)(m)(h,k) &= G^\Phi_c(K^\Phi_c(m,h),k) \\
&= G^\Phi_c\Big(m, H^\Phi_c(h,k)\Big),
\end{aligned}
$$

$$
\begin{aligned}
K^\Phi_c(m,h) &= -\bigg(\int_{S^1}\kappa_c\langle m,n\rangle ds\bigg)\frac{\partial_1\Phi(\ell,\kappa)}{\Phi(\ell,\kappa)}h \\
&+ \frac{\partial_2\Phi(\ell,\kappa)}{\Phi(\ell,\kappa)}\Big(\langle D_s^2(m),n\rangle - 2\kappa\langle D_s(m),v\rangle\Big)h \\
&+ \langle D_s(m),v\rangle h \\
H^\Phi_c(h,k) &= \frac{1}{\Phi(\ell,\kappa)}\bigg(- \Big(\kappa_c\int\partial_1\Phi(\ell,\kappa)\langle h,k\rangle ds\Big)n \\
&+ D_s^2\Big(\partial_2\Phi(\ell,\kappa)\langle h,k\rangle n\Big) + \\
&+ 2D_s\Big(\partial_2\Phi(\ell,\kappa)\kappa\langle h,k\rangle v\Big) - D_s\Big(\Phi(\ell,\kappa)\langle h,k\rangle v\Big)\bigg)
\end{aligned}
$$

Conserved momenta for G^Φ along any geodesic $t \mapsto c(\quad, t)$:

$$\boxed{\begin{array}{ll} \Phi(\ell_c,\kappa_c)\langle v, c_t\rangle |c_\theta|^2 \ \in \mathfrak{X}(S^1) & \text{reparam. mom.} \\ \int_{S^1} \Phi(\ell_c,\kappa_c) c_t ds \ \in \mathbb{R}^2 & \text{linear moment.} \\ \int_{S^1} \Phi(\ell_c,\kappa_c)\langle Jc, c_t\rangle ds \in \mathbb{R} & \text{angular moment.} \end{array}}$$

Setting the reparametrization momentum to 0 and doing symplectic reduction amounts exactly to investigating the quotient space

$B_i(S^1, \mathbb{R}^2) = \mathrm{Imm}(S^1, \mathbb{R}^2)/\operatorname{Diff}(S^1)$

and using horizontal geodesics for doing so; a horizontal geodesic is G^Φ-normal to the $\operatorname{Diff}(S^1)$-orbits. If it is normal at one time it is normal forever (since the reparametrization momentum is conserved).

Horizontality for G^Φ.
$T_c(c \circ \mathrm{Diff}(S^1)) = \{X.c_\theta : X \in C^\infty(S^1, \mathbb{R})\}$. Thus the bundle of horizontal vectors is

$$\begin{aligned}\mathcal{N}_c &= \{h \in C^\infty(S^1, \mathbb{R}^2) : \langle h, v\rangle = 0\}\\ &= \{a.n \in C^\infty(S^1, \mathbb{R}^2) : a \in C^\infty(S^1, \mathbb{R})\}\end{aligned}$$

A tangent vector $h \in T_c \mathrm{Imm}(S^1, \mathbb{R}^2) = C^\infty(S^1, \mathbb{R}^2)$ has an orthonormal decomposition

$$\begin{aligned}h &= h^\top + h^\perp \in T_c(c \circ \mathrm{Diff}^+(S^1)) \oplus \mathcal{N}_c\\ h^\top &= \langle h, v\rangle v \in T_c(c \circ \mathrm{Diff}^+(S^1)),\\ h^\perp &= \langle h, n\rangle n \in \mathcal{N}_c,\end{aligned}$$

into smooth tangential and normal components, independent of the choice of $\Phi(\ell, \kappa)$.

Consider a path $t \mapsto c(\cdot, t)$ in the manifold $\mathrm{Imm}(S^1, \mathbb{R}^2)$. It projects to a path $\pi \circ c$ in $B_i(S^1, \mathbb{R}^2)$ whose energy is called the *horizontal energy* of c:

$$E^{\mathsf{hor}}_{G^\Phi}(c) = \tfrac{1}{2}\int_a^b \int_{S^1} \Phi(\ell_c, \kappa_c)\langle c_t, n\rangle^2 \, d\theta dt$$
$$= \tfrac{1}{2}\int_{[a,b]\times S^1} \Phi(\ell_c, \kappa_c)\frac{|n_S^0|^2}{\sqrt{1-|n_S^0|^2}}\, d\mu_S$$

Here the final expression is only in terms of the surface S and its fibration over the time axis, and is valid for any path c. This anisotropic area functional has to be minimized in order to prove that geodesics exists between arbitrary curves (of the same degree) in $B_i(S^1, \mathbb{R}^2)$.

The horizontal geodesic equation.

Let $c(\theta,t)$ be a horizontal geodesic for the metric G^{Φ}. Then $c_t(\theta,t) = a(\theta,t).n(\theta,t)$. Denote the integral of a function over the curve with respect to arclength by a bar. Then the geodesic equation for horizontal geodesics is:

$$\boxed{\begin{aligned} a_t = \frac{1}{2\Phi}\Big(& \left(-\kappa\Phi + \kappa^2\partial_2\Phi\right) a^2 \\ & - D_s^2\left(\partial_2\Phi \cdot a^2\right) + 2\partial_2\Phi \cdot a D_s^2(a) \\ & - 2\partial_1\Phi \cdot \overline{(\kappa a)} \cdot a + \overline{(\partial_1\Phi \cdot a^2)} \cdot \kappa \Big) \end{aligned}}$$

Curvature on B_i for G^{Φ}.

Let $W(\theta_1, \theta_2) = h(\theta_1)m(\theta_2) - h(\theta_2)m(\theta_1)$
so that its second derivative
$\partial_2 W(\theta_1, \theta_1) = W_2(\theta_1, \theta_1) = h(\theta_1)m'(\theta_1) - h'(\theta_1)m(\theta_1)$
is the Wronskian of h and m.

$$
\begin{aligned}
&R_0^\Phi(m,h,m,h) = G_0^\Phi(R_0(m,h)m,h) = \\
&= \int \Big(\kappa.\Phi_2 - \frac{\Phi}{2} + \frac{\Phi_2.\Phi_2'' - 2(\Phi_2')^2 - (\Phi_2\kappa)^2}{2\Phi}\Big)(\theta_1) W_2(\theta_1,\theta_1)^2\, d\theta_1 \\
&+ \int \frac{\Phi_{22}(\theta_1)}{2} W_{22}(\theta_1,\theta_1)^2\, d\theta_1 \\
&+ \iint \Big(\frac{\Phi_1'\Phi_2}{\Phi} - \frac{\Phi_1\Phi_2\Phi_1'}{\Phi^2}\Big)(\theta_1) W_2(\theta_1,\theta_1) \int W(\theta_1,\theta_2)\kappa(\theta_2)\, d\theta_2\, d\theta_1 \\
&+ \iint \Big(\frac{\Phi_1\Phi_2}{\Phi} - \Phi_{12}\Big)(\theta_1) W_{22}(\theta_1,\theta_1) \int W(\theta_1,\theta_2)\kappa(\theta_2)\, d\theta_2\, d\theta_1 \\
&+ \iint \frac{\Phi_1(\theta_1)}{2}\Big(1 - \frac{\Phi_2.\kappa}{\Phi}(\theta_2)\Big) W_1(\theta_1,\theta_2)^2\, d\theta_2\, d\theta_1 \\
&+ \iint \Big(\frac{\Phi_2.\kappa^3 - \Phi_2''.\kappa}{4\Phi} - \frac{\kappa^2}{4} + \Big(\frac{\Phi_2'.\kappa}{2\Phi}\Big)' + \overline{\Big(\frac{\kappa^2}{8\Phi}\Big)}.\Phi_1\Big)(\theta_1) \\
&\qquad\qquad \Phi_1(\theta_2) W(\theta_1,\theta_2)^2\, d\theta_2\, d\theta_1 \\
&+ \iiint \Big(\Big(\frac{\Phi_{11}}{2} - \frac{\Phi_1^2}{4\Phi}\Big)(\theta_1) - \Phi_1(\theta_1)\frac{\Phi_1}{2\Phi}(\theta_2)\Big) \\
&\qquad\qquad \kappa(\theta_2)\kappa(\theta_3) W(\theta_1,\theta_2) W(\theta_1,\theta_3)\, d\theta_2\, d\theta_1\, d\theta_3
\end{aligned}
$$

Special case: the metric G^A.

If we choose $\Phi(\ell_c, \kappa_c) = 1 + A\kappa_c^2$ then we obtain the metric we have investigated before:

$$G_c^A(h,k) = \int_{S^1} (1 + A\kappa_c(\theta)^2)\langle h(\theta), k(\theta)\rangle ds.$$

The horizontal geodesic equation for the G^A-metric reduces to

$$\begin{aligned} a_t = \frac{1}{1 + A\kappa_c^2}\Big(& -\tfrac{1}{2}\kappa_c a^2 \\ & + A\Big(a^2(-D_s^2(\kappa_c) + \tfrac{1}{2}\kappa_c^3) \\ & \quad - 4D_s(\kappa_c) a D_s(a) - 2\kappa_c D_s(a)^2\Big)\Big) \end{aligned}$$

Along a geodesic $t \mapsto c(t, \quad)$ we have the following conserved quantities:

$$(1 + A\kappa_c^2)\langle v, c_t\rangle |c_\theta|^2 \in \mathfrak{X}(S^1) \qquad \text{reparam. mom.}$$
$$\int_{S^1} (1 + A\kappa_c^2) c_t ds \in \mathbb{R}^2 \qquad \text{linear momentum}$$
$$\int_{S^1} (1 + A\kappa_c^2)\langle Jc, c_t\rangle ds \in \mathbb{R} \qquad \text{angular momentum}$$

Lipschitz continuity of $\sqrt{\ell} : B_i \to \mathbb{R}_{\geq 0}$.
For C_0 and C_1 in $B_i = \mathrm{Imm} / \mathrm{Diff}(S^1)$ we have for $A > 0$:

$$\sqrt{\ell(C_1)} - \sqrt{\ell(C_0)} \leq \frac{1}{2\sqrt{A}} \mathrm{dist}_{G^A}^{B_i(S^1,\mathbb{R}^2)}(C_1, C_2).$$

Area swept out bound.

If c is any path from C_0 to C_1, then

$$\begin{pmatrix}\textit{area of the region}\\ \textit{swept out by the}\\ \textit{variation } c\end{pmatrix} \leq \max_t \sqrt{\ell(c(t,\cdot))} \cdot L^{hor}_{G^A}(c).$$

Maximum distance bound.

Consider $\epsilon < \min\{\sqrt{A\ell}/4, \ell^{3/4}/\sqrt{8}\}$ and let $\eta = 4(\ell^{3/4}A^{-1/4} + \ell^{1/4})\sqrt{\epsilon}$. Then for any path c starting at C_0 whose length $L^{hor}_{G^A}$ is ϵ, the final curve lies in the tubular neighborhood of C_0 of width η. More precisely, if we choose the path $c(t,\theta)$ to be horizontal, then

$\max_\theta |c(0,\theta) - c(1,\theta)| < \eta$.

Corollary.

For any $A > 0$, the map from $B_i(S^1, \mathbb{R}^2)$ in the G^A metric to the space $B_i^{cont}(S^1, \mathbb{R}^2)$ in the Frechet metric is continuous, and, in fact, uniformly continuous on every subset where the length ℓ is bounded. In particular, G^A is a separating metric on $B_i(S^1, \mathbb{R}^2)$. Moreover, the completion $\overline{B_i}(S^1, \mathbb{R}^2)$ of $B_i(S^1, \mathbb{R}^2)$ in this metric can be identified with a subset of $B_i^{lip}(S^1, \mathbb{R}^2)$.

Explicit equicontinuity bounds, under appropriate parameterization.

Corollary.

If a path $c(\theta, t), 0 \leq t \leq 1$ *satisfies:*

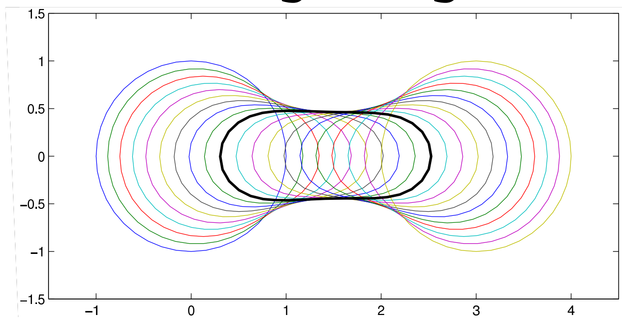
- $|c_\theta(\theta, t)| \equiv \ell(t)/2\pi$ *for all* θ, t,
- $\langle c_t, c_\theta \rangle(0, t) \equiv 0$ *in a base point* 0 *for all* t
- $\int_{C_t} (1 + A\kappa_{C_t}^2) |\langle c_t, ic_\theta \rangle|^2 d\theta / |c_\theta| \equiv L^2$ *for all* t,

then

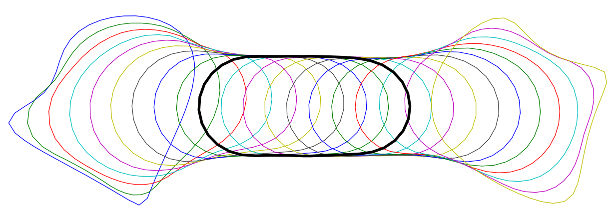
$$|c(\theta_1, t_1) - c(\theta_2, t_2)| \leq \frac{\ell_{\max}}{2\pi} |\theta_1 - \theta_2| + \\ + 7(\ell_{\max}^{3/4}/A^{1/4} + \ell_{\max}^{1/4}) \sqrt{L(t_1 - t_2)} \quad (1)$$

whenever $|t_1 - t_2| \leq \min(2\sqrt{A\ell_{\min}}, \ell_{\min}^{3/2})/(8L)$.

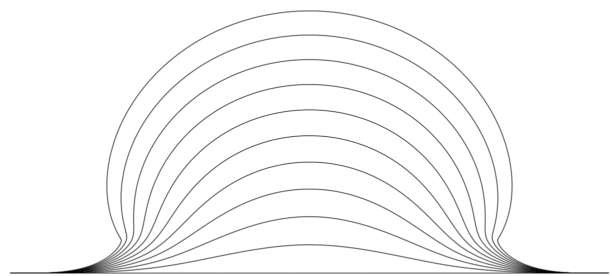
A numerical simulation of the geodesic connecting two circles. Minimize $E^{\text{hor}}_{G^1}(c)$ for variations c with initial and end curves unit circles at distance 3 produced the following image for the geodesic:



The geodesic joining 2 'random' shapes of size about 1 at distance 5 apart with $A = .25$ (using 20 time samples and a 48-gon approximation for all curves).

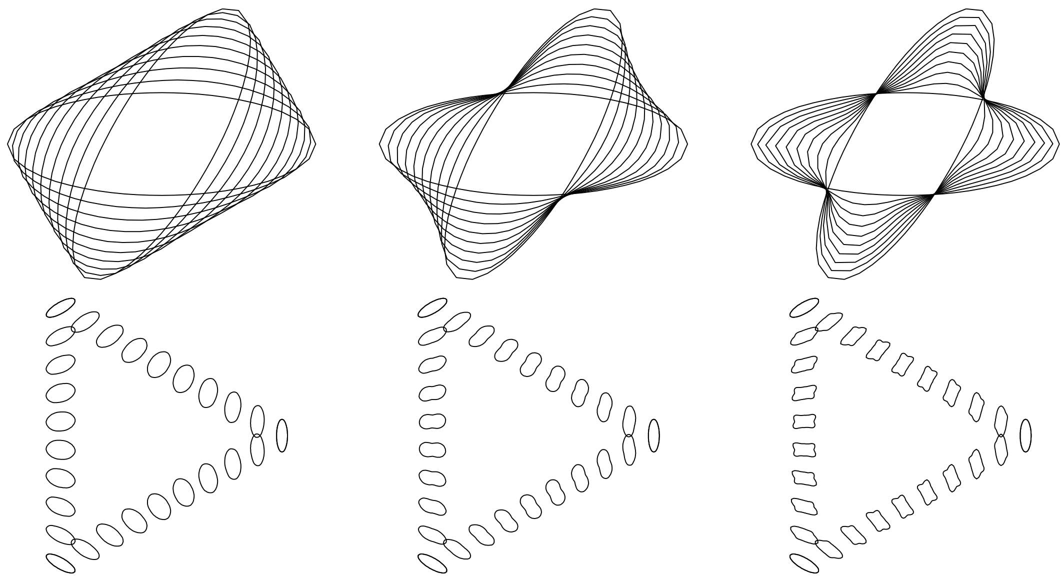


The forward integration of the geodesic equation when $A = 0$, starting from a straight line in the direction given by a smooth bump-like vector field. Note that two corner like singularities with curvature going to ∞ are about to form.



Top Row: Geodesics in 3 metrics joining the same two ellipses. Ellipses have eccentricity 3, same center and are rotated at 60° degree.

$A = 1$; $A = 0.1$; $A = 0.01$.



Bottom Row: Geodesic triangles in B_e formed by joining three ellipses at angles 0, 60 and 120 degrees, for the same three values of A. Here the intermediate shapes are just rotated versions of the geodesic in the top row but are laid out on a plane triangle for visualization purposes.

The sectional curvature on B_i

$$
\begin{aligned}
&R_0(a,b,a,b) = G_0^A(R_0(a,b)a,b) = \\
&= \int_{S^1} \left(\tfrac{1}{2}(A\kappa^2 - 1)(ab' - a'b)^2 + A(ab'' - a''b)^2 \right) d\theta \\
&+ \int_{S^1} \frac{A\kappa^2 - A^2\kappa^4 + 2A^2\kappa\kappa'' - 4A^2\kappa'^2}{1 + A\kappa^2}(ab' - a'b)^2 d\theta \\
&= \int_{S^1} \frac{-(A\kappa^2 - 1)^2 + 4A^2\kappa\kappa'' - 8A^2\kappa'^2}{2(1 + A\kappa^2)} W(a,b)^2 \, d\theta \\
&+ \int_{S^1} A \, W(a,b)'^2 \, d\theta
\end{aligned}
$$

where $W(a,b) = ab' - a'b$ is the Wronskian of a and b.

Special case: the conformal metrics

$\Phi(\ell(c),\kappa(c)) = \Phi(\ell(c))$, metric proposed by Menucci and Yezzi and, for Φ linear, independently by Shah:

$$G_c^\Phi(h,k) = \Phi(\ell_c)\int_{S^1}\langle h,k\rangle ds = \Phi(\ell_c)G_c^0(h,k).$$

All these metrics are conformally equivalent to the basic L^2-metric G^0.

As they show, the infimum of path lengths in this metric is positive so long as Φ satifies an inequality $\Phi(\ell) \geq C.\ell$ for some $C > 0$.

