Integrable field theories with boundaries and defects

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Contents

Integrable field theory in the presence of boundaries (one boundary or two), or defects (shocks), is an extensive subject. It is impossible to cover it comprehensively in a short series of lectures. The purpose here is to give (from a personal perspective) a flavour of some questions and techniques.

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The sine-Gordon field theory

From a physicist's perspective, began with Skyrme 1959-62.

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = -\frac{m^2}{\beta}\sin\beta u.$$

- c is a constant with the dimensions of velocity (usually set to unity),
- m is a constant with dimensions of inverse length ($\hbar m$ has the dimensions of mass);
- β is a dimensionless coupling constant.

All these constants can be removed by scaling t, x and u; important after quantization.

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For the following reasons the sine-Gordon nonlinear wave equation is the paradigm:

- it is (almost) the simplest (a single scalar field), relativistic, integrable nonlinear wave equation in two dimensions (one time, one space) (t, x);
- it is simple enough to allow direct computations in the classical or quantum domains;
- it is complicated enough to display a wide range of interesting phenomena;
- though originally studied on the range $-\infty < x < \infty$, or with periodic boundary conditions, there are new features when the model is restricted to a half-line ($x < 0$, say), or to an interval $x \in [-L, L]$, by suitable boundary conditions, or if there are 'impurities'.

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Expanding the right hand side of the sine-Gordon equation reveals....

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = -m^2 u + \dots$$

$$+\frac{m^2\beta^2}{3!}u^3 - \frac{m^2\beta^4}{5!}u^5 + \dots$$

The first three (linear) terms taken alone are simply the Klein-Gordon equation for a relativistic scalar particle with mass parameter m.

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Energy and momentum

The sine-Gordon equation provides the stationary points of an action given by the Lagrangian density:

$$\mathcal{L} = \frac{1}{2}\partial_\mu u \, \partial^\mu u - \frac{m^2}{\beta^2}(1 - \cos \beta u).$$

The corresponding conserved energy and momentum are given by

$$\mathcal{E} = \int_{-\infty}^{\infty} dx \, \left(\frac{1}{2}(u_t^2 + u_x^2) + \frac{m^2}{\beta^2}(1 - \cos \beta u) \right),$$

$$\mathcal{P} = -\int_{-\infty}^{\infty} dx \, u_t u_x.$$

Well-defined provided u is 'smooth' with $u_t, u_x \to 0, \ \beta u \to 2n\pi$, as $x \to \pm\infty$, where n is an integer or zero.

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A soliton

It is easy to check that the following gives an exact (real) solution to the sine-Gordon equation:

$$e^{i\beta u/2} = \frac{1+iE}{1-iE}, \quad E = e^{ax+bt+c},$$

where a, b are real constants satisfying

$$a^2 - b^2 = m^2,$$

and c is a constant that need not be real, but e^c is real.
Note:

- Useful to put $a = m\cosh\theta, \; b = m\sinh\theta$; and θ is the 'rapidity'.
- We take $a > 0$.

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Properties

Assume first $E > 0$ (ie $e^c > 0$).

- The spatial derivative u_x is given by

$$u_x = \frac{4a}{\beta} \frac{E}{1 + E^2},$$

  which implies u is monotonically increasing.
- As $x \to -\infty$, $e^{i\beta u/2} \to 1$; thus $u \to 0$ is a suitable choice for $x \to -\infty$.
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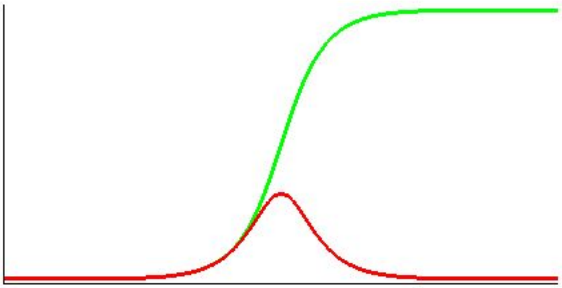
A soliton snapshot



The lower curve represents u_x (and is similar in general shape to the energy density) and the upper curve represents the soliton itself smoothly interpolating $u = 0$ to $u = 2\pi$.

The solution is changing rapidly within a small region in the neighbourhood of $x = 0$.

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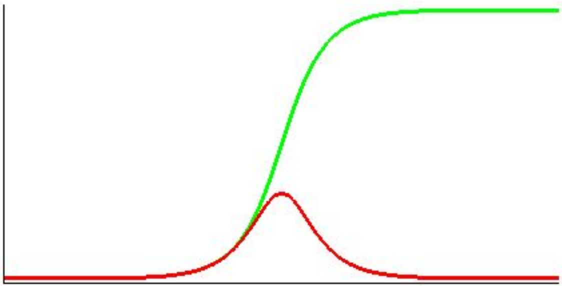

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The solution is changing rapidly within a small region in the neighbourhood of $x = 0$.