More precisely (Shah), if Area(c) is area swept over by the path c,

$$\begin{aligned}
\mathrm{dist}_{G^\ell}(C_0, C_1) &= \inf_c \mathrm{Area}(c) \\
\sqrt{Ae}.\inf_c \mathrm{Area}(c) &\leq \mathrm{dist}_{G^{e^{A\ell}}}(C_0, C_1) \leq \\
&\leq \sqrt{Ae}.e^{A\ell_{\max}} \inf_c \mathrm{Area}(c)
\end{aligned}$$

The horizontal geodesic equation reduces to:

$$a_t = -\frac{\kappa}{2}a^2 + \frac{\partial_1\Phi}{\Phi}\cdot\left(\tfrac{1}{2}\left(\int a^2.ds\right)\kappa - \left(\int \kappa.a.ds\right)a\right)$$

If we change variables and write $b(s,t) = \Phi(\ell(t)).a(s,t)$, then this equation simplifies to:

$$\boxed{b_t = -\frac{\kappa}{2\Phi}\left(b^2 - \frac{\partial_1\Phi}{\Phi}\int b^2\,ds\right)}$$

Along a geodesic $t \mapsto c(t, \quad)$ we have the following conserved quantities:

$\Phi(\ell_c)\langle v, c_t\rangle |c'(\theta)|^2 \in \mathfrak{X}(S^1)$ reparam. moment.

$\Phi(\ell_c)\int_{S^1} c_t ds \in \mathbb{R}^2$ linear moment.

$\Phi(\ell_c)\int_{S^1} \langle Jc, c_t\rangle ds \in \mathbb{R}$ angular moment.

Curvature on B_i for the conformal metrics. Sectional curvature has been computed by J. Shah. Let g, h be orthonomal, then

$$
\begin{aligned}
&\text{Curv. in plane } \langle g.h\rangle \\
&= \frac{\Phi}{2}\cdot\overline{(g.D_s(h)-h.D_s(g))^2} + \frac{\partial_1\Phi}{4\Phi}\cdot\Big(\overline{g^2.\kappa^2}+\overline{h^2.\kappa^2}\Big) \\
&+ \frac{3\partial_1\Phi^2 - 2\Phi.\partial_1^2\Phi}{4\Phi^2}\cdot\left(\overline{(g.\kappa)}^2+\overline{(h.\kappa)}^2\right) \\
&- \frac{\partial_1\Phi}{2\Phi}\cdot\left(\overline{D_s(g)^2}+\overline{D_s(h)^2}+\frac{\partial_1\Phi}{2\Phi^2}\cdot\overline{\kappa^2}\right)
\end{aligned}
$$

Note that the first two lines are positive while the last line is negative. The first term is the curvature term for the H^0-metric. The key point about this formula is how many positive terms it has.

Special case: the smooth scale invariant metric G^{SI}

$\Phi(\ell, \kappa) = \ell^{-3} + A\frac{\kappa^2}{\ell}$ gives the metric:

$$G_c^{SI}(h,k) = \int_{S^1} \left(\frac{1}{\ell_c^3} + A\frac{\kappa_c^2}{\ell_c}\right)\langle h, k\rangle ds.$$

The beauty of this metric is that (a) it is scale invariant and (b) $\log(\ell)$ is Lipschitz, hence the infimum of path lengths is always positive.

Horizontal geodesics in this metric as special case of the equation for G^{Φ}:

$$
\begin{aligned}
a_t = \frac{1}{1+A(\ell\kappa)^2}\Bigg(&\left(-1+A(\ell\kappa)^2\right)\frac{\kappa a^2}{2} \\
&- 2A\ell^2\kappa D_s(a)^2 - 4A\ell^2 D_s(\kappa) a D_s(a) \\
&+ \left(3+A(\ell\kappa)^2\right)\overline{(a\kappa)}\cdot a - \frac{3}{2}\overline{(a^2)}\cdot\kappa \\
&- \frac{A\ell^2}{2}\overline{(\kappa a)^2}\cdot\kappa \Bigg)
\end{aligned}
$$

where the “overline” stands now for the *average* of a function over the curve, i.e. $\int\cdots ds/\ell$.

Since this metric is scale invariant, there are now *four* conserved quantities, instead of three:

$\Phi(\ell,\kappa)\langle v, c_t\rangle \lvert c'(\theta)\rvert^2 \in \mathfrak{X}(S^1)$	reparam. mom.
$\int_{S^1} \Phi(\ell,\kappa) c_t ds \in \mathbb{R}^2$	linear moment.
$\int_{S^1} \Phi(\ell,\kappa)\langle Jc, c_t\rangle ds \in \mathbb{R}$	angular moment.
$\int_{S^1} \Phi(\ell,\kappa)\langle c, c_t\rangle ds \in \mathbb{R}$	scaling moment.

The Wasserstein metric and a related G^{Φ}-metric. The Wasserstein metric (also known as the Monge-Kantorovich metric) is a metric between probability measures on a common metric space. Let μ and ν be 2 probability measures on a metric space (X, d). Consider all measures ρ on $X \times X$ whose marginals under the 2 projections are μ and ν. Then:

$$d_{\text{wass}}(\mu, \nu) = \inf_{\rho} \iint_{X \times X} d(x, y) d\rho(x, y).$$

where inf is over all ρ with $\text{pr}_{1,*}(\rho) = \mu$ and $\text{pr}_{2,*}(\rho) = \nu$.

The Wasserstein norm is sandwiched between $G^{\ell^{-1}}$ and G^{Φ_W} for $\Phi_W = \frac{1}{\ell} + \frac{1}{12}\ell\kappa^2$.

Immersion-Sobolev metrics on $\mathrm{Imm}(S^1, \mathbb{R}^2)$ **and on** B_i

Note that $D_s = \frac{\partial_\theta}{|c_\theta|}$ is anti self-adjoint for the metric G^0, i.e., for all $h, k \in C^\infty(S^1 . \mathbb{R}^2)$ we have

$$\int_{S^1} \Big\langle D_s(h), k \Big\rangle ds = \int_{S^1} \Big\langle h, -D_s(k) \Big\rangle ds$$

The metric:

$$\begin{aligned} G_c^{\mathrm{imm},n}(h,k) &= \int_{S^1} \left(\langle h, k \rangle + A. \langle D_s^n h, D_s^n k \rangle \right) .ds \\ &= \int_{S^1} \langle L_n(h), k \rangle ds \qquad \text{where} \end{aligned}$$

$$L_n(h) \text{ or } L_{n,c}(h) = I + (-1)^n A. D_s^{2n}(h)$$

Geodesics in the $H^{\mathsf{imm},n}$-metric

$$\boxed{\begin{aligned}(L_n(c_t))_t &= -\langle L_n(c_t), D_s(c_t)\rangle v \\ &\quad - \frac{|c_t|^2\kappa(c)}{2} n - \langle D_s(c_t), v\rangle L_n c_t \\ &\quad + \frac{A}{2}. \sum_{j=1}^{2n-1} (-1)^{n+j} \langle D_s^{2n-j} c_t, D_s^j c_t\rangle \kappa(c) n\end{aligned}}$$

Existence of geodesics. Theorem
Let $n \geq 1$. For each $k \geq 2n+1$ the geodesic equation has unique local solutions in the Sobolev space of H^k-immersions. The solutions depend C^∞ on t and on the initial conditions $c(0,\ .\ )$ and $c_t(0,\ .\ )$. The domain of existence (in t) is uniform in k and thus this also holds in $\mathrm{Imm}(S^1, \mathbb{R}^2)$.

Sketch of Proof Flow equation of a smooth (C^∞) vector field on the H^2-open set $U^k \times H^k(S^1, \mathbb{R}^2)$ in the Sobolev space $H^k(S^1, \mathbb{R}^2) \times H^k(S^1, \mathbb{R}^2)$ where $U^k = \{c \in H^k : |c_\theta| > 0\} \subset H^k$ is H^2-open.

$$
\begin{aligned}
c_t &= u =: X_1(c,u) \\
u_t &= L_{n,c}^{-1}\Big(-\langle L_{n,c}(u), D_s(u)\rangle D_s(c) \\
&\quad - \frac{|c_t|^2\kappa(c)}{2} J D_s(c) - \langle D_s(u), D_s c\rangle u \\
&\quad + \frac{A}{2}. \sum_{j=1}^{2n-1} (-1)^{n+j} \langle D_s^{2n-j} u, D_s^j u\rangle \kappa(c) J D_s(c) \\
&\quad + (-1)^n A. \sum_{j=1}^{2n-1} D_s^j \Big(\langle D_s(u), D_s(c)\rangle D_s^{2n-j}(u)\Big) \Big) \\
&=: X_2(c,u)
\end{aligned}
$$

The conserved momenta of $G^{\mathrm{imm},n}$ along any geodesic $t \mapsto c(t, \quad)$:

$\langle c_\theta, L_{n,c}(c_t)\rangle \lvert c'(\theta)\rvert \ \in \mathfrak{X}(S^1)$	repar. moment.
$\displaystyle\int_{S^1} L_{n,c}(c_t)\, ds \ \in \mathbb{R}^2$	linear moment.
$\displaystyle\int_{S^1} \langle Jc, L_{n,c}(c_t)\rangle\, ds \in \mathbb{R}$	angular moment.

Horizontality for $G^{\text{imm},n}$ $h \in T_c\text{Imm}(S^1,\mathbb{R}^2)$ is $G_c^{\text{imm},n}$-orthogonal to the Diff(S^1)-orbit through c if and only if

$$0 = G_c^{\text{imm},n}(h, \zeta_X(c)) = \int_{S^1} X.\langle L_{n,c}(h), c_\theta\rangle\, ds$$

for all $X \in \mathfrak{X}(S^1)$. So the $G^{\text{imm},n}$-normal bundle is given by

$$\mathcal{N}_c^n = \{h \in C^\infty(S,\mathbb{R}^2) : \langle L_{n,c}(h), v\rangle = 0\}.$$

The G^n-orthonormal projection $T_c\text{Imm} \to \mathcal{N}_c^n$, denoted by $h \mapsto h^\perp = h^{\perp,G^n}$ and the complementary projection $h \mapsto h^\top \in T_c(c \circ \text{Diff}(S^1))$ are 1-dimensional pseudo-differential operators.

They are determined as follows:

$h^\top = X(h).v$ where $\langle L_{n,c}(h), v\rangle = \langle L_{n,c}(X(h).v), v\rangle$

Thus we are led to consider the linear differential operators associated to $L_{n.c}$

$$\begin{aligned} &L_c^\top, L_c^\perp : C^\infty(S^1) \to C^\infty(S^1), \\ &L_c^\top(f) = \langle L_{n,c}(f.v), v\rangle = \langle L_{n,c}(f.n), n\rangle, \\ &L_c^\perp(f) = \langle L_{n,c}(f.v), n\rangle = -\langle L_{n,c}(f.n), v\rangle. \end{aligned}$$

The operator $L_c^\top$ is of order $2n$ and also unbounded, self-adjoint and positive on $L^2(S^1, |c_\theta|\,d\theta)$. In particular, $L_c^\top$ is injective. $L_c^\perp$, on the other hand is of order $2n-1$ and is skew-adjoint. For example, if $n = 1$, then one finds that:

$$\begin{aligned} L_c^\top &= -A.D_s^2 + (1 + A.\kappa^2).I \\ L_c^\perp &= -2A.\kappa.D_s - A.D_s(\kappa).I \end{aligned}$$

The operator $L_c^\top : C^\infty(S^1) \to C^\infty(S^1)$ *is invertible.* This is by deformation invariance of the index.

We want to go back and forth between the 'natural' horizontal space of vector fields $a.n$ and the $G^{\text{imm},n}$-horizontal vector fields $\{h \mid \langle Lh, v\rangle = 0\}$: We use $C_c : C^\infty(S^1, \mathbb{R}^2) \to C^\infty(S^1)$ given by

$$C_c(h) := (L_c^\top)^{-1} \circ L_c^\perp,$$

a pseudo-differential operator of order -1 so that

$$a.n + C(a).v \quad \text{is } H^{\text{imm},n}\text{-horizontal}$$

The restriction of the metric $G^{\mathrm{imm},n}$ to horizontal vector fields $h_i = a_i.n + b_i.v$ can be computed like this:

$$\begin{aligned} G_c^{\mathrm{imm},n}(h_1, h_2) &= \int_{S^1} \langle L h_1, h_2 \rangle .ds \\ &= \int_{S^1} \Big(L^\top + L^\perp \circ C \Big) a_1.a_2.ds. \end{aligned}$$

Thus the metric restricted to horizontal vector fields is given by the pseudo differential operator $L^{\mathrm{red}} = L^\top + L^\perp \circ (L^\top)^{-1} \circ L^\perp$.

The metric on the cotangent space to B_i, is simple. On the smooth cotangent space $C^\infty(S^1,\mathbb{R}^2) \cong G^0_c(T_c\,\text{Imm}(S^1,\mathbb{R}^2)) \subset \mathcal{D}(S^1)^2$ the dual metric is given by convolution with the elementary kernel K_n.

$$\check{G}^n_c(a_1,a_2) = \iint_{S^1\times S^1} K_n(s_1-s_2).$$
$$.\langle n_c(s_1), n_c(s_2)\rangle.a_1(s_1).a_2(s_2).ds_1 ds_2.$$

Horizontal geodesics

For any smooth path c *in* $\mathrm{Imm}(S^1, \mathbb{R}^2)$ *there exists a smooth path* φ *in* $\mathrm{Diff}(S^1)$ *with* $\varphi(t, \quad) = \mathrm{Id}_{S^1}$ *depending smoothly on* c *such that the path* e *given by* $e(t,\theta) = c(t, \varphi(t,\theta))$ *is horizontal:* $\langle L_{n,c}(e_t), e_\theta \rangle = 0$*.*

We may specialize the general geodesic equation to horizontal paths and then take the v and n parts of the geodesic equation. For a horizontal path we may write $L_{n,c}(c_t) = \tilde{a}n$ for $\tilde{a}(t,\theta) = \langle L_{n,c}(c_t), n \rangle$. The v part of the equation turns out to vanish identically and then n part gives us

$$\boxed{\begin{aligned}\tilde{a}_t = & -\frac{|c_t|^2\kappa(c)}{2} - \langle D_s c_t, v\rangle \tilde{a} + \\ & + \frac{\kappa(c)}{2}\sum_{j=1}^{2n-1}(-1)^{n+j}\langle D_s^{2n-j}c_t, D_s^j c_t\rangle\end{aligned}}$$

A Lipschitz bound for arclength in $G^{\mathrm{imm},n}$

$$\boxed{|\sqrt{\ell(C_1)} - \sqrt{\ell(C_0)}| \le \frac{C(A,n)}{2}\,\mathrm{dist}_{G^n}^{B_i}(C_1, C_0)}$$

The scale invariant Sobolov H^1-metric and its relation to the Grassmannian of 2-planes in an infinite dimensional space, and Neretin geodesics.

$$
\begin{aligned}
G_c(h,k) &= \lim_{A\to\infty} \frac{1}{A} G_c^{\text{imm},\text{scal},1}(h,k) \\
&= \frac{1}{\ell(c)} \int_{S^1} \langle D_s h, D_s k\rangle \, ds \\
&= \frac{1}{\ell(c)} \int_{S^1} \langle h, -D_s^2 k\rangle \, ds
\end{aligned}
$$

on $\text{Imm}\,/\text{translations}$ or $\{c \in \text{Imm} : c(1) = 0\}$.

Geodesics in this metric

$$\begin{aligned} c_{tt} = &-\tfrac{1}{2}D_s^{-2}\Big(\kappa_c n_c\Big)\|c_t\|^2_{G_c} - \tfrac{1}{2}D_s^{-1}\Big(|D_s c_t|^2 v_c\Big) \\ &- \frac{1}{\ell_c}\int \kappa_c\langle c_t, n_c\rangle\, ds\cdot c_t - D_s^{-1}\Big(\langle D_s c_t, v_c\rangle D_s c_t\Big) \end{aligned}$$

The conserved momenta of $G^{\text{imm},n}$ along any geodesic $t \mapsto c(t, \quad)$:

$$\boxed{\begin{array}{ll} \dfrac{-1}{\ell(c)}\langle c_\theta, D_s^2(c_t)\rangle|c'(\theta)| \ \in \mathfrak{X}(S^1) & \text{repar. moment.} \\ \dfrac{-1}{\ell(c)}\displaystyle\int_{S^1} D_s^2(c_t)\, ds = 0 \ \in \mathbb{R}^2 & \text{linear moment.} \\ \dfrac{-1}{\ell(c)}\displaystyle\int_{S^1}\langle ic, D_s^2(c_t)\rangle\, ds \in \mathbb{R} & \text{angular moment.} \\ \dfrac{-1}{\ell(c)}\displaystyle\int_{S^1}\langle c, D_s^2(c_t)\rangle\, ds = \partial_t \log(\ell(t)) & \text{scaling moment.} \end{array}}$$

Thm. *For each $k \geq 3/2$ this geodesic equation has unique local solutions in the Sobolev space of H^k-immersions. The solutions depend C^∞ on t and on the initial conditions $c(0,\ .\ )$ and $c_t(0,\ .\ )$. The domain of existence (in t)*
is uniform in k and thus this also holds in $\mathrm{Imm}_* := \{c \in \mathrm{Imm}(S^1, \mathbb{R}^2) : c(1) = 0\}$.

Sphere, Stiefel, and Grassmannian

$V := \{f \in C^\infty(\mathbb{R}, \mathbb{R}) : f(x + 2\pi) = \mp f(x)\}$
below only $-$: odd case. $+$: even case.
$\|f\|^2 = \int_0^{2\pi} f^2\,dx$ weak inner product on V.

$\mathbf{Gr}(2, V)$ Grassmannian of oriented 2-planes.
$T_W\,\mathbf{Gr} = L(W, W^\perp)$ with metric
$\|v\|^2 = \mathrm{tr}(v^\top \circ v) = \|v(e)\|^2 + \|v(f)\|^2$,
e, f orthonormal basis of W.

For $W \in \mathbf{Gr}(2, V)$ let
$Z(V) = \{x : f(x) = 0 \forall f \in W\}$.
$\mathbf{Gr}^0(2, V) = \{W \in \mathbf{Gr}(2, V) : Z(W) = \emptyset\}$ open in $\mathbf{Gr}(2, V)$.

The Stiefel manifold $\mathrm{St}(2,V)$ of orthonormal pairs in V.
$\mathbf{St}^0(2,V) = \{(e,f) \in \mathbf{St} : Z(e,f) = \emptyset\}$ open in $\mathbf{St}$.
$T_{(e,f)}\mathbf{St} = \{(\delta e, \delta f) \in V^2 : 0 = \langle e, de\rangle = \langle f, \delta f\rangle = \langle e, \delta f\rangle + \langle f, \delta e\rangle\}$
Metric $\|(\delta e, \delta f)\|^2 = \|\delta e\|^2 + \|\delta f\|^2$.

$\mathbf{St}(2,V) \subset \mathbf{S}(V^2_{\text{open}})$ sphere of radius 2.
$V_{\text{open}} = C^\infty([0,2\pi],\mathbb{R})$.