- For $\theta < 0$ the soliton is travelling along the *x*-axis in a positive direction with velocity $b/a = \tanh\theta$.
- Its energy and momentum are calculated directly to be

$$(\mathcal{E}, \mathcal{P}) = \frac{8m}{\beta^2}(\cosh\theta, \sinh\theta).$$

  This expression is the energy-momentum of a relativistic particle ($c = 1$) of mass $M = 8m/\beta^2$.
- Note: assigning the units of action (ML) to the action requires $[u]^2 = ML$ and hence $[\beta^2] = 1/ML$ (which is why a physicist might prefer not to put $\beta = 1$). Since $[m] = 1/L$, this means that M has the same dimensions as $\hbar m$, and it corresponds to a classically generated mass.
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An anti-soliton

Return to the expression for a soliton:

$$e^{i\beta u/2} = \frac{1 + iE}{1 - iE}, \quad E = e^{ax+bt+c}$$

and replace c by $c + i\pi$ (equivalently, replace E by $-E$). Note

$$u_x = -\frac{4a}{\beta}\frac{E}{1 + E^2},$$

which is always negative - this time the solution interpolates from 0 to -2π, with identical energy-momentum.
Define a conserved ('topological') charge

$$Q = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx\, u_x = \frac{1}{2\pi}[u(t, \infty) - u(t, -\infty)].$$

Then $Q = 1$ for a soliton and $Q = -1$ for an anti-soliton.

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Multi- solitons

It is also possible to check directly (use Maple/Mathematica) that the following expression is also a solution and describes two solitons (stems from the 60s - see any soliton book):

$$e^{i\beta u/2} = \frac{1 + iE_1 + iE_2 - \Omega_{12} E_1 E_2}{1 - iE_1 - iE_2 - \Omega_{12} E_1 E_2}, \quad \Omega_{12} = \tanh^2\left(\frac{\theta_1 - \theta_2}{2}\right),$$

where

$$E_k = e^{a_k x + b_k t + c_k}, \; a_k = m\cosh\theta_k, b_k = m\sinh\theta_k, \quad k = 1, 2$$

Also

$$(\mathcal{E}, \mathcal{P}) = (\mathcal{E}_1, \mathcal{P}_1) + (\mathcal{E}_2, \mathcal{P}_2),$$

the sum of the individual soliton energies and momenta.

Generalises to any number of solitons (point to note, rapidities are all different).

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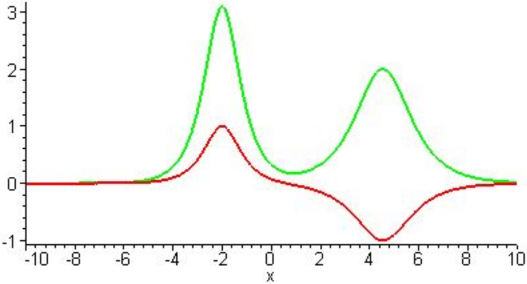
Again, u_x is positive and, taking as example $\theta_1 = 0,\ \theta_2 = 0.5$, two maxima are clearly seen in the regions where the solution is changing rapidly:



In this snapshot the moving soliton is to the left of the stationary one (and the red curve represents $\sin(u/2)$). Since the derivative is always positive, u increases from $0 \to 4\pi$.

Remarks:

- Either E_1 or E_2 or both can be replaced by $-E_1,\ -E_2$, respectively, to give solutions with soliton-anti-soliton, or two solitons.
- A simple time-periodic solution (known as a 'breather') may be constructed by setting

$$\theta_1 = i\lambda, \;\; \theta_2 = -i\lambda, \;\; c_1 = c_2.$$

- The energy-momentum of this breather is given by

$$(\mathcal{E}, \mathcal{P}) = \frac{16m}{\beta^2}(\cos\lambda, 0) \equiv 2M(\cos\lambda, 0).$$

Evidently, the energy of a breather is less than the mass of two solitons, indicating a bound-state - further evidence for Skyrme that this is an interesting model to analyse.

Further remarks

- A ‘real’ version of sine-Gordon is sinh-Gordon $\partial^2 u = -\sinh u$; it is at first sight less interesting because it has no solitons.
- It is sometimes convenient to use light-cone variables $z = t + x, \bar{z} = t - x$. Then the sinh-Gordon equation reads $4\partial\bar{\partial}u = -\sinh u$.
- The Liouville equation is simpler-looking: $4\partial\bar{\partial}u = -e^u$. It is also conformally invariant under the transformation

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Affine Toda field theory

The sinh/sine-Gordon model is the simplest of a large class of field theories based on Lie algebra data (the sinh/sine-Gordon model is based on the roots of a_1 or su(2)).

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Let the simple roots of the rank r semi-simple Lie algebra g be

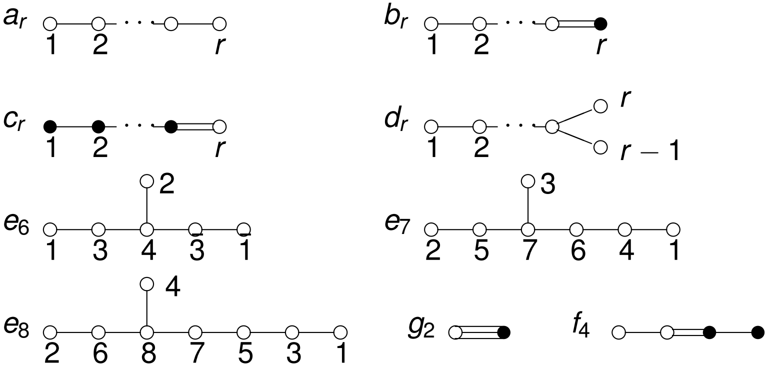
$$\alpha_1, \alpha_2, \ldots, \alpha_r$$

g can be any one of the set

$$a_r, b_r, c_r, d_r, e_6, e_7, e_8, f_4, g_2$$

and the associated simple roots are conveniently summarised by a Dynkin diagram:

Dynkin diagrams



Open circles denote ‘long roots’ (with convention $|\alpha|^2 = 2$); filled circles denote ‘short roots’ (with conventions $|\alpha|^2 = 1$ except for g_2 where $|\alpha|^2 = 2/3$).