The basic bijection

$$\Phi(e,f) = c(\theta) = \frac{1}{2}\int_0^{\theta}(e+if)^2\,dx$$

$$\begin{tikzcd}
\Phi : \mathbf{S}^0 \arrow[r, "2\text{-fold}"] & \dfrac{\mathrm{Imm}_{\mathrm{open}}}{\text{transl.,scalings}} \\
\Phi : \mathbf{St}^0 \arrow[u] \arrow[d] \arrow[r, "2\text{-fold}"] & \dfrac{\mathrm{Imm}_{odd}}{\text{transl.,scalings}} \arrow[u] \arrow[d] \\
\overline{\Phi} : \mathbf{Gr}^0 \arrow[d] \arrow[r, "\approx"] & \dfrac{\mathrm{Imm}_{odd}}{\text{transl.,rot.,scalings}} \arrow[d] \\
\overline{\overline{\Phi}} : \mathbf{Gr}^0/U(V) \arrow[r, "\approx"] & \dfrac{B_{i,odd}}{\text{transl.,rot.,scalings}}
\end{tikzcd}$$

Thm. Φ *is an isometry from the natural metric on* St^0 *to* Imm_{odd}*/translations with the metric* G.

Proof. $c_\theta = \frac{1}{2}(e+if)^2$, $ds = \frac{1}{2}|e+if|^2 d\theta$.
$\delta c = T_{(e,f)}\Phi.(\delta e, \delta f) = \int^\theta (\delta e + i\delta f)(e+if)dx$
$D_s(\delta c) = 2\frac{(\delta e + i\delta f)(e+if)}{|e+if|^2}$
$|D_s(\delta c)|^2\, ds = (|\delta e|^2 + |\delta f|^2)d\theta$.

The dictionary between pairs (e, f) and immersions c connects many properties. Curvature κ works out especially nicely. We list here some of the connections:

$$\frac{ds}{d\theta} = |c_\theta| = \tfrac{1}{2}(e^2 + f^2)$$

$$v = D_s(c) = \frac{(e + if)^2}{e^2 + f^2}$$

and if $W_\theta(e, f) = ef_\theta - fe_\theta$ is the Wronskian, then:

$$v_\theta = \left(\frac{(e + if)^2}{e^2 + f^2}\right)_\theta = 2\frac{W_\theta(e, f)}{(e^2 + f^2)} iv, \quad \text{hence}$$

$$\kappa = 2\frac{W_\theta(e, f)}{(e^2 + f^2)^2} \quad \text{for the curvature of } c.$$

Reparameterizations

Let $U(V)$ be the group of all unitary operators on V of the form $f \mapsto \sqrt{\varphi'}(f \circ \varphi)$ for all smooth $\varphi : \mathbb{R} \to \mathbb{R}$ with $\varphi'(x) > 0$ and $\varphi(x + 2\pi) = \varphi(x) + 2\pi$, i.e. lifts of $\bar{\varphi} \in \mathrm{Diff}^+(S^1)$.

The infinitesimal action on V of a periodic vector field X on $\mathbb{R}$ is $f \mapsto \frac{1}{2}X_\theta.f + X.f_\theta$.

Prop. $\Phi(e, f) \circ \bar{\varphi} = \Phi\Big(\sqrt{\varphi'}(e \circ \varphi), \sqrt{\varphi'}(f \circ \varphi)\Big)$.

A tangent vector $(\delta e, \delta f) \in T_{(e,f)}\mathbf{St}$ *is perpendicular to the rotation orbits iff* $\langle e, df\rangle_V = \langle f, de\rangle_V = 0$.

It is perpendicular to the reparameterization orbit iff $W_\theta(e, \delta e) + W_\theta(f, \delta f) = 0$ *where* $W_\theta(a, b) = a.b_\theta - a_\theta.b$ *is the Wronskian.*

Neretin geodesics on $\mathbf{Gr}(2, V)$

Y.A.Neretin: On Jordan angles and the triangle inequality in Grassmann manifolds, Geom. Dedicata 86 (2001)

If $W_0, W_1 \in \mathbf{Gr}(2, V)$, use the singular value decomposition of the orthonormal projection $p : W_0 \to W_1$. This gives ONB (e^0, f^0) of W_0 and (e^1, f^1) of W_1 such that $p(e^0) = \cos(\varphi)e^1$, $p(f^0) = \cos(\psi)f^1$, $e^0 \perp f^1$ and $f^0 \perp e^1$ for
$0 \leq \varphi, \psi \leq \pi/2$ — the *Jordan angles*.
The metric is then given by

$$\boxed{\mathrm{dist}(W^0, W^1) = \sqrt{\varphi^2 + \psi^2}}$$

and the geodesic by

$$W(t) = \left\{ \begin{array}{l} e(t) = \dfrac{\sin((1-t)\varphi)}{\sin(\varphi)}.e^0 + \dfrac{\sin(t\varphi)}{\sin(\varphi)}.e^1 \\ f(t) = \dfrac{\sin((1-t)\psi)}{\sin(\varphi)}.f^0 + \dfrac{\sin(t\varphi)}{\sin(\varphi)}.f^1 \end{array} \right\}$$

We apply this to compute the distance between curves in $\mathrm{Imm}_{\mathrm{od}}/(\mathrm{sim})$ and $B_{i,\mathrm{od}}/(\mathrm{sim})$. We write $\partial_\theta c^0 = r_0(\theta)e^{i\alpha^0(\theta)}$ and $\partial_\theta c^1 = r_1(\theta)e^{i\alpha^1(\theta)}$. We put

$$\begin{aligned} \bar{e}^0 &= \sqrt{2r_0}\cos\tfrac{\alpha^0}{2} & \qquad \bar{f}^0 &= \sqrt{2r_0}\sin\tfrac{\alpha^0}{2}, \\ \bar{e}^1 &= \sqrt{2r_1}\cos\tfrac{\alpha^1}{2} & \qquad \bar{f}^1 &= \sqrt{2r_1}\sin\tfrac{\alpha^1}{2}, \end{aligned}$$

lifting the curves to 2-planes in the Grassmannian. The 2×2 matrix $M(c^0, c^1)$ of the orthogonal projection from the space $\{\bar{e}^0, \bar{f}^0\}$ to $\{\bar{e}^1, \bar{f}^1\}$ in these bases is:

$$\begin{pmatrix} \int_{S^1} 2\sqrt{r^0.r^1}.\cos\frac{\alpha^0}{2}\cos\frac{\alpha^1}{2}\,d\theta & \int_{S^1} 2\sqrt{r^0.r^1}.\cos\frac{\alpha^0}{2}\sin\frac{\alpha^1}{2}\,d\theta \\ \int_{S^1} 2\sqrt{r^0.r^1}.\sin\frac{\alpha^0}{2}\cos\frac{\alpha^1}{2}\,d\theta & \int_{S^1} 2\sqrt{r^0.r^1}.\sin\frac{\alpha^0}{2}\sin\frac{\alpha^1}{2}\,d\theta \end{pmatrix}$$

Notations:

$$\begin{aligned} C_\pm &:= \int_{S^1} \sqrt{r^0.r^1} \cos \tfrac{\alpha^0 \pm \alpha^1}{2}\, d\theta \\ &= \tfrac{1}{2}\left(M(c^0,c^1)_{11}) \mp M(c^0,c^1)_{22}\right) \\ S_\pm &:= \int_{S^1} \sqrt{r^0.r^1} \sin \tfrac{\alpha^0 \pm \alpha^1}{2}\, d\theta \\ &= \tfrac{1}{2}\left(M(c^0,c^1)_{21} \pm M(c^0,c^1)_{12}\right) \end{aligned}$$

We have to diagonalize this matrix by rotating the curve c^0 by a constant angle β^0, i.e., the basis $\{\bar{e}^0, \bar{f}^0\}$ by the angle $\beta^0/2$; and similarly c^1 by a constant angle β^1. So replace α^0 by $\alpha^0 - \beta^0$ and α^1 by $\alpha^1 - \beta^1$ such that (for both signs)

$$\begin{aligned} 0 &= \int_{S^1} \sqrt{r^0.r^1} \sin\left(\frac{(\alpha^0 - \beta^0) \pm (\alpha^1 - \beta^1)}{2}\right) d\theta \\ &= S_\pm . \cos \tfrac{\beta^0 \pm \beta^1}{2} - C_\pm . \sin \tfrac{\beta^0 \pm \beta^1}{2} \end{aligned}$$

Thus

$$\beta_0 \pm \beta_1 = 2 \arctan \left(S_{\pm} / C_{\pm} \right).$$

In the newly aligned bases, the diagonal elements of the matrix will be the cosines of the Jordan angles. The following lemma gives you a formula for them:

If $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $C_{\pm} = \frac{1}{2}(a \mp d), S_{\pm} = \frac{1}{2}(c \pm b)$, *then the singular values of* M *are:*

$$\sqrt{C_-^2 + S_-^2} \pm \sqrt{C_+^2 + S_+^2}.$$

This gives the formula

$$\boxed{\begin{aligned} &D_{\mathrm{od,rot}}(c^0,c^1)^2 = \\ &\qquad = \arccos^2\left(\sqrt{S_+^2+C_+^2}+\sqrt{S_-^2+C_-^2}\right) \\ &\qquad + \arccos^2\left(\sqrt{S_-^2+C_-^2}-\sqrt{S_+^2+C_+^2}\right). \end{aligned}}$$

This is the distance in the space $\mathrm{Imm}_{\mathrm{od}}(S^1,\mathbb{C})/(\text{transl, rot., scalings})$.

Horizontal Neretin distances.
If we want the distance in the quotient space $B_{i,\mathrm{od}}/(\text{transl, rot., scalings})$ by the group $\mathrm{Diff}(S^1)$ we have to take the infimum of this distance over all reparametrizations.
To simplify, we assume that the initial curves c^0, c^1 are parametrized by arc length so that $r^0 \equiv r^1 \equiv 1/2\pi$.

Then consider a reparametrization $\phi \in \mathrm{Diff}(S^1)$ of one of the two curves, say $c^0 \circ \phi$:

$$\boxed{\begin{aligned} D_{\mathsf{sim,diff}}(c^0, c^1)^2 = \inf_{\phi} \Big(& \arccos^2(\lambda_e(c^0 \circ \phi, c^1)) \\ & + \arccos^2(\lambda_f(c^0 \circ \phi, c^1))\Big) \end{aligned}}$$

where now

$$\lambda_e(c^0 \circ \phi, c^1) = \sqrt{S_-^2(\phi) + C_-^2(\phi)} + \sqrt{S_+^2(\phi) + C_+^2(\phi)}$$

$$\lambda_f(c^0 \circ \phi, c^1) = \sqrt{S_-^2(\phi) + C_-^2(\phi)} - \sqrt{S_+^2(\phi) + C_+^2(\phi)}$$

$$S_\pm(\phi) := \frac{1}{2\pi} \int_{S^1} \sqrt{\phi_\theta} \sin \tfrac{(\alpha^0 \circ \phi) \pm \alpha^1}{2} \, d\theta,$$

$$C_\pm(\phi) := \frac{1}{2\pi} \int_{S^1} \sqrt{\phi_\theta} \cos \tfrac{(\alpha^0 \circ \phi) \pm \alpha^1}{2} \, d\theta.$$

To describe the inf, we can use that geodesics in B_i are horizontal geodesics in Imm.

Consider the Neretin geodesic $t \mapsto \{e(t), f(t)\}$ in $\mathbf{Gr}(2, V)$ described above

$$W(t) = \left\{ \begin{aligned} e(t) &= \frac{\sin((1-t)\varphi)}{\sin(\varphi)}.e^0 + \frac{\sin(t\varphi)}{\sin(\varphi)}.e^1 \\ f(t) &= \frac{\sin((1-t)\psi)}{\sin(\varphi)}.f^0 + \frac{\sin(t\varphi)}{\sin(\varphi)}.f^1 \end{aligned} \right\}$$

for

$$e^0 = \sqrt{\tfrac{\phi_\theta}{\pi}} \cos \tfrac{(\alpha^0 \circ \phi) - \beta^0}{2} \qquad e^1 = \tfrac{1}{\sqrt{\pi}} \cos \tfrac{\alpha^1 - \beta^1}{2},$$

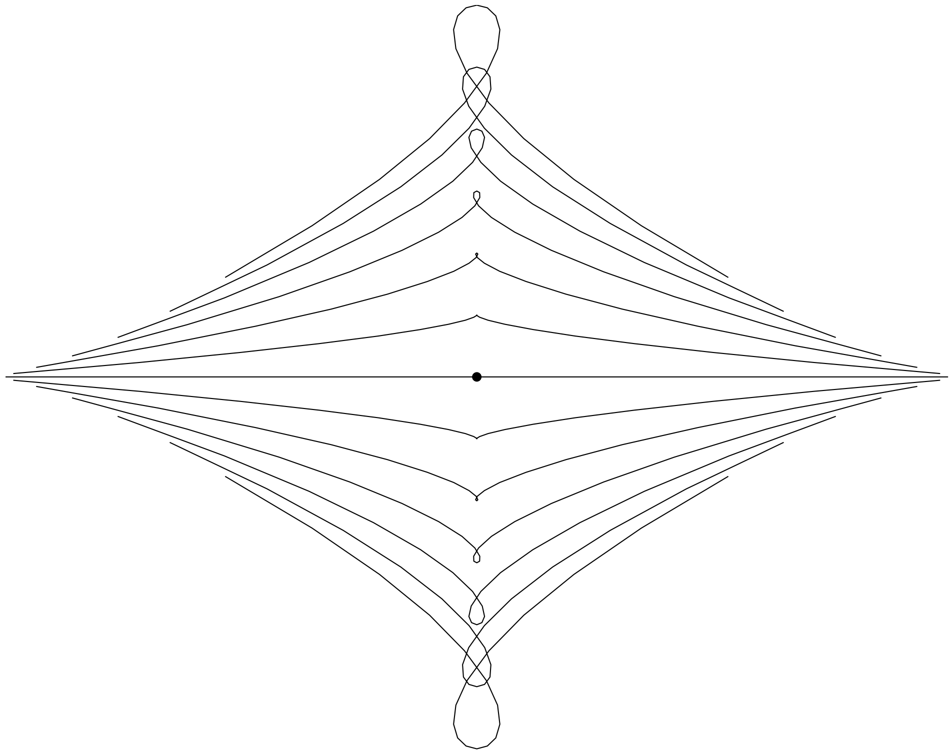
$$f^0 = \sqrt{\tfrac{\phi_\theta}{\pi}} \sin \tfrac{(\alpha^0 \circ \phi) - \beta^0}{2} \qquad f^1 = \tfrac{1}{\sqrt{\pi}} \sin \tfrac{\alpha^1 - \beta^1}{2},$$

where the rotations β^0 and β^1 must be computed from $c^0 \circ \phi$ and c^1.

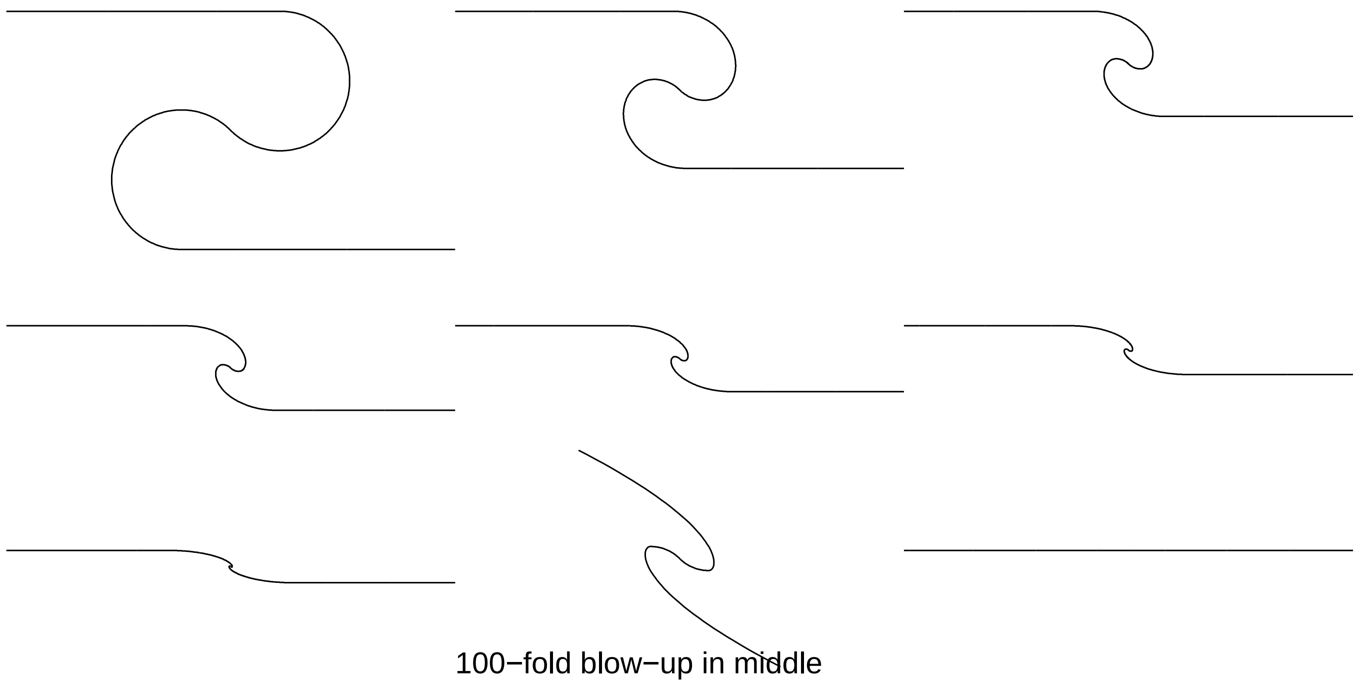
The geodesic is perpendicular to all Diff(S^1)-orbits if and only if the sum of Wronskians vanishes:

$$
\begin{aligned}
0 &= W_\theta(e^0, e_t(0)) + W_\theta(f^0, f_t(0)) = \\
&= -\frac{1}{\sqrt{\phi_\theta}}\Big\{\phi_{\theta\theta}\Big(\frac{\psi_e}{\sin\psi_e}\cos\tfrac{(\alpha^0\circ\phi)-\beta^0}{2}\cos\tfrac{\alpha^1-\beta^1}{2} \\
&\qquad + \frac{\psi_f}{\sin\psi_f}\sin\tfrac{(\alpha^0\circ\phi)-\beta^0}{2}\sin\tfrac{\alpha^1-\beta^1}{2}\Big) \\
&\quad - \phi_\theta\alpha^1_\theta\Big(\frac{\psi_e}{\sin\psi_e}\cos\tfrac{(\alpha^0\circ\phi)-\beta^0}{2}\sin\tfrac{\alpha^1-\beta^1}{2} \\
&\qquad - \frac{\psi_f}{\sin\psi_f}\sin\tfrac{(\alpha^0\circ\phi)-\beta^0}{2}\cos\tfrac{\alpha^1-\beta^1}{2}\Big) \\
&\quad + \phi^2_\theta(\alpha^0_\theta\circ\phi)\Big(\frac{\psi_e}{\sin\psi_e}\sin\tfrac{(\alpha^0\circ\phi)-\beta^0}{2}\cos\tfrac{\alpha^1-\beta^1}{2} \\
&\qquad - \frac{\psi_f}{\sin\psi_f}\cos\tfrac{(\alpha^0\circ\phi)-\beta^0}{2}\sin\tfrac{\alpha^1-\beta^1}{2}\Big)\Big\}
\end{aligned}
$$

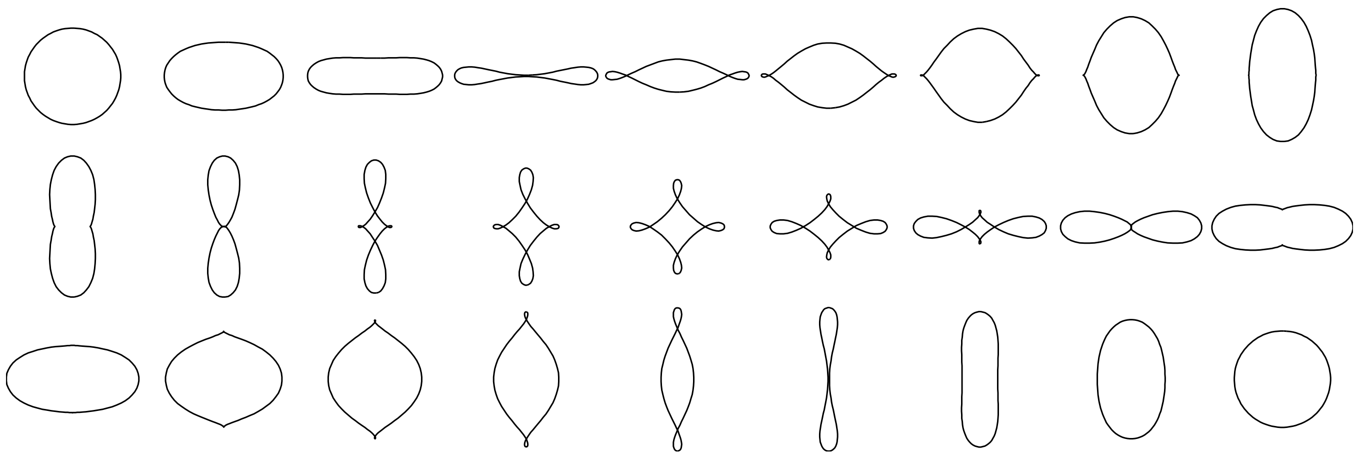
This is an ordinary differential equation for ϕ which is coupled to the (integral) equations for calculating the β's as functions of ϕ. If it is non-singular (i.e., the coefficient function of $\phi_{\theta\theta}$ does not vanish for any θ) then there is a solution ϕ, at least locally. But the non-existence of the inf described for open curves above will also affect closed curves and global solutions may actually not exist. However, for closed curves that do not double back on themselves too much geodesics do seem to usually exist.