A single line joining two roots denotes an angle $2\pi/3$ between them, a double line denotes $3\pi/4$, a triple line denotes $5\pi/6$; unjoined circles represent orthogonal roots.

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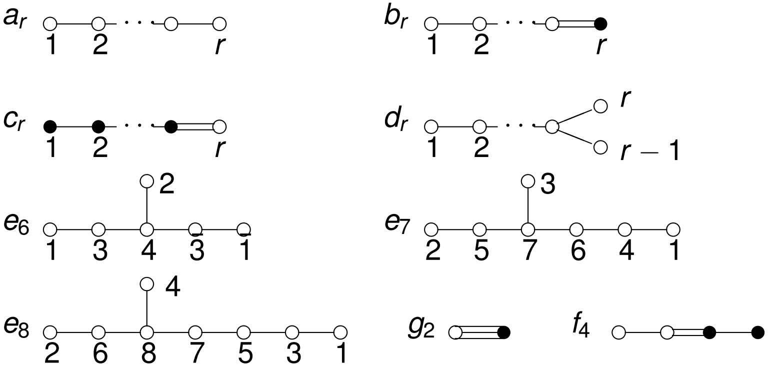

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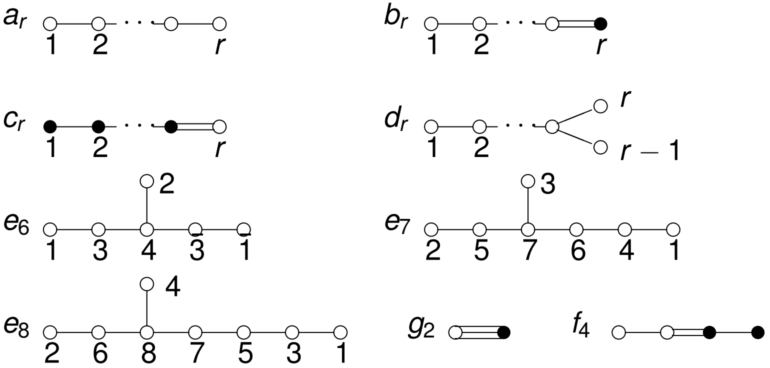

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Toda field theory

Use these roots to define a field theory with Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial^{\mu} u \cdot \partial_{\mu} u - \frac{m^2}{\beta^2}\sum_{k=1}^{r} m_k \, e^{\beta\alpha_k \cdot u},$$

where u is an r-vector and $\{m_k\}$ is a special set of integers to which we shall return.

Besides simple roots we shall need the set of fundamental weights $\{w_k, k = 1, \ldots, r\}$ satisfying

$$2\,\frac{w_k \cdot \alpha_l}{|\alpha|^2} = \delta_{kl}.$$

The vector ρ, defined by

$$\rho = \sum_{k=1}^{r} \frac{2w_k}{|\alpha_k|^2},$$

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Using light-cone coordinates and ρ we can check that every Toda field theory is conformal under the transformation

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For the Lie algebra a_1, with one simple root, this is the Liouville model.

What generalizes the sine/sinh-Gordon model?

In each case, add one more carefully chosen 'root',

$$\alpha_0 = -\sum_{i=1}^{r} n_i \alpha_i$$

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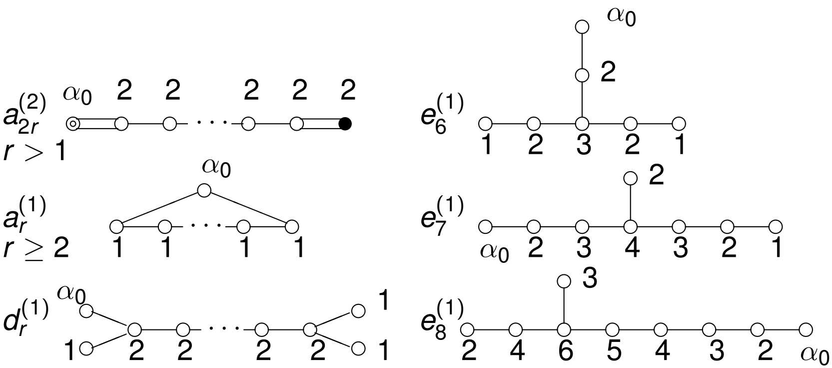
Kač-Dynkin diagrams

These are the 'self-dual' diagrams (invariant under $\alpha \to 2\alpha/|\alpha|^2$)



Note: in $a_{2r}^{(2)}$ $|\alpha|^2 = 4, 2, 1$; $a_2^{(2)}$ omits all medium roots.

In each case, there is an additional vector α_0 whose inner product with the other roots is indicated. In terms of the other roots α_0 is given by the special linear combination

$$\alpha_0 = -\sum_{k=1}^{r} m_k \alpha_k,$$

where the integers m_k are indicated on the diagrams (and were mentioned before).

Except for $a_{2r}^{(2)}$, the extra root is the 'lowest' root (ie subtracting any other root from it fails to provide another root of the Lie algebra).

There is another collection of root systems that fall into dual pairs

$$(b_r^{(1)}, a_{2r-1}^{(2)}),\ (c_r^{(1)}, d_{r+1}^{(2)}),\ (g_2^{(1)}, d_4^{(3)}),\ (f_4^{(1)}, e_6^{(2)})$$

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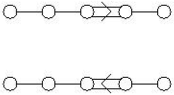
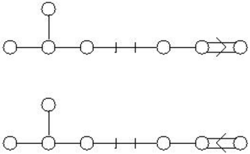


B



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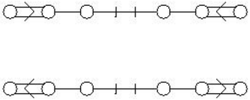

Affine Toda field theory

For any set of extended roots $\alpha_0, \alpha_1, \ldots, \alpha_r$ we may start with a Lagrangian

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One way to discuss (classical) integrability for a field theory is to make use of the Lax pair idea. For a relativistic field theory (including all affine Toda field theories) this consists of suitably constructing a two-dimensional matrix-valued gauge field A_μ with the following property:

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] = 0 \;\Leftrightarrow\; \partial^2 u = -\nabla_u V(u).$$

Then, A_1 can be used to construct a (countably) infinite set of independent conserved quantities in involution (ie their mutual Poisson brackets are zero).

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In particular, we use the commutators

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The next key observation is the following. Take the 'path-ordered' exponential

$$T(a,b,\lambda) = \mathcal{P}exp\left(\int_a^b dxA_1\right),$$

and note that under quite mild asymptotic conditions the quantity

$$Q(\lambda) = \mathrm{tr}\, T(\lambda), \quad T(\lambda) \equiv T(-\infty, \infty, \lambda)$$

is time-independent. Its formal Laurent expansion in powers of λ has time-independent coefficients that serve as conserved charges (the coefficients of $\lambda^{\pm 1}$ being $E \pm P$). Proving the charges are in involution is more complicated but one way uses Sklyanin's classical r-matrix with the property:

$$\{T(\lambda) \overset{\otimes}{,} T(\mu)\} = [r(\lambda/\mu), T(\lambda) \otimes T(\mu)].$$

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The simplest models (one field) are the sine or sinh-Gordon ($a_1^{(1)}$), and the Tzitzéica ($a_2^{(2)}$) equations.

Note, the latter follows from a 'folding' of $a_2^{(1)}$ on setting

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Comment: Many properties of the Tzitzéica model (and the same goes for other models obtained by folding) are inherited from the $a_2^{(1)}$ model, although, historically, many properties (Lax pair, conserved quantities, etc.) were discovered before the affine Toda models were studied systematically as a group.

A natural next question concerns the solitons of the general class; how do they generalise the sine-Gordon soliton?

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Affine Toda solitons

The set of field equations for the scalar field u is:

$$\partial^2 u = -\frac{m^2}{\beta} \sum_{k=0}^{r} m_k \alpha_k \, e^{\beta \alpha_k \cdot u},$$

and note, $u = 0$ is a solution since $\sum_{k=0}^{r} m_k \alpha_k = 0$.

Other constant solutions have the form

$$u_k = \frac{2\pi i}{\beta} \frac{2 w_k}{|\alpha_k|^2}, \quad k = 1, 2, \ldots, r$$

where the w_k are the fundamental weights introduced previously. Most generally, the constant solutions are proportional to an integer linear combination of fundamental weights (ie any element of the weight lattice associated with g); they are all pure imaginary.

Soliton solutions are considered to be those (generally complex) smooth solutions to the field equations with the property:

$$u(-\infty, t) = 0,\ u(\infty, t) = \sum_{1}^{r} l_k u_k,\ l_k \in \mathbb{Z}$$

and we refer to the weight representing $u(\infty, t)$ as 'topological charge', (cf sine-Gordon). The elementary static solitons should be time-independent.

Questions:

- Which weights can correspond to static solitons?
- What is the energy (ie mass) of these solitons?

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Bäcklund transformations

Return for a while to the sine-Gordon equation we began with

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = -\frac{m^2}{\beta}\sin\beta u,$$

or, alternatively, scaling away all constants, $u_{tt} - u_{xx} = -\sin u$.

A remarkable observation of Bäcklund (1882) concerns two solutions to the sine-Gordon equation related by first order differential equations:

$$\begin{aligned} u_x &= v_t + \lambda\sin\left(\frac{u+v}{2}\right) + \lambda^{-1}\sin\left(\frac{u-v}{2}\right) \\ v_x &= u_t - \lambda\sin\left(\frac{u+v}{2}\right) + \lambda^{-1}\sin\left(\frac{u-v}{2}\right). \end{aligned}$$

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Eliminating v gives the sine-Gordon equation for u, and vice-versa.

The first interesting remark concerns the choice $v = 0$. With this choice u satisfies:

$$\begin{aligned} u_x &= \left(\lambda + \lambda^{-1}\right) \sin\left(\frac{u}{2}\right) \\ u_t &= \left(\lambda - \lambda^{-1}\right) \sin\left(\frac{u}{2}\right), \end{aligned}$$

whose solution is precisely the single soliton we had at the beginning provided we identify $\lambda = e^{\theta}$, where θ is the soliton's rapidity. That is, u is given by

$$e^{iu/2} = \frac{1 + iE}{1 - iE}, \quad E = e^{ax+bt+c},$$

with $a = \cosh\theta,\ b = \sinh\theta$.

The second point concerns energy and momentum, which are each clearly seen to be boundary terms. For example:

$$\mathcal{P} = -\int_{-\infty}^{\infty} dx\, u_t u_x = -\int_{-\infty}^{\infty} dx \left(\lambda - \lambda^{-1}\right) \sin\left(\frac{u}{2}\right)\, u_x.$$

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A similar argument yields the energy as a boundary contribtion

$$\mathcal{E} = -\left(\lambda + \lambda^{-1}\right)\left[\cos\left(\frac{u}{2}\right)\right]_{-\infty}^{\infty} = 4\cosh\theta.$$

A third point is that the Bäcklund transformation can be used to generate multiple solitons. For example, letting v be a single soliton and solving for u leads to a double-soliton solution, and so on.