The generic way in which a family of open immersions crosses the hypersurface where $Z \neq \emptyset$. The parametrized straight line in the middle of the family has velocity with a double zero at the black dot, hence is not an immersion.



This is a geodesic of open curves running from the curve with the kink at the top left to the straight line on the bottom right. A blow up of the next to last curve is shown to reveal that the kink never goes away – it merely shrinks. Thus this geodesic is not continuous in the C^1-topology on B_{open}. The straight line is parametrized so that it stops for a whole interval of time when it hits the middle point and thus it is C^1-continuous in $\mathrm{Imm}_{\text{open}}$.



A great circle geodesic on B_{od}. The geodesic begins at the circle at the top left, runs from left to right, then to the second row and finally the third. It leaves B_{od} twice: at the top right and bottom left, in both of which the singularity of the first figure occurs in 2 places. The index of the curve changes from $+1$ to -3 in the middle row.

Curvature. Let $W \in \mathbf{Gr}(2,V)$ be a fixed 2-plane. Let $\eta : V \to V$ be the isomorphism which equals -1 on W and 1 on $W^\perp$ satisfying $\eta = \eta^{-1}$. Then $\mathbf{Gr}$ is the symmetric space $O(V)/(O(W) \times O(W^\perp))$ with involutive automorphism $\sigma : O(V) \to O(V)$ given by $\sigma(U) = \eta . U \eta$. For the Lie algebra in the $V = W \oplus W^\perp$-decomposition we have

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x & -y^T \\ y & U \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} x & y^T \\ y & U \end{pmatrix}$$

Here $x \in L(W,W), y \in L(W,W^\perp)$. The fixed point group is $O(V)^\sigma = O(W) \times O(W^\perp)$.

The reductive decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is given by

$$\left\{\begin{pmatrix} x & -y^T \\ y & U \end{pmatrix}\right\} = \left\{\begin{pmatrix} x & 0 \\ 0 & U \end{pmatrix}, x \in \mathfrak{so}(2)\right\}$$
$$+ \left\{\begin{pmatrix} 0 & -y^T \\ y & 0 \end{pmatrix}, y \in L(W, W^\perp)\right\}$$

For the sectional curvature we have (where we assume that Y_1, Y_2 is orthonormal):

$$\begin{aligned}
&k_{\mathrm{span}(Y_1,Y_2)} = -B(Y_2, [[Y_1, Y_2], Y_1]) \\
&= \mathrm{tr}_W(y_2^T y_2 y_1^T y_1 + y_2^T y_1 y_1^T y_2 - 2y_2^T y_1 y_2^T y_1) \\
&= \tfrac{1}{2}\|y_2^T y_1 - y_1^T y_2\|^2_{L^2(W,W)} \\
&\qquad + \tfrac{1}{2}\|y_2 y_1^T - y_1 y_2^T\|^2_{L^2(W^\perp,W^\perp)} \geq 0.
\end{aligned}$$

where L^2 stands for the space of Hilbert-Schmidt operators. Note that there are many orthonormal pairs Y_1, Y_2 on which sectional curvature vanishes and that its maximum value 2 is attained when y_i are isometries and $y_2 = Jy_1$ where J is rotation through angle $\pi/2$ in the image plane of y_1.

We obtain the expression of the curvature in Imm/(sim

$$k^{\mathrm{Imm,sim}}_{\mathrm{span}(h_1,h_2)} = \left(\int_C \det(D_s h_1, D_s h_2) ds\right)^2$$

$$+ \iint_{C\times C} \frac{1+\cos(\alpha(x)-\alpha(y))}{2} \cdot$$

$$\cdot \left(\begin{array}{l} \langle D_s h_1(x), D_s h_2(y)\rangle \\ -\langle D_s h_2(x), D_s h_1(y)\rangle \end{array} \right)^2 ds(x)ds(y)$$

$$+ \iint_{C\times C} \frac{1-\cos(\alpha(x)-\alpha(y))}{2} \cdot$$

$$\cdot \left(\begin{array}{l} \det(D_s h_1(x), D_s h_2(y)) \\ -\det(D_s h_2(x), D_s h_1(y)) \end{array} \right)^2 ds(x)ds(y)$$

A major consequence of the calculation for the curvature on the Grassmannian is:

Thm. *The sectional curvature on* $B_i/(\mathrm{sim})$ *is* ≥ 0.

Proof. We apply O'Neill's formula to the Riemannian submersion

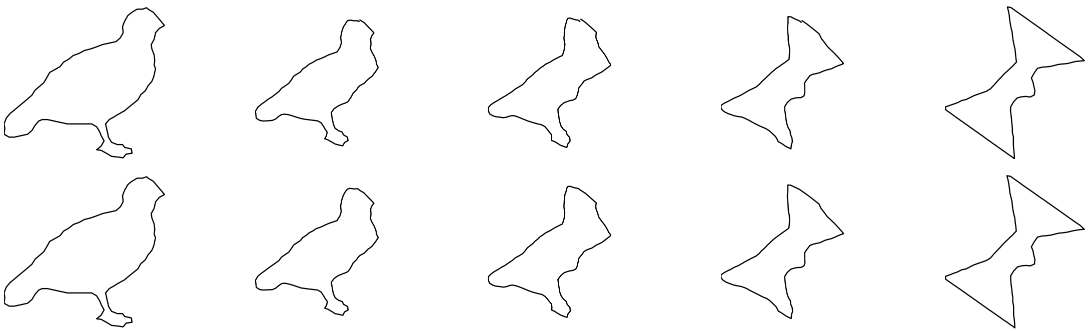
$$\pi : \mathbf{Gr}^0 \to \mathbf{Gr}^0 / U(V) \cong B_i / \operatorname{Diff}^+(S^1)$$
$$k_{\pi(W)}^{\mathbf{Gr}^0/U(V)}(X,Y) = k_W^{\mathbf{Gr}^0}(X^{\mathrm{hor}}, Y^{\mathrm{hor}}) + \tfrac{3}{4}\|[X^{\mathrm{hor}}, X^{\mathrm{hor}}]^{\mathrm{ver}}|_W\|^2 \geq 0$$

where X^{hor} is a horizontal vector field projecting to X at $\pi(W)$. The horizontal and vertical projections exist and are pseudo differential operators.

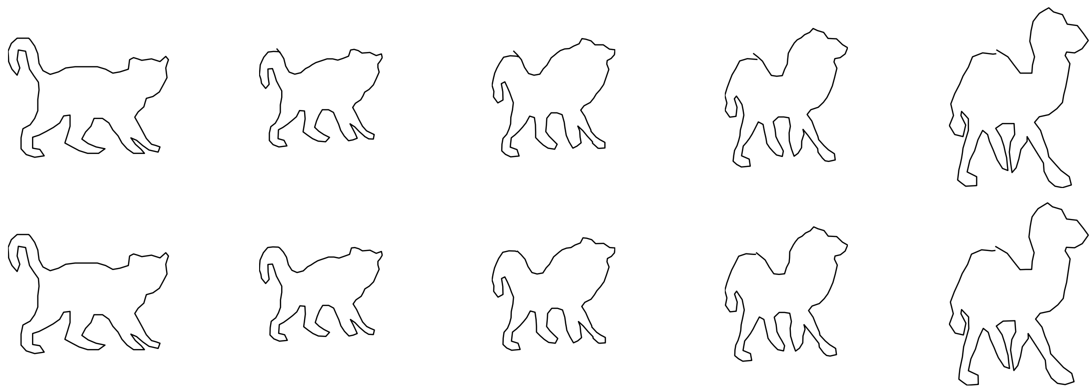
We have explicit formulas for the O'Neill term and thus for the sectional curvature $k^{\mathsf{B_i}/(\mathrm{sim})}_{\mathrm{span}(h_1,h_2)}$ at a curve $C \in \mathsf{B_i}/(\mathrm{sim})$ and tangent vector h_i. We also have an explicit upper bound for this as a function of h_1. This shows that geodesics have at least a small interval before they meet another geodesic. The size of this interval can be controlled by an upper bound that involves the supremum norm of the first two derivatives of h_1.

See the paper for this.

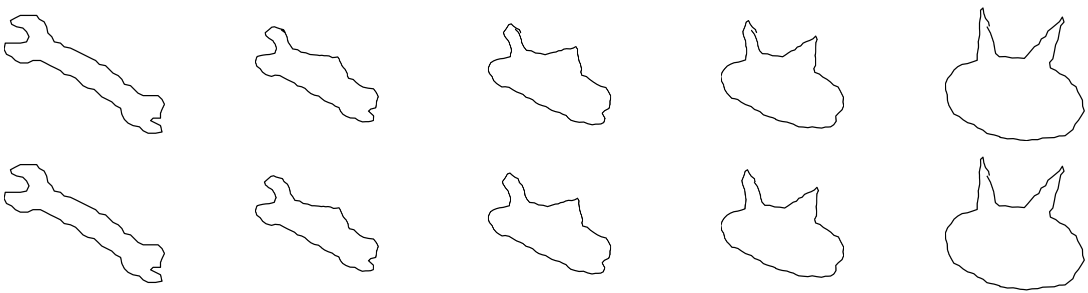
Some numerical experiments:



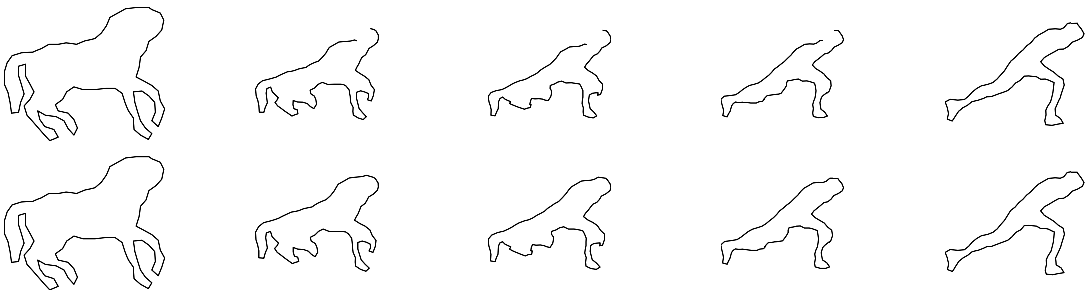
Curve evolution with and without the closedness constraint. Lower and upper bounds for the geodesic distance: 0.443 and 0.444



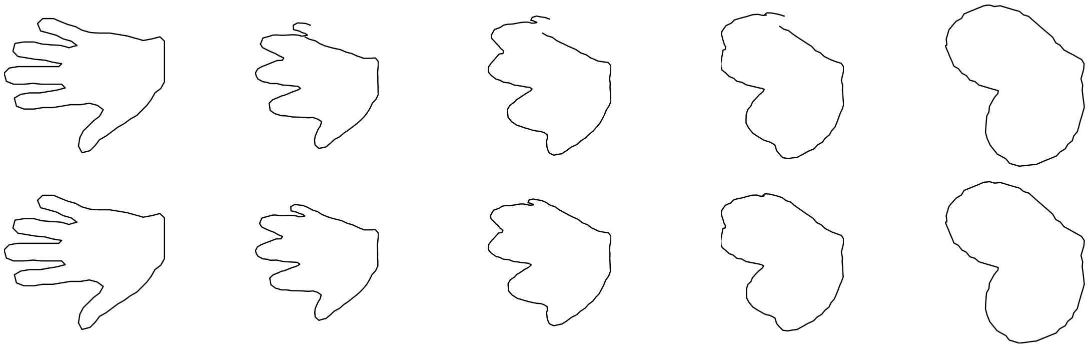
Curve evolution with and without the closedness constraint. Lower and upper bounds for the geodesic distance: 0.462 and 0.464



Curve evolution with and without the closedness constraint. Lower and upper bounds for the geodesic distance: 0.433 and 0.439



Curve evolution with and without the closedness constraint. Lower and upper bounds for the geodesic distance: 0.498 and 0.532



Curve evolution with and without the closedness constraint. Lower and upper bounds for the geodesic distance: 0.513 and 0.528

Shape spaces as quotients of diffeomorphism groups.

Sobolev metrics on Diff$(\mathbb{R}^2)$ **and its quotients** Emb$(S^1,\mathbb{R}^2)$ **and** $B_e(S^1,\mathbb{R}^2)$

Right invariant metric on the Lie group Diff$(\mathbb{R}^2)$ induced by the inner product

$$H^n(X,Y)=\int_{\mathbb{R}^2}\langle LX,Y\rangle\,dx \qquad \text{where}$$

$$L=L_{A,n}=(1-A\Delta)^n, \qquad \Delta=\partial^2_{x^1}+\partial^2_{x^2}.$$

with fundamental solution $L_{A,n}(F_{A,n})=\delta_0$ given by

$$\begin{aligned}F_{A,n}(x)&=\frac{1}{2\pi}\int_{\mathbb{R}^2}e^{i\langle x,\xi\rangle}\frac{1}{(1+A|\xi|^2)^n}\,d\xi\\&=\frac{c}{A^{(n-1)/2}}.|x|^{n-1}.K_{n-1}(\frac{|x|}{\sqrt{A}}),\end{aligned}$$

for the classical modified Bessel functions K_r.

The geodesic equation on Diff($\mathbb{R}^2$) is V.Arno'ld's equation EPDiff:

$$t \mapsto \varphi(t, \quad) \in \mathrm{Diff}(\mathbb{R}^2)$$

$$v(t) = (\partial_t \varphi) \circ \varphi^{-1} \in \mathfrak{X}(\mathbb{R}^2), \qquad u(t) = L(v(t)),$$

$$\frac{\partial u_i}{\partial t} + \sum_j \left(v^j . \frac{\partial u_i}{\partial x^j} + u^j . \frac{\partial v^j}{\partial x^i} \right) + \mathrm{div}\, v . u_i = 0.$$

The quotient Emb$(S^1, \mathbb{R}^2)$**.**

Diff$(\mathbb{R}^2) \to$ Emb$(S^1, \mathbb{R}^2)$

$\varphi \mapsto \varphi \circ i$, where $i : S^1 \subset \mathbb{R}^2$.

If $c = \varphi \circ i$, the fiber through φ is

$\varphi.\{\psi : \psi \circ i = i\} = \{\psi : \psi \circ c = c\}.\varphi$.

The tangent space to the fiber is (right translated by φ)

$\{X \in X(\mathbb{R}^2) : X \circ c = 0\}$.

The horizontal subspace is the translate by φ of

$\{Y : \int_{\mathbb{R}^2} \langle LY, X \rangle \, dx = 0, \text{ if } X \circ c = 0\}$.

If Y is C^∞ then $Y = 0$. So we need

$LY = c_*(p(\theta).ds)$ for $p \in C^\infty(S^1, \mathbb{R}^2)$, a distribution carried by c. Thus

$Y(x) = \int_{S^1} F(x - c(\theta))p(\theta) \, ds$

$Y(x) = \int_{S^1} F(x - c(\theta))p(\theta)\,ds$
Mapped to $T_c\,\mathrm{Emb}$ we get

$$
\begin{aligned}
(Y \circ c)(\theta) &= \int_{S^1} F(c(\theta) - c(\theta_1)).p(\theta_1).|c'(\theta_1)|d\theta_1 \\
&=: (F_c * p)(\theta) \qquad \text{where} \\
F_c(\theta_1, \theta_2) &:= F(c(\theta_1) - c(\theta_2))
\end{aligned}
$$

is an elliptic pseudo differential operator kernel of order $-2n+1$ which is real and positive, so the operator $p \mapsto F_c * p$ is self-adjoint and positive, so injective, and by index deformation it is bijective between the Sobolev spaces on S^1. The inverse operator $(F_c * \quad)^{-1}$ has kernel $L_c(\theta, \theta_1)$ which is a pseudo differential operator kernel of order $2n-1$.

Write $h = Y \circ c \in T_c \operatorname{Emb}$ and express the horizontal lift $Y = Y_h$ in terms of h:
$h = Y_h \circ c = F * (c_*(p.ds)) = F_c * p$ so $p = L_c * h$

$$Y = Y_h = F * (c_*((L_c * h).ds))$$

$$Y_h(x) = \int_{S^1} F(x - c(\theta)).$$

$$\cdot \int_{S^1} L_c(\theta, \theta_1) h(\theta_1) |c'(\theta_1)| d\theta_1 |c'(\theta)| d\theta$$

Finally the metric:

$$G_c^{\mathrm{diff},n}(h, k) = \int_{\mathbb{R}^2} \langle LY_h, Y_k \rangle dx$$

$$= \iint_{S^1 \times S^1} L_c(\theta, \theta_1) \langle h(\theta_1), k(\theta) \rangle ds_1 \, ds.$$

We can now compute K and H and the geodesic equation. It becomes simpler if written for the 1-current $L_c * c_t = p.|c_\theta| =: q$:

$$\boxed{q_t(\theta_0) = -\int_{S^1} F'_c(\theta_0, \theta_1)\langle q(\theta_0), q(\theta_1)\rangle \, d\theta_1}$$

where $F'_c(\theta_1, \theta_2) = \operatorname{grad} F(c(\theta_1) - c(\theta_2))$.