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• Question: does the idea extend to all the other Toda field theories?

Partial answer: Fordy and Gibbons (1980) generalised the sine-Gordon Bäcklund transformation to the field theories based on the root data of a_r;

Liao, Olive and Turok (1993) used this result to demonstrate the topological nature of the energy-momentum of the (complex) $a_r^{(1)}$ solitons, and to generate formulae for multi-solitons in these models.

There seem to be no similar Bäcklund-type formulae for other Toda models (except for $a_2^{(2)}$ - where the formulae for the Bäcklund transformations that have been found are quite messy - Sharipov and Yamilov (1991), Yang and Li (1996)).

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An almost physical example - a shock

Bowcock, EC, Zambon (2002)

Typical shock (or bore) in fluid mechanics:

- flow flips from supersonic to subsonic,

- abrupt change of depth in a channel.

- Velocity field changes rapidly over a small distance,
- Model by a discontinuity in $\mathbf{v}(\mathbf{x}, t)$,
- Nevertheless, there are conserved quantities - mass, momentum, for example.

- Are shocks allowed in integrable QFT?
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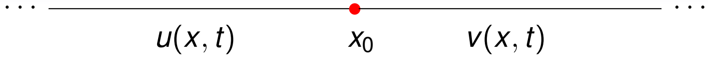

Start with a single selected point on the x-axis, say $x = 0$, and denote the field to the left of it ($x < 0$) by u, and to the right ($x > 0$) by v, with field equations in their respective domains:

$$\partial^2 u = -\frac{\partial U}{\partial u}, \quad x < 0, \quad \partial^2 v = -\frac{\partial V}{\partial v}, \quad x > 0$$

- How can the fields be 'sewn' together preserving integrability?

One natural choice (δ-impurity) would be to put

$$u(0,t) = v(0,t), \quad u_x(0,t) - v_x(0,t) = \mu\, u(0,t),$$

- but, integrability is lost (eg Goodman, Holmes and Weinstein (2002)).

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• Potential problem: there is a distinguished point, translation symmetry is lost and the conservation laws - at least some of them - (for example, momentum), are violated unless the impurity has the property of adding by itself compensating terms.

Consider the field contributions to momentum:

$$\mathcal{P} = -\int_{-\infty}^{0} dx\, u_t u_x - \int_{-\infty}^{0} dx\, v_t v_x .$$

Then, using the field equations, $2\dot{\mathcal{P}}$ is given by

$$\begin{aligned}
&= -\int_{-\infty}^{0} dx\, \left[u_t^2 + u_x^2 - 2U(u)\right]_x - \int_{0}^{\infty} dx\, \left[v_t^2 + v_x^2 - 2V(v)\right]_x \\
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If there are 'sewing' conditions for which the last step is valid then $\mathcal{P} + \mathcal{P}_s$ will be conserved, with $\mathcal{P}_s$ a function of u, v - and possibly derivatives - evaluated at $x = 0$.

Next, consider the energy density and calculate

$$\dot{\mathcal{E}} = [u_x u_t]_0 - [v_x v_t]_0.$$

Setting $u_x = v_t + X(u, v), \;\; v_x = u_t + Y(u, v)$ we find

$$\dot{\mathcal{E}} = u_t X - v_t Y.$$

This is a total time derivative provided

$$X = -\frac{\partial S}{\partial u}, \;\; Y = \frac{\partial S}{\partial v},$$

for some S. Then

$$\dot{\mathcal{E}} = -\frac{dS}{dt},$$

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meaning $\mathcal{E} + S$ is conserved, with S a function of the fields evaluated at the shock.

This argument strongly suggests that the only chance will be sewing conditions of the form

$$u_x = v_t - \frac{\partial S}{\partial u}, \quad v_x = u_t + \frac{\partial S}{\partial v},$$

where S depends on both fields evaluated at $x = 0$, leading to

$$\dot{\mathcal{P}} = v_t \frac{\partial S}{\partial u} + u_t \frac{\partial S}{\partial v} - \frac{1}{2}\left(\frac{\partial S}{\partial u}\right)^2 + \frac{1}{2}\left(\frac{\partial S}{\partial v}\right)^2 + (U - V).$$

This is a total time derivative provided the first piece is a perfect differential and the second piece vanishes. Thus....

$$\frac{\partial S}{\partial u} = -\frac{\partial \mathcal{P}_s}{\partial v}, \quad \frac{\partial S}{\partial v} = -\frac{\partial \mathcal{P}_s}{\partial u}.$$

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In other words....

$$\frac{\partial^2 S}{\partial v^2} = \frac{\partial^2 S}{\partial u^2}, \quad \frac{1}{2}\left(\frac{\partial S}{\partial u}\right)^2 - \frac{1}{2}\left(\frac{\partial S}{\partial v}\right)^2 = (U - V).$$

• By setting $S = f(u+v) + g(u-v)$ and differentiating the left hand side of the functional equation with respect to u and v one finds:

$$f'''g' = g'''f'.$$

If neither of f or g is constant we also have

$$\frac{f'''}{f'} = \frac{g'''}{g'} = \gamma^2,$$

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....the possibilities for f, g are restricted to:

$$\begin{aligned} f'(u+v) &= f_1 e^{\gamma(u+v)} + f_2 e^{-\gamma(u+v)} \\ g'(u-v) &= g_1 e^{\gamma(u-v)} + g_2 e^{-\gamma(u-v)}, \end{aligned}$$

for $\gamma \neq 0$, and quadratic polynomials for $\gamma = 0$. Various choices of the coefficients will provide sine-Gordon, Liouville, massless free ($\gamma \neq 0$); or, massive free ($\gamma = 0$).