Existence of geodesics. Theorem.
Let $n \geq 1$. For each $k > 2n - \frac{1}{2}$ the geodesic equation has unique local solutions in the Sobolev space of H^k-embeddings. The solutions are C^∞ in t and in the initial conditions $c(0, \,.\,)$ and $c_t(0, \,.\,)$. The domain of existence (in t) is uniform in k and thus this also holds in $\mathrm{Emb}(S^1, \mathbb{R}^2)$.

Conserved momenta: Along a geodesic c,

$$G_c^{\text{diff},n}(c_\theta.X, c_t) =$$
$$= \iint_{S^1\times S^1} L_c(\theta,\theta_1)\langle c_\theta(\theta_1)X(\theta_1), c_t(\theta)\rangle ds_1\, ds$$

is conserved for every vector field X on S^1; the conserved **reparametrization momentum** is $\langle c_\theta, L_c * c_t\rangle = \langle c_\theta, q\rangle$.
Also $\iint_{(S^1)^2} L_c(\theta,\theta_1)c_t(\theta)\rangle ds_1\, ds = \int_{S^1} q(\theta)\, ds$ is the conserved **linear momentum**.

$$\iint_{S^1\times S^1} L_c(\theta,\theta_1)\langle Jc(\theta_1), c_t(\theta)\rangle ds_1\, ds =$$
$$= \int_{S^1}\langle Jc(\theta), q(\theta)\rangle ds$$

is the conserved **angular momentum**.

Horizontal geodesics.

A field h along c is horizontal if $\langle L_c * h, c_\theta \rangle = 0$. For a horizontal path we have $\langle q, c_\theta \rangle = 0$, so let $q = \tilde{a}.n$. Then the horizontal geodesic equation is

$$\boxed{\begin{aligned}&\tilde{a}_t(\theta) = \langle q_t, n \rangle(\theta) = \\ &= -\int_{S^1} \langle F_c'(\theta, \theta_1), n(\theta) \rangle \tilde{a}(\theta) \tilde{a}(\theta_1) \langle n(\theta), n(\theta_1) \rangle \, d\theta_1\end{aligned}}$$

Note that also $n = Jc_\theta / |c_\theta|$ appears. It is a strange equation, but it is well-posed byt the theorem above.

Geometry of landmark space and of spaces of currents

The diffeomorphism group
$\mathrm{Diff} = \mathrm{Diff}_{\mathcal{S}}(\mathbb{R}^n)$: the regular Lie group of all diffeomorphisms which are rapidly falling towards the identity.
Its Lie algebra is the space $\mathfrak{X}_{\mathcal{S}}(\mathbb{R}^n)$ of all smooth vector fields which decrease rapidly, with the negative of the usual bracket as Lie bracket.
We consider $\mathfrak{X}_{\mathcal{S}}(\mathbb{R}^n)$ as pre Hilbert space H^L with inner product

$$\langle X, Y\rangle_{H^L} = \int_{\mathbb{R}^n} \langle LX, Y\rangle \, dx$$

where $L : \mathfrak{X}_{\mathcal{S}}(\mathbb{R}^n) \to \mathfrak{X}_{\mathcal{S}}(\mathbb{R}^n)$ is an invertible linear (elliptic) scalar differential operator or pseudo-differential operator which is self-adjoint with respect to the weak inner product

$$G^0(X, Y) = \int_{\mathbb{R}^n} \langle X, Y\rangle \, dx$$

on $\mathfrak{X}_{\mathcal{S}}(\mathbb{R}^n)$ and which is applied to each component of a vector field separately.

For example:

For the Laplacian $\Delta = \sum \partial_i^2$ and constant A, let

$$L = (1 - A\Delta)^l = \sum_{|\alpha| \leq l} \frac{(-A)^{|\alpha|} l!}{\alpha!\,(l-\alpha)!} \partial^{2\alpha}$$

$$= \sum_{\alpha_1 + \cdots + \alpha_n \leq l} \frac{(-A)^{\alpha_1 + \cdots + \alpha_n}\, l!}{\alpha_1! \ldots \alpha_n!\,(l-\alpha_1)! \ldots (l-\alpha_n)!} \partial_{x_1}^{2\alpha_1} \ldots \partial_{x_n}^{2\alpha_n}$$

The Fourier transform is $\widehat{Lu} = (1 + A|\xi|^2)^l \hat{u}(\xi)$. Thus the fundamental solution K of $LK = \delta$ in the space $\mathcal{S}'(\mathbb{R}^n)$ of tempered distributions is

$$K(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\langle x, \xi \rangle} \frac{1}{(1 + |\xi|^2)^l} d^n\xi$$

which can be expressed in terms of the classical modified Bessel functions $K_{l-1}(|x|/\sqrt{A})$. It satisfies

$(L^{-1}u)(x) = \int_{\mathbb{R}^n} K(x-y)u(y)d^n y$ for each tempered distribution u.

Or:

We consider a kernel function $K : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ with good properties (for example smooth and rapidly decreasing off the diagonal) and its associated operator $K(f)(x) = \int_{\mathbb{R}^n} K(x,y) f(y)\, dy$ which we assume to be invertible on $C^\infty_{\mathcal{S}}(\mathbb{R}^n)$ on the space of of smooth functions with compact support, and then we put $L = K^{-1}$.

Landmark space as homogeneus space

A *landmark* $q = (q_1, \dots, q_N)$: N-tuple of distinct points in $\mathbb{R}^n$.

$\mathrm{Land}^N \subset (\mathbb{R}^n)^N$: the open subset of all landmarks.

$q^0 = (q_1^0, \dots, q_N^0)$ a fixed standard template landmark.

Then we have the the surjective mapping

$$
\begin{aligned}
&\mathrm{ev}_{q^0} : \mathrm{Diff}(\mathbb{R}^n) \to \mathrm{Land}^N, \\
&\varphi \mapsto \mathrm{ev}_{q^0}(\varphi) = \varphi(q^0) = (\varphi(q_1^0), \dots, \varphi(q_N^0)).
\end{aligned}
$$

The fiber of ev_{q^0} over a landmark $q = \varphi_0(q^0)$ is

$$
\begin{aligned}
\{\varphi \in \mathrm{Diff}(\mathbb{R}^n) : \varphi(q^0) = q\} &= \\
&= \varphi_0 \circ \{\varphi \in \mathrm{Diff}(\mathbb{R}^n) : \varphi(q^0) = q^0\} \\
&= \{\varphi \in \mathrm{Diff}(\mathbb{R}^n) : \varphi(q) = q\} \circ \varphi_0;
\end{aligned}
$$

We shall use the latter representation.

The tangent space to the fiber is

$$\{X \circ \varphi_0 : X \in \mathfrak{X}_{\mathcal{S}}(\mathbb{R}^n), X(q_i) = 0 \text{ for all } i\}.$$

A tangent vector $Y \circ \varphi_0 \in T_{\varphi_0} \operatorname{Diff}_{\mathcal{S}}(\mathbb{R}^n)$ is $G^L_{\varphi_0}$-perpendicular to the fiber over q if

$$\int_{\mathbb{R}^n} \langle LY, X \rangle \, dx = 0 \quad \forall X \text{ with } X(q) = 0.$$

If we require Y to be smooth then $Y = 0$. So we assume that $LY = \sum_i P_i.\delta_{q_i}$, a distributional vector field (current) with support in q. Here $P_i \in T_{q_i}\mathbb{R}^n$. But then

$$\begin{aligned} Y(x) = L^{-1}\Big(\sum_i P_i.\delta_{q_i}\Big) &= \int_{\mathbb{R}^n} K(x-y) \sum_i P_i.\delta_{q_i}(y)\, dy \\ &= \sum_i K(x - q_i).P_i \end{aligned}$$

$$T_{\varphi_0} \mathsf{ev}_{q^0} .(Y \circ \varphi_0) = Y(q_k)_k = \sum_i (K(q_k - q_i).P_i)_k$$

Consider a tangent vector $P = (P_k) \in T_q \operatorname{Land}^N$. Its horizontal lift with footpoint φ_0 is $P^{\text{hor}} \circ \varphi_0$ where the vector field P^{hor} on $\mathbb{R}^n$ is given as follows: Let $K^{-1}(q)_{ki}$ be the inverse of the $(N \times N)$-matrix $K(q)_{ij} = K(q_i - q_j)$. Then

$$P^{\text{hor}}(x) = \sum_{i,j} K(x - q_i) K^{-1}(q)_{ij} P_j$$

$$L(P^{\text{hor}}(x)) = \sum_{i,j} \delta(x - q_i) K^{-1}(q)_{ij} P_j$$

Note that P^{hor} is a vector field of class H^{2l-1}.

The Riemannian metric on Land^N induced by the g^L-metric on $\mathrm{Diff}_{\mathcal{S}}(\mathbb{R}^n)$ is

$$
\begin{aligned}
g^L_q(P,Q) &= G^L_{\varphi_0}(P^{\mathrm{hor}}, Q^{\mathrm{hor}}) = \int_{\mathbb{R}^n} \langle L(P^{\mathrm{hor}}), Q^{\mathrm{hor}}\rangle \, dx \\
&= \int_{\mathbb{R}^n} \Big\langle \sum_{i,j} \delta(x-q_i) K^{-1}(q)_{ij} P_j, \sum_{k,l} K(x-q_k) K^{-1}(q)_{kl} Q_l \Big\rangle \, dx \\
&= \sum_{i,j,k,l} K^{-1}(q)_{ij} K(q_i - q_k) K^{-1}(q)_{kl} \langle P_j, Q_l \rangle
\end{aligned}
$$

So the metric is given by:

$$
\boxed{g^L_q(P,Q) = \sum_{k,l} K^{-1}(q)_{kl} \langle P_k, Q_l \rangle.}
$$

Recall: $K^{-1}(q)_{ki}$ is the inverse of the $(N \times N)$-matrix $K(q)_{ij} = K(q_i - q_j)$.

Lemma *Let* $X, Y \in \mathfrak{X}_{\mathcal{S}}(\mathbb{R}^n)$ *be a vector fields with support in a compact box* $B \subset \mathbb{R}^n$*. Let* $q_1, q_2, q_3, \dots$ *be an equidistributed sequence in* B*: For each Borel subset* $U \subset B$ *we require*

$$\lim_{N\to\infty} \frac{\#\{i \le N : q_i \in U\}}{N} = \frac{\mathrm{Vol}(U)}{\mathrm{Vol}(B)}.$$

For each N *consider the initial part* $q^N = (q_1, \dots, q_N)$ *as a point in the landmark space* Land^N *of* N *points in* $\mathbb{R}^n$*. Then we have*

$$\lim_{N\to\infty} \frac{\mathrm{Vol}(B)^2}{N^2} \sum_{i,j=1}^{N} K^{-1}(q)_{i,j} \langle X(q_i), Y(q_j)\rangle =$$

$$= \int_{\mathbb{R}^n} \langle LX, Y\rangle \, dx.$$

The geodesic equation on $T^*\mathrm{Land}^N(\mathbb{R}^n)$.

Elements of the cotangent bundle $T^*\mathrm{Land}^N(\mathbb{R}^n) = \mathrm{Land}^N(\mathbb{R}^n) \times ((\mathbb{R}^n)^N)^*$ are denoted by

$$
\begin{aligned}
(q,\alpha) &= \left((q_1, \ldots, q_N), \begin{pmatrix} \alpha^1 \\ \vdots \\ \alpha^N \end{pmatrix} \right) \\
&= \left(\begin{pmatrix} q_1^1 & \ldots & q_N^1 \\ \ldots & & \\ q_1^n & \ldots & q_N^n \end{pmatrix}, \begin{pmatrix} \alpha_1^1 & \ldots & \alpha_n^1 \\ \ldots & & \\ \alpha_1^N & \ldots & \alpha_n^N \end{pmatrix} \right)
\end{aligned}
$$

and we shall use this as global coordinates. The metric looks like

$$
(g^L)_q^{-1}(\alpha,\beta) = \sum_{i,j} K(q)_{ij} \langle \alpha_i, \beta_j \rangle,
$$

$$
K(q)_{ij} = K(q_i - q_j).
$$

We consider the the energy function

$$E(q,\alpha) = \tfrac{1}{2}(g^L)_q^{-1}(\alpha,\alpha) = \tfrac{1}{2}\sum_{i,j} K(q)_{ij}\langle\alpha_i,\beta_j\rangle$$
$$= \tfrac{1}{2}\sum_{i,j} K(q)_{ij}\langle\alpha_i,\beta_j\rangle$$

and its Hamiltonian vector field (using $\mathbb{R}^n$-valued derivatives to save notation)

$$H_E(q,\alpha) = \frac{1}{2}\sum_{i,j,k=1}^{N}\Big(\frac{\partial K(q)_{ij}\langle\alpha_i,\alpha_j\rangle}{\partial\alpha^k}\frac{\partial}{\partial q_k} - \frac{\partial K(q)_{ij}\langle\alpha_i,\alpha_j\rangle}{\partial q_k}\frac{\partial}{\partial\alpha^k}\Big).$$
$$= \sum_{i,k=1}^{N}\Big(K(q_k-q_i)\alpha_i\frac{\partial}{\partial q_k} + \operatorname{grad} K(q_i-q_k)\langle\alpha_i,\alpha_k\rangle\frac{\partial}{\partial\alpha^k}\Big).$$

So the geodesic equation is the flow of this vector field:

$$\boxed{\begin{aligned}\dot q_k &= \sum_i K(q_i - q_k)\alpha^i \\ \dot\alpha^k &= \sum_i \langle \alpha^k, \alpha^i\rangle \,\mathrm{grad}\, K(q_i - q_k)\end{aligned}}$$

A covariant formula for curvature and its relations to O'Neill's curvature formulas.
Mario Micheli in his 2008 thesis derived the the coordinate version of the following formula for the sectional curvature expression, which is valid for **closed** 1-forms α, β on a Riemannian manifold (M, g), where we view $g : TM \to T^*M$ and so g^{-1} is the

dual inner product on T^*M. Here $\alpha^\sharp = g^{-1}(\alpha)$.

$$\boxed{\begin{aligned} g^L\Big(R(\alpha^\sharp,\beta^\sharp)\alpha^\sharp,\beta^\sharp\Big) &= -\tfrac{1}{4}\|d(g^{-1}(\alpha,\beta))\|^2 \\ &\quad + \tfrac{1}{4}g^{-1}\Big(d(\|\alpha\|^2), d(\|\beta\|^2)\Big) \\ &\quad + \tfrac{3}{4}g\Big([\alpha^\sharp,\beta^\sharp],[\alpha^\sharp,\beta^\sharp]\Big) \\ &\quad - \tfrac{1}{2}\alpha^\sharp\alpha^\sharp(\|\beta\|^2) - \tfrac{1}{2}\beta^\sharp\beta^\sharp(\|\alpha\|^2) \\ &\quad + \tfrac{1}{2}(\alpha^\sharp\beta^\sharp + \beta^\sharp\alpha^\sharp)g^{-1}(\alpha,\beta) \end{aligned}}$$

Mario's formula in coordinates.

Assume that $\alpha = \alpha_i dx^i, \beta = \beta_i dx^i$ where the coefficients α_i, β_i are *constants*, hence α, β are closed. Then $\alpha^\sharp = g^{ij}\alpha_i\partial_j, \beta^\sharp = g^{ij}\beta_i\partial_j$ and we have:

$$\boxed{\begin{aligned}&4g\Big(R(\alpha^\sharp,\beta^\sharp)\beta^\sharp,\alpha^\sharp\Big)\\ &= (\alpha_i\beta_k - \alpha_k\beta_i)\cdot(\alpha_j\beta_l - \alpha_l\beta_j)\cdot\\ &\cdot\Big(2g^{is}(g^{jt}g^{kl}_{,t})_{,s} - \tfrac{1}{2}g^{ij}_{,s}g^{st}g^{kl}_{,t} - 3g^{is}g^{kp}_{,s}g_{pq}g^{jt}g^{lq}_{,t}\Big)\end{aligned}}$$

Covariant curvature and O'Neill's formula, finite dimensional.

Let $p : (E, g_E) \to (B, g_B)$ be a Riemannian submersion between finite dimensional manifolds, i.e., for each $b \in B$ and $x \in E_b := p^{-1}(b)$ the g_E-orthogonal splitting

$$T_x E = T_x(E_{p(x)}) \oplus T_x(E_{p(x)})^\perp =: T_x(E_{p(x)}) \oplus \mathrm{Hor}_x(p)$$

has the property that $T_x p : (\mathrm{Hor}_x(p), g_E) \to (T_b B, g_B)$ is an isometry. Each vector field $X \in \mathfrak{X}(E)$ is decomposed as $X = X^{\mathrm{hor}} + X^{\mathrm{ver}}$ into horizontal and vertical parts. Each vector field $\xi \in \mathfrak{X}(B)$ can be uniquely lifted to a smooth horizontal field $\xi^{\mathrm{hor}} \in \Gamma(\mathrm{Hor}(p)) \subset \mathfrak{X}(E)$.

O'Neill's formula says that for any two horizontal vector fields X, Y on M and any $x \in E$, the sectional curvatures of E and B are related by:

$$\boxed{\begin{aligned} &g_{p(x)}(R^B(p_*(X_x), p_*(Y_x))p_*(Y_x), p_*(X_x)) \\ &= g_x(R^E(X_x, Y_x)Y_x, X_x) + \tfrac{3}{4}\|[X,Y]^{ver}\|_x^2. \end{aligned}}$$

Comparing Mario's formula on E and B gives an immediate proof of this fact. Namely: If $\alpha \in \Omega^1(B)$, then the vector field $(p^*\alpha)^\sharp$ is horizontal and we have $Tp \circ (p^*\alpha)^\sharp = \alpha^\sharp \circ p$. Therefore $(p^*\alpha)^\sharp$ equals the horizontal lift $(\alpha^\sharp)^{\mathrm{hor}}$. For each $x \in E$ the mapping $(T_x p)^* : (T^*_{p(x)}B, g_B^{-1}) \to (T^*_x E, g_E^{-1})$ is an isometry. We also use:

$$\|[(p^*\alpha)^\sharp, (p^*\beta)^\sharp]^{\mathrm{hor}}\|^2_{g_E} = p^*\|[\alpha^\sharp, \beta^\sharp]\|^2_{g_B}$$

Curvature via the cotangent bundle Mario's formula for **closed** 1-forms α, β on landmark space, where $\alpha_k^\sharp = \sum_i K(q_k - q_i)\alpha^i$. We shall use *constant* 1-forms below.

$$
\begin{aligned}
&4g^L\Big(R(\alpha^\sharp,\beta^\sharp)\alpha^\sharp,\beta^\sharp\Big) = \\
&= -2\alpha^\sharp\alpha^\sharp(\|\beta\|^2) - 2\beta^\sharp\beta^\sharp(\|\alpha\|^2) + 2(\alpha^\sharp\beta^\sharp + \beta^\sharp\alpha^\sharp)g^{-1}(\alpha,\beta) \\
&\quad - \|d(g^{-1}(\alpha,\beta))\|^2 + g^{-1}\Big(d(\|\alpha\|^2), d(\|\beta\|^2)\Big) + 3g\Big([\alpha^\sharp,\beta^\sharp],[\alpha^\sharp,\beta^\sharp]\Big) \\
&= \Bigg(-2\sum_{i,j,k,l}\Big\langle dq_j \cdot \Big(d^2K(q_i-q_j)(dq_l,dq_k)(K(q)_{il}-K(q)_{jl})(K(q)_{ik}-K(q)_{jk}) \\
&\qquad\qquad + dK(q_i-q_j)(dq_k)\Big(dK(q_i-q_k)(dq_l)(K(q)_{il}-K(q)_{kl}) \\
&\qquad\qquad\qquad - dK(q_j-q_k)(dq_l)(K(q)_{jl}-K(q)_{kl})\Big)\Big), dq_i\Big\rangle \\
&\quad + \sum_{i,j,k,l} K(q)_{ik}\langle dK(q_i-q_j), dK(q_k-q_l)\rangle\langle dq_i,dq_j\rangle\langle dq_k,dq_l\rangle(R_{3124}+R_{1324}) \\
&\quad + 3\sum_{k,l,i,j,m,n} K^{-1}(q)_{kl}\Big(K(q)_{kj}-K(q)_{ij}\Big)\,dK(q_k-q_i)(dq_j)
\end{aligned}
$$

$$\Big(K(q)_{kn} - K(q)_{mn}\Big)\Big\langle dq_i, dK(q_k - q_m)(dq_n)\, dq_m \Big\rangle\Bigg)\Big((\alpha \wedge \beta) \otimes (\alpha \wedge \beta)\Big)$$

Notation for the coordinate formula:

$$\begin{aligned}
A = &\text{ indices of landmark points in } \mathbb{R}^n \\
a, b, c, \cdots = &\text{ elements of } A \\
\alpha, \beta = &\{\alpha_a | a \in A\}, \{\beta_a | a \in A\}, \text{ cotangent vectors to } \mathcal{L} \\
\alpha^\sharp, \beta^\sharp = &\text{ the dual tangent vectors, e.g.} \\
\alpha^\sharp_a = &\sum_b K(P_a - P_b)\alpha_b \\
K(\vec{x}) = &k(\|\vec{x}\|), \text{ the kernel defining the metric} \\
&\text{note: } \nabla K(\vec{x}) = k'(\|\vec{x}\|)\frac{\vec{x}}{\|\vec{x}\|} \\
d_{ab} = &\|P_a - P_b\|, \vec{u}_{ab} = (P_a - P_b)/d_{ab}, \\
&\text{the unit vector between landmarks} \\
K_{ab} = &k(d_{ab}), \nabla K_{ab} = DK(P_a - P_b) = k'(d_{ab})\vec{u}_{ab} \\
\widetilde{K}''_{ab} = &\frac{k''(d_{ab})}{k'^2(d_{ab})} - \frac{1}{d_{ab}k'(d_{ab})}
\end{aligned}$$

Four expressions in the skew form $\alpha \wedge \beta$:

$$\sigma_{ab,cd}(\alpha,\beta) = \langle \alpha_a, \nabla K_{cd}\rangle \beta_b - \langle \beta_a, \nabla K_{cd}\rangle \alpha_b$$

$$\sigma^*_{bcd}(\alpha,\beta) = \sum_a (K_{ac} - K_{ad})\sigma_{ab,cd}(\alpha,\beta)$$

$$= \langle \alpha^\sharp_c - \alpha^\sharp_d, \nabla K_{cd}\rangle \beta_b - \langle \beta^\sharp_c - \beta^\sharp_d, \nabla K_{cd}\rangle \alpha_b$$

(Note that the terms in angle brackets are discrete strains)

$$\tau_{ab,cd}(\alpha,\beta) = \langle (\alpha_a \otimes \beta_c) - (\beta_a \otimes \alpha_c),\ (\alpha_b \otimes \beta_d) - (\beta_c \otimes \alpha_d)\rangle,$$

(Bracket in $\mathbb{R}^n \otimes \mathbb{R}^n$, points a, b on left, c, d on right)

$$\tau^*_{bd}(\alpha,\beta) = \Big\langle (\alpha^\sharp_b - \alpha^\sharp_d) \otimes (\beta^\sharp_b - \beta^\sharp_d) - (\beta^\sharp_b - \beta^\sharp_d) \otimes (\alpha^\sharp_b - \alpha^\sharp_d),$$
$$\alpha_b \otimes \beta_d - \beta_b \otimes \alpha_d \Big\rangle$$

With these notations, we get the following formula:

$$\boxed{\begin{aligned} R(\alpha,\beta,\alpha,\beta) &= \tfrac{1}{2}\sum_{bd} \widetilde{K}''_{bd}\langle \sigma^*_{bbd}(\alpha,\beta), \sigma^*_{dbd}(\alpha,\beta)\rangle \\ &\quad + \tfrac{1}{2}\sum_{bcd}\langle \left(\sigma^*_{bcb}(\alpha,\beta) - \sigma^*_{bcd}(\alpha,\beta)\right), \sigma_{cd,bd}(\alpha,\beta)\rangle \\ &\quad - \tfrac{3}{4}\|\sum_{b}\sigma^*_{bb\cdot}(\alpha,\beta)\|^2_{K^{-1}} + \tfrac{1}{2}\sum_{cd}\frac{k'_{bd}}{d_{ab}}\cdot \tau^*_{cd}(\alpha,\beta) \\ &\quad - \tfrac{1}{8}\sum_{abcd}(K_{ab}-K_{ad}-K_{cb}+K_{cd})\langle \nabla K_{ac}, \nabla K_{bd}\rangle \cdot \tau_{ab,cd}(\alpha,\beta). \end{aligned}}$$

Sobolev metrics on $\mathrm{Diff}(\mathbb{R}^2)$ **and its quotients** $\mathrm{Emb}(S^1, \mathbb{R}^2)$ **and** $B_e(S^1, \mathbb{R}^2)$

Right invariant metric on the Lie group $\mathrm{Diff}(\mathbb{R}^2)$ induced by the inner product

$$H^n(X, Y) = \int_{\mathbb{R}^2} \langle LX, Y \rangle \, dx \qquad \text{where}$$

$$L = L_{A,n} = (1 - A\Delta)^n, \qquad \Delta = \partial^2_{x^1} + \partial^2_{x^2}.$$

with fundamental solution $L_{A,n}(K_{A,n}) = \delta_0$ given by

$$\begin{aligned} K_{A,n}(x) &= \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{i\langle x,\xi\rangle} \frac{1}{(1 + A|\xi|^2)^n} \, d\xi \\ &= \frac{C}{A^{(n-1)/2}} . |x|^{n-1} . K_{n-1}\Big(\frac{|x|}{\sqrt{A}}\Big), \end{aligned}$$

for the classical modified Bessel functions K_r.

The geodesic equation on $\mathrm{Diff}(\mathbb{R}^2)$ is V.Arno'ld's equation EPDiff:

$$t \mapsto \varphi(t, \quad) \in \mathrm{Diff}(\mathbb{R}^2)$$
$$v(t) = (\partial_t \varphi) \circ \varphi^{-1} \in \mathfrak{X}(\mathbb{R}^2), \qquad u(t) = L(v(t)),$$
$$\frac{\partial u_i}{\partial t} + \sum_j \left(v^j . \frac{\partial u_i}{\partial x^j} + u^j . \frac{\partial v^j}{\partial x^i} \right) + \mathrm{div}\, v . u_i = 0.$$

The quotient $\mathrm{Emb}(S^1, \mathbb{R}^2)$**.**

$\mathrm{Diff}(\mathbb{R}^2) \to \mathrm{Emb}(S^1, \mathbb{R}^2)$

$\varphi \mapsto \varphi \circ i$, where $i : S^1 \subset \mathbb{R}^2$.

If $c = \varphi \circ i$, the fiber through φ is

$\varphi.\{\psi : \psi \circ i = i\} = \{\psi : \psi \circ c = c\}.\varphi.$

The tangent space to the fiber is (right translated by φ)

$\{X \in X(\mathbb{R}^2) : X \circ c = 0\}.$

The horizontal subspace is the translate by φ of

$\{Y : \int_{\mathbb{R}^2} \langle LY, X \rangle \, dx = 0, \text{ if } X \circ c = 0\}.$

If Y is C^∞ then $Y = 0$. So we need

$LY = c_*(p(\theta).ds)$ for $p \in C^\infty(S^1, \mathbb{R}^2)$, a distribution carried by c. Thus

$Y(x) = \int_{S^1} K(x - c(\theta))p(\theta) \, ds$

$Y(x) = \int_{S^1} K(x - c(\theta))p(\theta)\,ds$
Mapped to T_cEmb we get

$$\begin{aligned}(Y \circ c)(\theta) &= \int_{S^1} K(c(\theta) - c(\theta_1)).p(\theta_1).|c'(\theta_1)|d\theta_1 \\ &=: (K_c * p)(\theta) \qquad \text{where}\end{aligned}$$

$$K_c(\theta_1, \theta_2) := K(c(\theta_1) - c(\theta_2))$$

is an elliptic pseudo differential operator kernel of order $-2n + 1$ which is real and positive, so the operator $p \mapsto K_c * p$ is self-adjoint and positive, so injective, and by index deformation it is bijective between the Sobolev spaces on S^1. The inverse operator $(K_c * \quad)^{-1}$ has kernel $L_c(\theta, \theta_1)$ which is a pseudo differential operator kernel of order $2n - 1$.

Write $h = Y \circ c \in T_c \operatorname{Emb}(S^1, \mathbb{R}^2)$ and express the horizontal lift $Y = Y_h$ in terms of h:
$h = Y_h \circ c = K * (c_*(p.ds)) = K_c * p$ so $p = L_c * h$

$Y = Y_h = K * (c_*((L_c * h).ds))$
$Y_h(x) =$
$$= \int_{S^1} K(x - c(\theta)). \int_{S^1} L_c(\theta, \theta_1) h(\theta_1) |c'(\theta_1)| d\theta_1 |c'(\theta)| d\theta$$
Finally the metric:
$$G_c^{\mathrm{diff},n}(h,k) = \int_{\mathbb{R}^2} \langle LY_h, Y_k \rangle dx$$
$$= \iint_{S^1 \times S^1} L_c(\theta, \theta_1) \langle h(\theta_1), k(\theta) \rangle ds_1 \, ds.$$
This formula looks innocent, but there is an inversion of the (nice) operator $K_c *$ in it to get $L_c * = (K_c * \quad)^{-1}$

We can now compute K and H and the geodesic equation. It becomes simpler if written for the 1-current $L_c * c_t = p.|c_\theta| =: \alpha$:

$$\boxed{\alpha_t(\theta_0) = -\int_{S^1} K_c'(\theta_0, \theta_1)\langle \alpha(\theta_0), \alpha(\theta_1)\rangle \, d\theta_1}$$

where $K_c'(\theta_1, \theta_2) = \operatorname{grad} K(c(\theta_1) - c(\theta_2))$.

Existence of geodesics. Theorem.
Let $n \geq 1$. For each $k > 2n - \frac{1}{2}$ the geodesic equation has unique local solutions in the Sobolev space of H^k-embeddings. The solutions are C^∞ in t and in the initial conditions $c(0,\ .\ )$ and $c_t(0,\ .\ )$. The domain of existence (in t) is uniform in k and thus this also holds in $\operatorname{Emb}(S^1, \mathbb{R}^2)$.

Conserved momenta: Along a geodesic c,

$$G_c^{\mathrm{diff},n}(c_\theta.X, c_t) =$$
$$= \iint_{S^1\times S^1} L_c(\theta,\theta_1)\langle c_\theta(\theta_1)X(\theta_1), c_t(\theta)\rangle ds_1\, ds$$

is conserved for every vector field X on S^1; the conserved **reparametrization momentum** is $\langle c_\theta, L_c * c_t\rangle = \langle c_\theta, \alpha\rangle$.

Also $\iint_{(S^1)^2} L_c(\theta,\theta_1)c_t(\theta)\rangle ds_1\, ds = \int_{S^1} \alpha(\theta)\, ds$ is the conserved **linear momentum**.

$$\iint_{S^1\times S^1} L_c(\theta,\theta_1)\langle Jc(\theta_1), c_t(\theta)\rangle ds_1\, ds =$$
$$= \int_{S^1} \langle Jc(\theta), \alpha(\theta)\rangle ds$$

is the conserved **angular momentum**.

Horizontal geodesics.

A field h along c is horizontal if $\langle L_c * h, c_\theta \rangle = 0$. For a horizontal path we have $\langle \alpha, c_\theta \rangle = 0$, so let $\alpha = \tilde{a}.n$. Then the horizontal geodesic equation is

$$\boxed{\begin{aligned}&\tilde{a}_t(\theta) = \langle \alpha_t, n \rangle(\theta) = \\ &\ = -\int_{S^1} \langle K_c'(\theta,\theta_1), n(\theta)\rangle \tilde{a}(\theta)\tilde{a}(\theta_1)\langle n(\theta), n(\theta_1)\rangle \, d\theta_1\end{aligned}}$$

Note that also $n = Jc_\theta/|c_\theta|$ appears. It is a strange equation, but it is well-posed by the theorem above.

Requirements for infinite dimensional manifolds

Let (M,g) be a *weak Riemannian manifold* modelled on convenient locally convex vector spaces. For $x\in M$ the metric $g_x : T_xM \to T_x^*M$ is usually only injective (weak metric). The image $g(TM) \subset T^*M$ is called the *smooth cotangent bundle* associated to g. Now $\Omega^1_g(M) := \Gamma(g(TM))$ and $\alpha^\sharp = g^{-1}\alpha \in \mathfrak{X}(M), X^\flat = gX$ are as above. The exterior derivative restricts to

$d : \Omega^1_g(M) \to \Omega^2(M) = \Gamma(L^2_{\text{skew}}(TM;\mathbb{R}))$

since the embedding $g(TM) \subset T^*M$ is a smooth fiber linear mapping.

Further requirements need to be imposed on (M,g). $g : TM \to T^*M$ is only injective in general, so the Levi-Civita covariant derivative might not exist in TM. Existence of ∇^g is equivalent to: The metric itself admits gradients with respect to itself: We express this is locally. So let for the moment M be a c^∞-open subset of a convenient vector space V_M. Then we assume that we can write

$$\boxed{\begin{aligned} D_{x,Z}g_x(X,Y) &= g_x(Z, \operatorname{grad}_1 g(x)(X,Y)) \\ &= g_x(\operatorname{grad}_2 g(x)(Z,X), Y) \end{aligned}}$$

where $\operatorname{grad}_1 g, \operatorname{grad}_2 g : M \times V_M \times V_M \to V_M$,
$(x,X,Y) \mapsto \operatorname{grad}_{1,2} g(x)(X,Y)$,
are smooth and bilinear in $X,Y \in V_M$.
Then the derivation of Mario's formula goes through and the final formula for curvature holds in both the finite and infinite dimensional cases.

Some constructions above encountered a second problem: they lead to vector fields whose values do not lie in T_xM, but in the Hilbert space completion $\overline{T_xM}$ with respect to $\| \quad \|_{g_x}$. To manipulate these as in the finite dimensional case, we need to know that $\bigcup_{x\in M} \overline{T_xM}$ forms a smooth vector bundle over M. In other words, in each coordinate chart on an open subset $U \subset M$, $TM|_U$ is a trivial bundle $U \times V$ and all the inner products $g_x, x \in U$ define inner products on one and the same topological vector space V. We assume that they are all bounded with respect to each other, so that the completion $\overline{V}$ of V with respect to g_x does not depend on x and $\bigcup_{x\in U} \overline{T_xM} \cong U \times \overline{V}$.

This means that $\bigcup_{x \in M} \overline{T_x M}$ forms a smooth vector bundle over M with trivialisations the linear extensions of the trivialisations of the bundle $TM \to M$. These two properties will be sufficient for all the constructions we need so we make them into a definition:

Definition. A convenient weak Riemannian manifold (M, g) will be called a *robust* Riemannian manifold if

(1) The metric g_x admits gradients in the above two senses,

(2) The completions $\overline{T_x M}$ form a vector bundle as above.

Covariant curvature and O'Neill's formula in infinite dimensions. Let $p:(E,g_E)\to(B,g_B)$ be a Riemann submersion between infinite dimensional robust Riemann manifolds; i.e., for each $b\in B$ and $x\in E_b:=p^{-1}(b)$ the tangent mapping $T_xp:(T_xE,g_E)\to(T_bB,g_B)$ is a surjective metric quotient map so that

$$\|\xi_b\|_{g_B}:=\inf\Big\{X_x\in T_xE:T_xp.X_x=\xi_b\Big\}.$$

The infinimum need not be attained in T_xE but will be in the completion $\overline{T_xE}$. The orthogonal subspace $\{Y_x:g_E(Y_x,T_x(E_b))=0\}$ will therefore be taken in $\overline{T_x(E_b)}$ in T_xE.

If $\alpha_b = g_B(\alpha_b^\sharp, \quad) \in g_B(T_bB) \subset T_b^*B$ is an element in the g_B-smooth dual,
then $p^*\alpha_b := (T_xp)^*(\alpha_b) = g_B(\alpha_b^\sharp, T_xp \quad) : T_xE \to \mathbb{R}$ is in T_x^*M but in general it is not an element in the smooth dual $g_E(T_xE)$. It is, however, an element of the Hilbert space completion $\overline{g_E(T_xE)}$ of the g_E-smooth dual $g_E(T_xE)$ with respect to the norm $\| \quad \|_{g_E^{-1}}$, and the element
$g_E^{-1}(p^*\alpha_b) =: (p^*\alpha_b)^\sharp$ is in the $\| \quad \|_{g_E}$-completion $\overline{T_xE}$ of T_xE. We can call $g_E^{-1}(p^*\alpha_b) =: (p^*\alpha_b)^\sharp$ the *horizontal lift* of $\alpha_b^\sharp = g_B^{-1}(\alpha_b) \in T_bB$.

The metric $(g_E)_x$ can be evaluated at elements in the completion $\overline{T_x E}$. Moreover, for any smooth sections $X, Y \in \Gamma(\overline{TE})$ the mapping

$$g_E(X, Y) : M \to \mathbb{R}$$

is still smooth, by the smooth uniform boundedness theorem.

Lemma. *If α is a smooth 1-form on an open subset U of B with values in the g_B-smooth dual $g_B(TB)$, then $p^*\alpha$ is a smooth 1-form on $p^{-1}(U) \subset E$ with values in the $\| \quad \|_{g_E^{-1}}$-completion of the g_E-smooth dual $g_E(TE)$. Thus also $(p^*\alpha)^\sharp$ is smooth from E into the g_E-completion of TE, and it has values in the g_E-orthogonal subbundle to the vertical bundle in the g_E-completion. We may continuously extend $T_x p$ to the $\| \quad \|_{g_E}$-completion, and then we have $Tp \circ (p^*\alpha)^\sharp = \alpha^\sharp \circ p$. Moreover, the Lie bracket of two such forms, $[(p^*\alpha)^\sharp, (p^*\beta)^\sharp]$, is defined. The exterior derivative $d(p^*\alpha)$ is defined and is applicable to vector fields with values in the completion like $(p^*\beta)^\sharp$.*

That the Lie bracket is defined, is also a non-trivial statement: We have to differentiate in directions which are not tangent to the manifold.