In the latter case, setting $U(u) = m^2 u^2/2$, $V(v) = m^2 v^2/2$, the shock function S turns out to be

$$S(u,v) = \frac{m\sigma}{4}(u+v)^2 + \frac{m}{4\sigma}(u-v)^2,$$

where σ is a free parameter analogous to the Bäcklund transformation parameter.

Note: the Tzitzéica potential is **not** is not included in the list.

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It is also worth noting there is a Lagrangian description of this type of ‘shock’:

$$\mathcal{L} \;=\; \theta(-x)\mathcal{L}(u) + \delta(x)\left(\frac{uv_t - u_t v}{2} - S(u,v)\right) + \theta(x)\mathcal{L}(v)$$

The usual E-L equations provide both the field equations for u, v in their respective domains **and** the ’sewing’ conditions.

Questions:

- In the free case, what happens to a wave incident from (say) the left half-line?

Show that if

$$u = \left(e^{ikx} + Re^{-ikx}\right) e^{-iwt}, \quad v = u = Te^{ikx} e^{-iwt}, \quad w^2 = k^2 + m^2,$$

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sine-Gordon

Choosing u, v to be sine-Gordon fields (and scaling the coupling and mass parameters to unity), we take:

$$S(u,v) = 2\left(\sigma\cos\frac{u+v}{2} + \sigma^{-1}\cos\frac{u-v}{2}\right)$$

to find

$$\begin{aligned}
x < x_0: \quad \partial^2 u &= -\sin u, \\
x > x_0: \quad \partial^2 v &= -\sin v, \\
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The last two expressions are a Bäcklund transformation frozen at $x = x_0$.

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Solitons and shocks

Consider a soliton incident from $x < 0$, then it will not be possible to satisfy the sewing conditions (in general) unless a similar soliton emerges into the region $x > 0$:

$$e^{iu/2} = \frac{1 + iE}{1 - iE}, \quad e^{iv/2} = \frac{1 + izE}{1 - izE}, \quad E = e^{ax+bt+c},$$
$$a = \cosh\theta, \quad b = -\sinh\theta,$$

where z is to be determined. It is also useful to set $\lambda = e^{-\eta}$.

- We find

$$z = \coth\left(\frac{\eta - \theta}{2}\right).$$

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- $\eta < \theta$ implies $z < 0$; ie the soliton emerges as an anti-soliton.

-The final state will contain a discontinuity of magnitude 4π at $x = 0$.

- $\eta = \theta$ implies $z = \infty$ and there is **no** emerging soliton.

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Comments and questions....

• The shock is local so there could be several shocks located at $x = x_1 < x_2 < x_3 < \cdots < x_n$; these behave independently as far as a soliton is concerned, each contributing a factor z_i for a total 'delay' of $z = z_1 z_2 \dots z_n$.

• When several solitons pass a defect each component is affected separately.

- This means that at most one of them can be 'filtered out' (since the components of a multisoliton in the sine-Gordon model must have different rapidities).

• Since a soliton can be absorbed, can a starting configuration with $u = 0$, $v = 2\pi$ decay into a soliton?

- No, there is no way to tell the time at which the decay would occur (and quantum mechanics would be needed to provide the probability of decay as a function of time).

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• The $a_r^{(1)}$ Toda models have Bäcklund transformations, do they support defects?

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• What about the other Toda field theories?

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Integrable boundary conditions

• The study of integrable boundary conditions started twenty years ago with Cherednik and Sklyanin.

Time is short so follow a more direct path due to Ghoshal and Zamolodchikov (1994).

After scaling away all the constants the sine-Gordon model with a boundary at $x = 0$ is

$$x < 0 : \quad u_{tt} - u_{xx} = -\sin u; \quad x = 0 : \quad u_x = -\frac{\partial \mathcal{B}}{\partial u}.$$

This follows from the action density

$$\mathcal{L} = \theta(-x)\mathcal{L}_u - \delta(x)\mathcal{B},$$

and $\mathcal{B}$ represents the boundary. The interesting question is: what possible choices for $\mathcal{B}$ are there that preserve integrability?

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G-Z examine the 'energy-like' combination of spin ± 3 conservation laws.

It is useful to think for a moment in light-cone coordinates, where the densities for spin s conserved quantities obey

$$\partial_{\mp} T_{\pm(s+1)} = \partial_{\mp} \Theta_{\pm(s-1)}.$$

In terms of these, an 'energy-like' quantity, possibly conserved if modified suitably, and associated with spin s is:

$$P_s = \int_{-\infty}^{0} dx \left(T_{s+1} - \Theta_{s-1} + T_{-s-1} - \Theta_{-s+1} \right).$$

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• Can the latter expression be the time derivative of a functional of the field at $x = 0$?

Consider $s = 3$ and the density T_4. It should have the form

$$T_4 = \frac{1}{4}(\partial_+ u)^4 + a^2(\partial_{++} u)^2,$$

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$$\begin{aligned}\partial_- T_4 &= \partial_{+-} u\,(\partial_+ u)^3 + 2a^2 \partial_{-++} u\,\partial_{++} u\\ &= -U'\,(\partial_+ u)^3 - 2a^2 U'' \partial_+ u\,\partial_{++} u\\ &= -U'\,(\partial_+ u)^3 - \partial_+(a^2 U''\,(\partial_+ u)^2) + a^2 U'''\,(\partial_+ u)^3.\end{aligned}$$

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Thus, using the boundary condition (and the equation of motion) to substitute for $u_{xx}(0,t)$,

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\begin{aligned}
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&\quad = F(u)\, u_t + 2\left(4a^2\mathcal{B}''' - \mathcal{B}'\right)\, u_t^3
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This is a total time derivative provided

$$4a^2\mathcal{B}''' = \mathcal{B}'.$$

For sine-Gordon (our conventions), $a^2 = -1$ and (up to additive constants)

$$\mathcal{B} = \epsilon e^{iu/2} + \bar{\epsilon} e^{-iu/2},$$

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• Comment: the boundary condition that allows a conserved 'spin 3' charge has two arbitrary parameters.

• Question: do higher spin charges further constrain the boundary 'potential' $\mathcal{B}$?

No!

• Question: are there similar boundary potentials for all other affine Toda field theories?

Yes!

(For example, it could be a nice exercise to check spin 2 charges for the $a_r^{(1)}$ collection.)

• To answer these questions we need to adapt the Lax pair to accommodate boundary conditions

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have a Lax pair form (with constants scaled to unity),

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$$F_{tx} = \partial_t a_x - \partial_x a_t + [a_t, a_x] = 0,$$

Reminder....the affine Toda equations

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For theories with a boundary, construct a field theory on two overlapping pieces of the x-axis, $R_{\pm}$, as follows:

- the half-line R_{-} consists of the portion $-\infty < x \le b$;
- the half-line R_{+} is the portion $a \le x < \infty$, where $a < 0 < b$.

Clearly the two portions overlap on the region $[a, b]$;

- the field in $x \ge b$ is defined in terms of the field in $x \le a$ via a reflection principle,

$$u(x) = u(a + b - x), \qquad x \ge b.$$

For a defect, no such reflection principle would be needed.

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Next, define a new Lax pair

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R_- : \quad & \hat{a}_t^- = a_t - \frac{1}{2}\theta(x-a)(\partial_x u + \frac{\partial \mathcal{B}}{\partial u}) \cdot H, \\
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- Check this works and gives the boundary conditions besides the field equations.

In the overlapping region $\hat{a}_t^\pm$ are independent of x (since $\hat{a}_x^\pm$ vanish), therefore zero curvature demands there is a gauge transformation

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Then, provided this is the case, the quantity

$$Q = \mathrm{tr}\left(P \exp\left\{ \int_{-\infty}^{a} dx\, a_x^- \right\} \mathcal{K}\, P \exp\left\{ \int_{b}^{\infty} dx\, a_x^+ \right\} \right),$$

will be conserved.

Its Laurent expansion in powers of λ provides (formally) a set of conserved quantities.

• Sklyanin 1988 started with this expression and developed an equation for $\mathcal{K}$ in terms of the classical r-matrix.

However, it is possible to tackle the problem differently, calculate $\mathcal{K}$ perturbatively, and deduce the general form of $\mathcal{B}$.

• Suppose $\mathcal{K}$ does not depend on the fields, and $\partial_0 \mathcal{K} = 0$.

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Making these two assumptions and using the explicit expressions for the two gauge components $a_t^{\pm}$, the gauge transformation becomes the following

$$\frac{1}{2}\left[\mathcal{K},\ \frac{\partial\mathcal{B}}{\partial u}\cdot H\right]_{+} = -\left[\mathcal{K},\ \sum_{i=0}^{r} m_i\left(\lambda E_{\alpha_i} - \frac{1}{\lambda}E_{-\alpha_i}\right)e^{\alpha_i\cdot u/2}\right]_{-}.$$

Note, there is an anti-commutator on the left and a commutator on the right; although $\mathcal{K}$ depends upon the spectral parameter λ, the boundary potential $\mathcal{B}$ and u do not.

First, if $\mathcal{K} = 1$ the commutator on the right hand side vanishes identically, while the anti-commutator on the left hand side vanishes only provided

$$\frac{\partial\mathcal{B}}{\partial u_a} = 0.$$

Thus $\mathcal{K} = 1$ is equivalent to the Neumann condition

$$\partial_x u_a = 0.$$

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In that case, the group element $\mathcal{K}$ should have an expansion of the form

$$\mathcal{K} = e^{\sum_{n=1}^{\infty} \lambda^n k_n}.$$

Using this, $\mathcal{K}$ can be determined iteratively (and often exactly).

- For a_1, take $\alpha_1 = \alpha = -\alpha_0$ and work directly to find

$$\mathcal{K}(\lambda) = I + \frac{\lambda}{1-\lambda^4} \begin{pmatrix} 0 & b_1 - \lambda^2 b_0 \\ b_0 - \lambda^2 b_1 & 0 \end{pmatrix}$$

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- Exercise: check this result.

Hint: use the basis

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- It was shown by MacIntyre (1995) that the general boundary potential derived above for sinh/sine-Gordon actually follows from Sklyanin's approach.

- For other models, the analysis provides a similar result. In all cases, the boundary potential has the form

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We conclude with two examples

- Tzitzéica equation:

$$\mathcal{B} = b_1 e^{u} + b_0 e^{-u/2}$$

where

$$b_0(b_1^2 - 2) = 0.$$

Ie there are two families of boundary potentials.