Theorem. *Let* $p : (E, g_E) \to (B, g_B)$ *be a Riemann submersion between infinite dimensional robust Riemann manifolds. Then for 1-forms* $\alpha, \beta \in \Omega^1_g(B)$ *O'Neill's formula holds in the form:*

$$\boxed{\begin{aligned} &g_B\Big(R^B(\alpha^\sharp,\beta^\sharp)\beta^\sharp,\alpha^\sharp\Big) = \\ &\qquad = g_E\Big(R^E((p^*\alpha)^\sharp,(p^*\beta)^\sharp)(p^*\beta)^\sharp,(p^*\alpha)^\sharp\Big) \\ &\qquad\quad + \tfrac{3}{4}\|[(p^*\alpha)^\sharp,(p^*\beta)^\sharp]^{ver}\|^2_{g_E} \end{aligned}}$$

Curvature computations

In terms of the dual momenta

$\alpha = (L_c * h)\, ds = (L_c * h)|c_\theta|\, d\theta$

in $L_c * T_c \operatorname{Emb}(S^1, \mathbb{R}^2) \subset \mathcal{D}'(S^1)^2 \otimes \mathbb{R}^2$, the metric looks particularly simple:

$$(G^{\text{diff},n})_c^{-1}(\alpha, \beta) = \iint_{S^1 \times S^1} K_c(\theta, \theta_1) \langle \alpha(\theta_1), \beta(\theta) \rangle$$

We use again the cotangent expression of curvature for constant (not depending on c) 1-forms α, β in $L_c * C^\infty(S^1, \mathbb{R}^2) \subset \mathcal{D}'(S^1)^2 \otimes \mathbb{R}^2$, where $\alpha^\sharp = K_c * \alpha$, etc

$$\begin{aligned}
&4G^{\text{diff},n}\big(R(\alpha^\sharp, \beta^\sharp)\alpha^\sharp, \beta^\sharp\big) = \\
&= G^{-1}\big(d(\|\alpha\|^2), d(\|\beta\|^2)\big) - \|d(G^{-1}(\alpha, \beta))\|^2 + 3G\big([\alpha^\sharp, \beta^\sharp], [\alpha^\sharp, \beta^\sharp]\big) \\
&\quad - 2\alpha^\sharp\alpha^\sharp(\|\beta\|^2) - 2\beta^\sharp\beta^\sharp(\|\alpha\|^2) + 2(\alpha^\sharp\beta^\sharp + \beta^\sharp\alpha^\sharp)G^{-1}(\alpha, \beta)
\end{aligned}$$

$$
\begin{aligned}
&4G^{\mathsf{diff},n}\big(R(\alpha^\sharp,\beta^\sharp)\alpha^\sharp,\beta^\sharp\big) = \\
&= \iiiint_{(S^1)^4} \Big(\det \begin{pmatrix} \langle \alpha(\theta_1),\alpha(\theta_2)\rangle & \langle \alpha(\theta_1),\beta(\theta_2)\rangle \\ \langle \alpha(\theta_3),\beta(\theta_4)\rangle & \langle \beta(\theta_3),\beta(\theta_4)\rangle \end{pmatrix} \\
&\qquad \Big\langle \operatorname{grad} K(c(\theta_1)-c(\theta_2)), \operatorname{grad} K(c(\theta_3)-c(\theta_4))\Big\rangle \\
&\qquad \cdot \big(K_c(\theta_1,\theta_3) - 2K_c(\theta_1,\theta_4) + K_c(\theta_2,\theta_4)\big) \\
&+ 3 \iint (S^1)^2 L_c(\theta_3,\theta_4) \\
&\qquad \Big\langle \int_{S^1} \big(\Big\langle \operatorname{grad} K(c(\theta_3)-c(\theta_1)), \alpha^\sharp(\theta_3)-\alpha^\sharp(\theta_1)\Big\rangle \beta(\theta_1) \\
&\qquad\qquad - \Big\langle \operatorname{grad} K(c(\theta_3)-c(\theta_1)), \beta^\sharp(\theta_3)-\beta^\sharp(\theta_1)\Big\rangle \alpha(\theta_1)\big), \\
&\qquad \int_{S^1} \big(\Big\langle \operatorname{grad} K(c(\theta_4)-c(\theta_2)), \alpha^\sharp(\theta_4)-\alpha^\sharp(\theta_2)\Big\rangle \beta(\theta_2) \\
&\qquad\qquad - \Big\langle \operatorname{grad} K(c(\theta_3)-c(\theta_1)), \beta^\sharp(\theta_3)-\beta^\sharp(\theta_1)\Big\rangle \alpha(\theta_1)\big) \Big\rangle \\
&\quad + \iint_{(S^1)^2} \Big(-2\langle \beta(\theta_1),\beta(\theta_2)\rangle\, d^2K(c(\theta_1)-c(\theta_2)) \\
&\qquad\qquad\qquad \big(\alpha^\sharp(\theta_1)-\alpha^\sharp(\theta_2), \alpha^\sharp(\theta_1)-\alpha^\sharp(\theta_2)\big)
\end{aligned}
$$

$$- 2\langle \alpha(\theta_1), \alpha(\theta_2) \rangle \, d^2K(c(\theta_1) - c(\theta_2))\big(\beta^\sharp(\theta_1) - \beta^\sharp(\theta_2), \beta^\sharp(\theta_1) - \beta^\sharp(\theta_2$$

$$+ 4\langle \alpha(\theta_1), \beta(\theta_2) \rangle \, d^2K(c(\theta_1) - c(\theta_2))\big(\alpha^\sharp(\theta_1) - \alpha^\sharp(\theta_2), \beta^\sharp(\theta_1) - \beta^\sharp(\theta_2$$

High dimensional shape space $\operatorname{Imm}(M,N)/\operatorname{Diff}(M)$.

M, a compact smooth connected manifold of dimension $m \geq 1$.
(N,g) a connected Riemannian manifold of dimension $n > m$.

$\operatorname{Diff}(M)$, the regular Lie group of all diffeomorphisms of M.

$\operatorname{Diff}_{x_0}(M)$, the subgroup of diffeomorphisms fixing $x_0 \in M$.

$\operatorname{Emb} = \operatorname{Emb}(M,N)$, the manifold of all smooth embeddings $M \to N$.

$\mathrm{Imm} = \mathrm{Imm}(M, N)$, the manifold of all smooth immersions $M \to N$.

$\mathrm{Imm}_{\mathrm{free}} = \mathrm{Imm}_{\mathrm{free}}(M,N)$, the manifold of all smooth free immersions $M \to N$ (those with trivial isotropy group for the right action of $\mathrm{Diff}(M)$ on $\mathrm{Imm}(M,N)$).

$B_e = B_e(M,N) = \mathrm{Emb}(M,N)/\,\mathrm{Diff}(M)$, the manifold of submanifolds of type M in N, base of a smooth principal bundle.

$B_i = B_i(M,N) = \mathrm{Imm}(M,N)/\,\mathrm{Diff}(M)$, an infinite dimensional 'orbifold'.

$B_{i,f} = B_{i,}(M,N) = \mathrm{Imm}_f(M,\mathbb{R}^2)/\,\mathrm{Diff}(M)$, a manifold, the base of a principal fiber bundle.

Free immersions

An immersion $f : M \to N$ is called *free* if $\mathrm{Diff}(M)$ acts freely on it, i.e., $f \circ \varphi = f$ for $\varphi \in \mathrm{Diff}(M)$ implies $\varphi = \mathrm{Id}$. We have the following results:

- *If* $\varphi \in \mathrm{Diff}(M)$ *has a fixed point and if* $f \circ \varphi = f$ *for any immersion* f *then* $\varphi = \mathrm{Id}$.
- *If for* $f \in \mathrm{Imm}(M, N)$ *there is a point* $x \in c(M)$ *with only one preimage then* f *is a free immersion.* There exist free immersions without such points.

We might view $\mathrm{Imm}_f(M, N)$ as the nonlinear Stiefel manifold of parametrized submanifolds of type M in N and consequently $B_{i,f}(M, N)$ as the nonlinear Grassmannian of unparametrized submanifolds of type M.

Non free immersions. Since M is compact, the orbit space $B_i(M,N) = \operatorname{Imm}(M,N)/\operatorname{Diff}(M)$ is Hausdorff. For any immersion f the isotropy group $\operatorname{Diff}(M)_f$ is a finite group which acts as group of covering transformations for a finite covering $q_f : M \to \bar{M}$ such that f factors over q_f to a free immersion $\bar{f} : \bar{M} \to N$ with $\bar{f} \circ q_f = f$.
For each $f \in \operatorname{Imm}$ there exist a slice $\mathcal{Q}(f)$ in a strong sense:
• $\mathcal{Q}(f)$ is invariant under the isotropy group $\operatorname{Diff}(M)_f$.
• If $(Q(f) \circ \varphi) \cap \mathcal{Q}(f) \neq \emptyset$ for $\varphi \in \operatorname{Diff}(M)$ then φ is in the isotropy group $\varphi \in \operatorname{Diff}(M)_f$.
• $\mathcal{Q}(f) \circ \operatorname{Diff}(M)$ is an invariant open neighbourhood of the orbit $f \circ \operatorname{Diff}(M)$ in $\operatorname{Imm}(M,N)$ admitting a smooth retraction r onto the orbit. The fiber $r^{-1}(f \circ \varphi)$ equals $\mathcal{Q}(f \circ \varphi)$.

We do not have a principal bundle and thus no principal connections, but we can prove the main consequence, the existence of horizontal paths, directly:

Lemma. *For any smooth path f in* $\operatorname{Imm}(M,N)$ *there exists a smooth path φ in* $\operatorname{Diff}(M)$ *with $\varphi(t, \quad) = \mathrm{Id}_M$ depending smoothly on f such that the path h given by $h(t,\theta) = c(t,\varphi(t,\theta))$ is horizontal: $g(h_t, Th) = 0$.*

Volumes of an immersion. For an immersion $f \in \mathrm{Imm}(M,N)$, we consider the volume density $\mathrm{vol}^g(f) = \mathrm{vol}(f^*g) \in \mathrm{Vol}(M)$ on M given by
$\mathrm{vol}^g(f)|_U = \sqrt{\det((f^*g)_{ij})}|du^1 \wedge \cdots \wedge du^m|$
for any chart $(U, u : U \to \mathbb{R}^m)$ of M.

Lemma. *The derivative of* $\mathrm{vol}^g : \mathrm{Imm}(M,N) \to \mathrm{Vol}(M)$ *is*

$$\begin{aligned} d\,\mathrm{vol}^g(f)(h) = -\,\mathrm{Tr}^{f^*g}(g(S^f, h^\perp))\,\mathrm{vol}(f^*g) + \\ + \,\mathrm{div}^{f^*g}(h^\top)(f^*g))\,\mathrm{vol}(f^*g). \end{aligned}$$

The second summand vanishes when integrated over M.

The metric on Imm. Let $h, k \in C^\infty_f(M, TN)$ be tangent vectors with foot point $f \in \text{Imm}(M,N)$, i.e., vector fields along f. We consider the following weak Riemannian metric on
$\text{Imm}(M,N)$, for a constant $A \geq 0$:

$$G^A_f(h,k) :=$$
$$= \int_M (1 + A\| \operatorname{Tr}^{f^*g}(S^f)\|^2_{g^{N(f)}}) g(h,k) \operatorname{vol}(f^*g)$$

where $\| \operatorname{Tr}^{f^*g}(S^f)\|_{g^{N(f)}}$ is the norm of the mean curvature. The metric G^A is invariant for the action of $\text{Diff}(M)$. This makes the map $\pi : \text{Imm}(M,N) \to B_i(M,N)$ into a *Riemannian submersion* (off the singularities of $B_i(M,N)$).

The tangent vectors to the orbits are $T_f(f\circ \mathrm{Diff}(M)) = \{Tf.\xi : \xi \in \mathfrak{X}(M)\}$. The bundle $\mathcal{N} \to \mathrm{Imm}(M,N)$ of tangent vectors normal to the $\mathrm{Diff}(M)$-orbits is independent of A:

$$\begin{aligned}\mathcal{N}_f &= \{h \in C^\infty(M,TN) : g(h,Tf) = 0\}\\ &= \Gamma(f^*(TN|_M/Tf.TM)) = \Gamma(f^*TN/TM),\end{aligned}$$

the space of sections of the normal bundle.
A tangent vector
$h \in T_f\,\mathrm{Imm}(M,N) = C^\infty_f(M,TN) = \Gamma(f^*TN)$ has an orthonormal decomposition

$$h = h^\top + h^\perp \in T_f(f\circ \mathrm{Diff}^+(M)) \oplus \mathcal{N}_f$$

into smooth tangential and normal components.

The metric G^A on $\operatorname{Imm}(M,N)$ is invariant under $\operatorname{Diff}(M)$ and induces a metric on the quotient $B_i(M,N)$ For any $F_0, F_1 \in B_i$, consider all liftings $f_0, f_1 \in \operatorname{Imm}$ such that $\pi(f_0) = F_0, \pi(f_1) = F_1$ and all smooth curves $t \mapsto f(t, \quad)$ in $\operatorname{Imm}(M,N)$ with $f(0,\cdot) = f_0$ and $f(1,\cdot) = f_1$. The length of $t \mapsto \pi(f(t,\cdot))$ in $B_i(M,N)$ is given by

$$
\begin{aligned}
&L^{\text{hor}}_{G^A}(f) := L_{G^A}(\pi(f(t,\cdot))) = \\
&= \int_0^1 \sqrt{G^A_{\pi(f)}(T_f\pi.f_t, T_f\pi.f_t)}\,dt = \int_0^1 \sqrt{G^A_f(f_t^\perp, f_t^\perp)}\,dt \\
&= \int_0^1 \Big(\int_M (1 + A\| \operatorname{Tr}^{f^*g}(S^f)\|_g^2) g(f_t^\perp, f_t^\perp) \operatorname{vol}(f^*g) \Big)^{\frac{1}{2}} dt
\end{aligned}
$$

In fact the last computation only makes sense on $B_{i,f}(M,N)$ but we take it as a motivation.

The metric on $B_i(M,N)$ is defined by taking the infimum of this over all paths f (and all lifts f_0, f_1):

$$\operatorname{dist}^{B_i}_{G^A}(F_1,F_2) = \inf_f L^{\text{hor}}_{G^A}(f).$$

Theorem. *For $f_0, f_1 \in \operatorname{Imm}(M,N)$ there exists always a path $t \mapsto f(t,\cdot)$ in $\operatorname{Imm}(M,N)$ with $f(0,\cdot) = f_0$ and $\pi(f(1,\cdot)) = \pi(f_1)$ such that $L^{hor}_{G^0}(f)$ is arbitrarily small.*

So the lowest order metric is **not** suitable for vision. Sketch the proof!

Lipschitz continuity of $\sqrt{\mathrm{Vol}^g} : B_i(M,N) \to \mathbb{R}_{\geq 0}$.
For F_0 and F_1 in $B_i(M,N) = \mathrm{Imm}(M,N)/\mathrm{Diff}(M)$ we have for $A > 0$:

$$\sqrt{\mathrm{Vol}^g(F_1)} - \sqrt{\mathrm{Vol}^g(F_0)} \leq \frac{1}{2\sqrt{A}} \mathrm{dist}^{B_i}_{G^A}(F_1, F_2).$$

Area swept out bound. *If f is any path from F_0 to F_1, then*

$$\begin{pmatrix} (m+1)-\textit{volume of the region swept} \\ \textit{out by the variation } f \end{pmatrix} \leq$$

$$\leq \max_t \sqrt{\mathrm{Vol}^g(f(t,\quad))} \cdot L_{G^A}^{hor}(f).$$

Together with Lipschitz continuity this shows that the geodesic distance $L_{G^A}^{B_i}$ separates points at least on $B_e(M,N)$, if $A > 0$.

Horizontal energy of a path as anisotropic volume We consider a path $t \mapsto f(t, \quad)$ in $\mathrm{Imm}(M, N)$. It projects to a path $\pi \circ f$ in B_i whose energy is:

$$\begin{aligned} E_{G^A}(\pi \circ f) &= \tfrac{1}{2} \int_a^b G^A_{\pi(f)}(T\pi.f_t, T\pi.f_t)\, dt = \\ &= \tfrac{1}{2} \int_a^b G^A_f(f_t^\perp, f_t^\perp)\, dt = \\ &= \tfrac{1}{2} \int_a^b \int_M (1 + A\| \operatorname{Tr}^{f^*g}(S^f)\|_g^2) g(f_t^\perp, f_t^\perp) \operatorname{vol}(f^*g)\, dt. \end{aligned}$$

We now consider the graph $\gamma_f : [a,b] \times M \ni (t,x) \mapsto (t, f(t,x)) \in [a,b] \times N$ of the path f and its image Γ_f, an immersed submanifold with boundary of $\mathbb{R} \times N$.

$$
\begin{aligned}
&E^{\mathsf{hor}}_{G^A}(\pi\circ f) = \\
&= \frac{1}{2}\int_{[a,b]\times M} (1 + A\|\operatorname{Tr}^{f^*g}(S^f)\|^2_{g^{N(f)}}) \\
&\qquad\qquad \times \frac{\|f_t^\perp\|^2}{\sqrt{1+\|f_t^\perp\|^2_g}}\operatorname{vol}(\gamma_f^*(dt^2+g))
\end{aligned}
$$

This is intrinsic for the graph Γ_f and the fibration $\mathrm{pr}_1 : \mathbb{R}\times N \to \mathbb{R}$. To find a geodesic between the shapes $\pi(f(a,\quad))$ and $\pi(f(b,\quad))$ we look for an immersed surface which is critical for $E^{\mathsf{hor}}_{G^A}$. This is a Plateau-problem with anisotropic volume.

The geodesic equation of G^0 in $\mathrm{Imm}(M,N)$

$$\boxed{\begin{aligned}&\nabla^g_{\partial_t} f_t + \mathrm{div}^{f^*g}(f_t^\top) f_t - g(f_t^\perp, \mathrm{Tr}^{f^*g}(S^f)) f_t + \\ &+ \tfrac{1}{2} Tf.\,\mathrm{grad}^{f^*g}(\|f_t\|^2_g) + \tfrac{1}{2}\|f_t\|^2_g\, \mathrm{Tr}^{f^*g}(S^f) = 0\end{aligned}}$$

We restrict to geodesics $t \mapsto f(t, \quad)$ in $\mathrm{Imm}(M, N)$ which are horizontal: $g(f_t, Tf) = 0$. Then $f_t^\top = 0$ and $f_t = f_t^\perp$, so the equation splits into a vertical (tangential) part which vanishes identically, and a horizontal (normal) part which is **the geodesic equation in** B_i for G^0:

$$\boxed{\begin{aligned} &\nabla^{N(f)}_{\partial_t} f_t - g(f_t, \mathrm{Tr}^{f^*g}(S^f))f_t + \\ &\qquad\qquad + \tfrac{1}{2}\|f_t\|^2_g \, \mathrm{Tr}^{f^*g}(S^f) = 0. \\ &\quad g(Tf, f_t) = 0. \end{aligned}}$$

The sectional curvature for G^0 in $B_i(M,N)$

$$k_f(P(m,h)) = -\frac{G^0_f(R(m,h)m,h)}{\|m\|^2\|h\|^2 - G^0_a(m,h)^2}.$$

We get then for $x, y \in \Gamma(\mathcal{N}_f)$:

$$R_f(x,y,x,y) = G^0_f(R_f(x,y)x,y) =$$

$$= \int_M \mathrm{vol}(f^*g)\Bigg($$

$$\begin{aligned}
&- \tfrac{1}{2}\widetilde{\mathrm{Tr}}(L^f \circ L^f)(x \wedge y) && \leq 0\\
&- \tfrac{1}{4}\|\,\mathrm{Tr}(L^f_x)y - \mathrm{Tr}(L^f_y)x\|^2_g && \leq 0\\
&+ \tfrac{1}{4}\|x \wedge y\|^2\|\,\mathrm{Tr}^g(S^f)\|^2 && \geq 0\\
&+ g(R^g(x,y)x,y) &&\\
&+ \|x \wedge y\|^2\,\mathrm{Ric}(TM, \mathrm{span}(x,y)) &&\\
&- \tfrac{1}{2}\|(g(x,\nabla^\perp y) - g(y,\nabla^\perp x)\|^2_{\Omega^1_M} && \leq 0
\end{aligned}$$

$$+\tfrac{1}{2}\|x\wedge\nabla^{\perp}y-y\wedge\nabla^{\perp}x\|^2_{\Omega^1_M\otimes\wedge^2 N(f)}\Bigg)\qquad\geq 0.$$

Corollary. *If M has codimension 1 in N then all sectional curvatures are non-negative. For any codimension, sectional curvature in the plane spanned by x and y is non-negative if x and y are parallel, i.e., $x\wedge y=0$ in $\bigwedge^2 T^*N$.*

Vanishing geodesic distance on groups of diffeomorphisms:

(N, g) a connected Riemannian manifold.

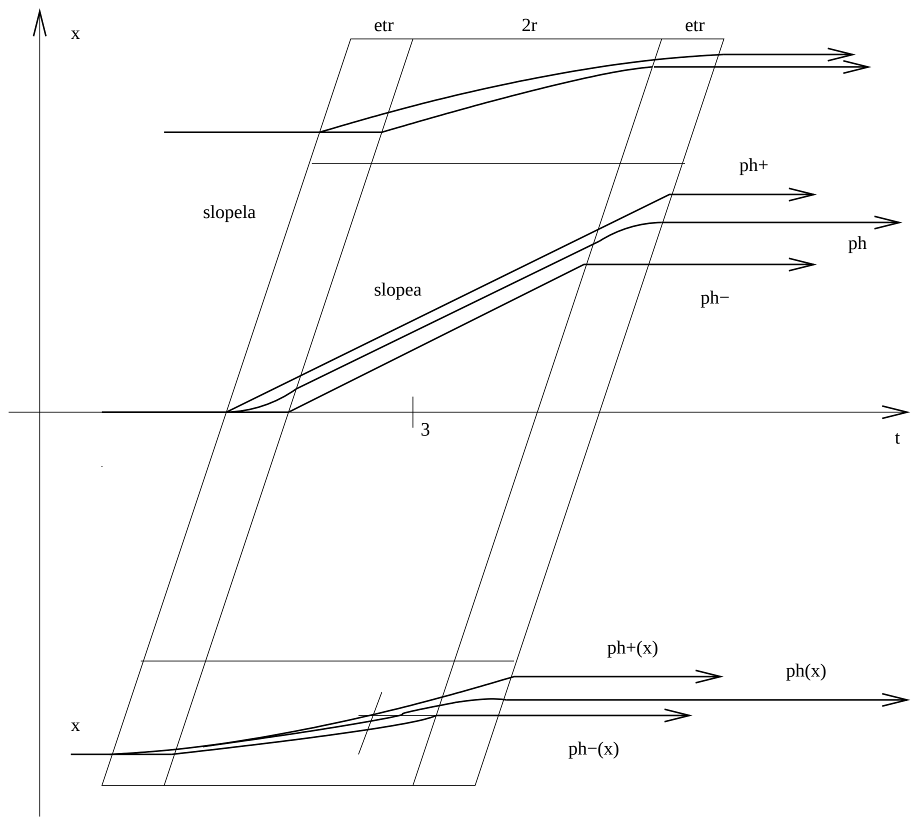
$\mathrm{Diff}_c(N)$ the group of all diffeomorphisms with compact support on N,

$\mathrm{Diff}_0(N)$ the subgroup of those which are diffeotopic in $\mathrm{Diff}_c(N)$ to the identity; this is the connected component of the identity in $\mathrm{Diff}_c(N)$, which a regular Lie group. The Lie algebra is $\mathfrak{X}_c(N)$, the space of all smooth vector fields with compact support on N. Moreover, $\mathrm{Diff}_0(N)$ is a **simple group** (has no nontrivial normal subgroups).

The *right invariant* H^0-metric on $\mathrm{Diff}_0(N)$ is then given as follows, where $h, k : N \to TN$ are vector fields with compact support along φ and where $X = h \circ \varphi^{-1}, Y = k \circ \varphi^{-1} \in \mathfrak{X}_c(N)$:

$$\begin{aligned} G^0_\varphi(h,k) &= \int_N g(h,k)\,\mathrm{vol}(\varphi^* g) \\ &= \int_N g(X\circ\varphi, Y\circ\varphi)\varphi^*\mathrm{vol}(g) \\ &= \int_N g(X,Y)\,\mathrm{vol}(g) \end{aligned}$$

Theorem. *Geodesic distance on* $\mathrm{Diff}_0(N)$ *with respect to the* H^0*-metric vanishes.*


x
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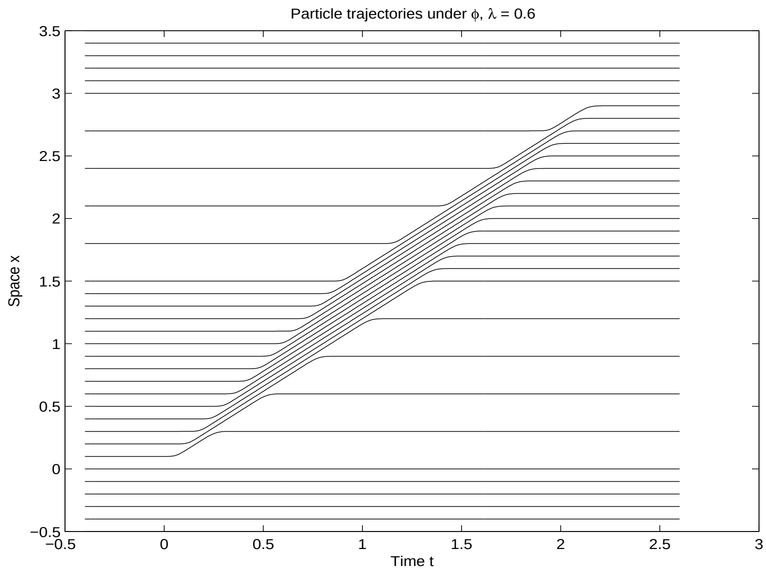
Particle trajectories under ϕ, λ = 0.6
Space x
Time t
3.5
3
2.5
2
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1
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2
2.5
3

Geodesics and sectional curvature on $\mathrm{Diff}(N)$: For a right invariant weak Riemannian metric G on an (possibly infinite dimensional) Lie group the geodesic equation and the curvature are given in terms of the dual operator (if it exists) $\mathrm{ad}(X)^*$ of the adjoint $\mathrm{ad}(X) : \mathfrak{g} \to \mathfrak{g}$ on the Lie algebra by: following formulas:

$$u_t = -\,\mathrm{ad}(u)^* u, \quad u = \varphi_t \circ \varphi^{-1}$$

$$\begin{aligned}
4G(R(X,Y)X,Y) &= 3G([X,Y],[X,Y]) \\
&\quad - 2G(X,[Y,[X,Y]]) - 2G(Y,[X,[Y,X]]) \\
&\quad + 4G(\mathrm{ad}(X)^*X, \mathrm{ad}(Y)^*Y) \\
&\quad - G(\mathrm{ad}(X)^*Y + \mathrm{ad}(Y)^*X, \mathrm{ad}(X)^*Y + \mathrm{ad}(Y)^*X)
\end{aligned}$$

In our case, for $\mathrm{Diff}_0(N)$, we have
$\mathrm{ad}(X)Y = -[X,Y]$

$$G^0(X,Y) = \int_N g(X,Y)\,\mathrm{vol}(g)$$

$$G^0(\mathrm{ad}(Y)^* X, Z) = G^0(X, -[Y,Z]) =$$

$$= \int_N g\Big(\mathcal{L}_Y X + (g^{-1}\mathcal{L}_Y g)X + \mathrm{div}^g(Y)X, Z\Big)\mathrm{vol}(g)$$

$$\mathrm{ad}(Y)^* = \mathcal{L}_Y + g^{-1}\mathcal{L}_Y(g) + \mathrm{div}^g(Y) = \mathcal{L}_Y + \beta(Y),$$

where the tensor field
$\beta(Y) = g^{-1}\mathcal{L}_Y(g) + \mathrm{div}^g(Y) : TN \to TN$
is self adjoint with respect to g.

Thus the geodesic equation for G^0 is

$$u_t = -(g^{-1}\mathcal{L}_u(g))(u) - \mathrm{div}^g(u)u = -\beta(u)u,$$
$$u = \varphi_t \circ \varphi^{-1}.$$

The main part of the sectional curvature is given by:

$$\begin{aligned}
&4G(R(X,Y)X,Y) = \\
&= \int_N \Big(-\|\beta(X)Y - \beta(Y)X + [X,Y]\|_g^2 \\
&\qquad\qquad - 4g([\beta(X),\beta(Y)]X,Y) \Big) \,\mathrm{vol}(g)
\end{aligned}$$

So sectional curvature consists of a part which is visibly non-negative, and another part which is difficult to decompose further.

Example. For $(N,g) = (\mathbb{R}, \text{can})$ or (S^1, can) the geodesic equation is *Burgers' equation*, a completely integrable infinite dimensional system,

$$u_t = -3u_x\, u, \qquad u = \varphi_t \circ \varphi^{-1},$$

to which corresponds vanishing geodesic distance. and we get $G^0(R(X,Y)X,Y) = -\int [X,Y]^2\, dx$ so that all sectional curvatures are non-negative.

Example. For $(N,g) = (\mathbb{R}^n, \text{can})$ or $((S^1)^n, \text{can})$:

$$(\text{ad}(X)Y)^k = \sum_i ((\partial_i X^k)Y^i - X^i(\partial_i Y^k))$$

$$G^0(\text{ad}(X)Y, Z) = \int_{\mathbb{R}^n} \langle dX.Y - dY.X, Z\rangle dx$$

$$= \int_{\mathbb{R}^n} \sum_{i,k} Y^k \Big((\partial_k X^i)Z^i + (\partial_i X^i)Z^k + X^i(\partial_i Z^k)\Big) dx$$

$$(\text{ad}(X)^* Z)^k =$$

$$= \sum_i \Big((\partial_k X^i)Z^i + (\partial_i X^i)Z^k + X^i(\partial_i Z^k)\Big),$$

so that the geodesic equation is given by

$$\partial_t u^k = -(\text{ad}(u)^\top u)^k =$$

$$= -\sum_i \Big((\partial_k u^i)u^i + (\partial_i u^i)u^k + u^i(\partial_i u^k)\Big),$$

the n-dimensional analogon of Burgers' equation, called the basic Euler-Poincaré equation (EPDiff) by Holm. Also here we have vanishing geodesic distance.

Stronger metrics on $\mathrm{Diff}_0(N)$.

A very small strengthening of the weak Riemannian H^0-metric on $\mathrm{Diff}_0(N)$ makes it into a true metric. We define the stronger right invariant weak Riemannian metric by the formula:

$$G^A_\varphi(h,k) = \int_N (g(X,Y) + A\,\mathrm{div}_g(X).\,\mathrm{div}_g(Y))\,\mathrm{vol}(g).$$

Theorem. *For any distinct diffeomorphisms φ_0, φ_1, the infimum of the lengths of all paths from φ_0 to φ_1 with respect to G^A is positive.*

Example We consider the groups $\mathrm{Diff}_c(\mathbb{R})$ or $\mathrm{Diff}(S^1)$ with Lie algebras $\mathfrak{X}_c(\mathbb{R})$ or $\mathfrak{X}(S^1)$ with Lie bracket $\mathrm{ad}(X)Y = -[X,Y] = X'Y - XY'$. The G^A-metric equals the H^1-metric on $\mathfrak{X}_c(\mathbb{R})$, and we have:

$$\begin{aligned} G^A(X,Y) &= \int_{\mathbb{R}} (XY + AX'Y')dx \\ &= \int_{\mathbb{R}} X(1-\partial_x^2)Y\,dx, \\ \mathrm{ad}(X)^* &= (1-\partial_x^2)^{-1}(2X' + X\partial_x)(1-A\partial_x^2) \end{aligned}$$

so that the geodesic equation in Eulerian representation $u = (\partial_t\varphi)\circ\varphi^{-1} \in \mathfrak{X}_c(\mathbb{R})$ or $\mathfrak{X}(S^1)$ is

$$\begin{aligned} \partial_t u &= -\,\mathrm{ad}(u)^* u \\ &= -(1-\partial_x^2)^{-1}(3uu' - 2Au''u' - Au'''u), \\ u_t - u_{txx} &= Au_{xxx}.u + 2Au_{xx}.u_x - 3u_x.u, \end{aligned}$$

which for $A = 1$ is the *Camassa-Holm equation*, another completely integrable infinite dimensional Hamiltonian system. Here geodesic distance is a metric.

Virasoro-Bott group. Let Diff denote any of the groups $\mathrm{Diff}(S^1)$, $\mathrm{Diff}_c(\mathbb{R})$ (diffeomorphisms with compact support), or $\mathrm{Diff}_{\mathcal{S}}(\mathbb{R})$. Then

$$
\begin{aligned}
c &: \mathrm{Diff} \times \mathrm{Diff} \to \mathbb{R} \\
c(\varphi, \psi) &:= \frac{1}{2} \int \log((\varphi \circ \psi)') d \log(\psi') \\
&= \frac{1}{2} \int \log(\varphi' \circ \psi) d \log(\psi')
\end{aligned}
$$

satisfies $c(\varphi, \varphi^{-1}) = 0$, $c(\mathrm{Id}, \psi) = 0$, $c(\varphi, \mathrm{Id}) = 0$, and is a smooth Hochschild group cocycle, i.e.,

$$
c(\varphi_2, \varphi_3) - c(\varphi_1 \circ \varphi_2, \varphi_3) + c(\varphi_1, \varphi_2 \circ \varphi_3) - c(\varphi_1, \varphi_2) = 0,
$$

called the Bott cocycle.

The corresponding central extension group $\mathbb{R} \times_c$ Diff, called the Virasoro-Bott group, is a regular Lie group with operations

$$\binom{\varphi}{\alpha}\binom{\psi}{\beta} = \binom{\varphi \circ \psi}{\alpha + \beta + c(\varphi, \psi)}, \qquad \binom{\varphi}{\alpha}^{-1} = \binom{\varphi^{-1}}{\alpha^{-1}}$$

for $\varphi, \psi \in$ Diff and $\alpha, \beta \in \mathbb{R}$.

The Lie algebra of the Virasoro-Bott Lie group is the central extension $\mathbb{R}\times_\omega \mathfrak{X}$ of $\mathfrak{X}$, called the Virasoro Lie algebra, with bracket:

$$\left[\binom{X}{a},\binom{Y}{b}\right]=\binom{-[X,Y]}{\omega(X,Y)}=\binom{X'Y-XY'}{\omega(X,Y)}$$

$$\omega(X,Y)=\omega(X)Y=\int X'dY'=\int X'Y''dx=$$

$$=\tfrac{1}{2}\int \det\begin{pmatrix}X' & Y'\\ X'' & Y''\end{pmatrix}dx,$$

is the *Gelfand-Fuchs Lie algebra cocycle* $\omega:\mathfrak{X}\times\mathfrak{X}\to\mathbb{R}$, which is a bounded skew-symmetric bilinear mapping satisfying the cocycle condition

$$\omega([X,Y],Z)+\omega([Y,Z],X)+\omega([Z,X],Y)=0.$$

It is a generator of the 1-dimensional bounded Chevalley cohomology $H^2(\mathfrak{X},\mathbb{R})$ for any of the Lie algebras

$\mathfrak{X} = \mathfrak{X}(S^1)$, $\mathfrak{X}_c(\mathbb{R})$, or $\mathcal{S}(\mathbb{R})\partial_x$.

We shall use the L^2-inner product on $\mathbb{R}\times_\omega\mathfrak{X}$, where $\mathfrak{X}=\mathfrak{X}(S^1),\mathfrak{X}_c(\mathbb{R}),\mathcal{S}(\mathbb{R})\partial_x$:
$\left\langle\binom{X}{a},\binom{Y}{b}\right\rangle_0 := \int XY\,dx+ab.$
Integrating by parts we get

$$\begin{aligned}
\left\langle \operatorname{ad}\binom{X}{a}\binom{Y}{b},\binom{Z}{c}\right\rangle_0 &= \left\langle\binom{X'Y-XY'}{\omega(X,Y)},\binom{Z}{c}\right\rangle_0\\
&=\int(X'YZ-XY'Z+cX'Y'')\,dx\\
&=\int(2X'Z+XZ'+cX''')Y\,dx\\
&=\left\langle\binom{Y}{b},\operatorname{ad}\binom{X}{a}^{\top}\binom{Z}{c}\right\rangle_0,\quad \text{where}
\end{aligned}$$

$$\operatorname{ad}\binom{X}{a}^{\top}\binom{Z}{c}=\binom{2X'Z+XZ'+cX'''}{0}.$$

The H^0 geodesic equation on the Virasoro-Bott group (Ovsienko-Khesin):

$$\binom{u_t}{a_t} = -\operatorname{ad}\binom{u}{a}^{\top}\binom{u}{a} = \binom{-3u_x u - a u_{xxx}}{0} \qquad \text{where}$$

$$\begin{aligned}
\binom{u(t)}{a(t)} &= \partial_s \binom{\varphi(s)}{\alpha(s)} . \binom{\varphi(t)^{-1}}{-\alpha(t)}\Big|_{s=t} \\
&= \partial_s \binom{\varphi(s)\circ\varphi(t)^{-1}}{\alpha(s)-\alpha(t)+c(\varphi(s),\varphi(t)^{-1})}\Big|_{s=t} \\
&= \binom{\varphi_t\circ\varphi^{-1}}{\alpha_t - \int \frac{\varphi_{tx}\varphi_{xx}}{2\varphi_x^2}dx}
\end{aligned}$$

Thus a is a constant in time and the geodesic equation is hence the *Korteweg-de Vries equation*

$$u_t + 3u_x u + a u_{xxx} = 0.$$

with its natural companions

$$\varphi_t = u \circ \varphi, \qquad \alpha_t = a + \int \frac{\varphi_{tx}\varphi_{xx}}{2\varphi_x^2} dx.$$

I do not know whether the right invariant L^2-metric on the Virasoro-Bott group has vanishing geodesic distance?

On Mondays I think: YES

On Tuesdays I think: NO

. . .

At the end of last main lecture:
Many thanks to the organizers (except one of them) for a great conference, and for the fine weather and great snow.