- a-d-e series:

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(a) $b_0 = b_1 = \cdots = b_r = 0$, or

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- The curious nature of these has persisted for more than a decade.

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Adapt Bowcock, EC, Dorey, Rietdijk, 1995.

Two regions overlapping the shock location: $x > a,\ x < b$ with $a < x_0 < b$.

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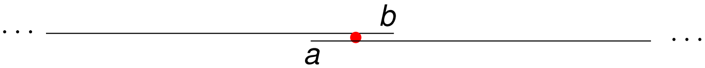

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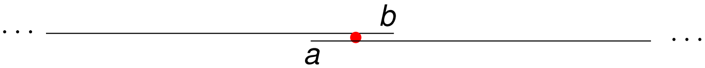

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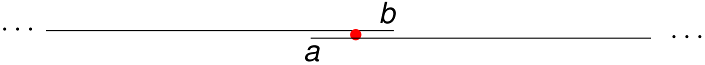

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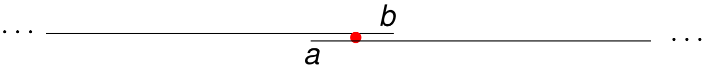

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Where,

$$\begin{aligned} a_t^{(a)} &= u_x H/2 + \sum_i e^{\alpha_i u/2} \left(\lambda E_{\alpha_i} - \lambda^{-1} E_{\alpha_i} \right) \\ a_x^{(a)} &= u_t H/2 + \sum_i e^{\alpha_i u/2} \left(\lambda E_{\alpha_i} + \lambda^{-1} E_{\alpha_i} \right), \end{aligned}$$

$\alpha_0 = -\alpha_1$ are the two roots of the extended $su(2)$ (ie $a_1^{(1)}$) algebra, and H, E_{α_i} are the usual generators of $su(2)$.

There are similar expressions for $a_t^{(b)}, a_x^{(b)}$.

Then

$$\partial_t a_x^{(a)} - \partial_x a_t^{(a)} + \left[a_t^{(a)}, a_x^{(a)} \right] = 0 \;\Leftrightarrow\; \text{sine Gordon}$$

The zero curvature condition for the components of the Lax pairs $\hat{a}_t, \hat{a}_x$ in the two regions imply:

- The field equations for u, v in $x < a$ and $x > b$, respectively,
- The shock conditions at a, b,
- For $a < x < b$ the fields are constant,
- For $a < x < b$ there should be a 'gauge transformation' κ so that

$$\partial_t \kappa = \kappa a_t^{(b)} - a_t^{(a)} \kappa$$

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$$S(u, v) = \sigma \sum_0^1 e^{\alpha_i (u+v)/2} + \sigma^{-1} \sum_0^1 e^{\alpha_i (u-v)/2}.$$

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Assume $\sigma > 0$ then...

- Expect Pure transmission compatible with the bulk S-matrix;
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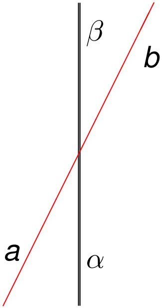

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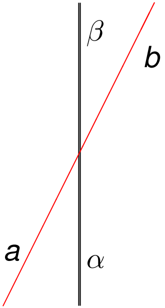

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Schematic triangle relation Delfino, Mussardo, Simonetti 1994

$$S^{cd}_{ab}(\Theta)\, T^{f\beta}_{d\alpha}(\theta_a) T^{e\gamma}_{c\beta}(\theta_b) = T^{d\beta}_{b\alpha}(\theta_b) T^{c\gamma}_{a\beta}(\theta_a) S^{ef}_{cd}(\Theta)$$

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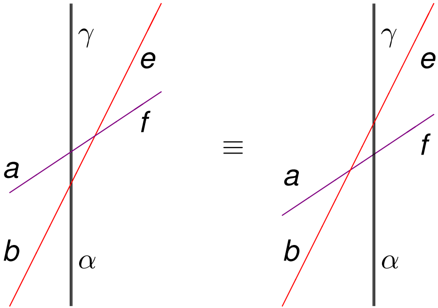

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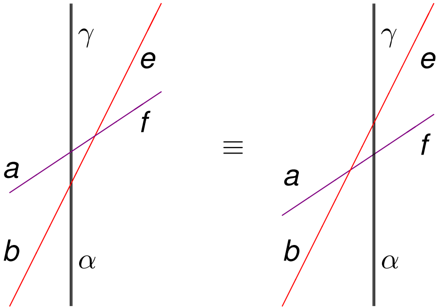

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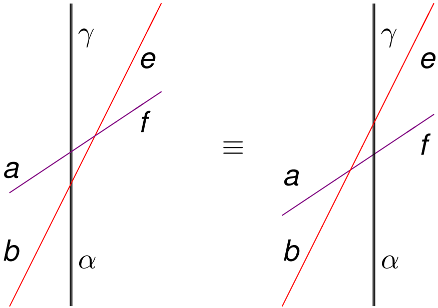

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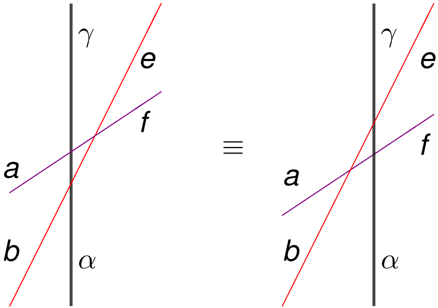

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Zamolodchikov's sine-Gordon S-matrix - reminder

$$S_{ab}^{cd}(\Theta) = \rho(\Theta) \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & C & B & 0 \\ 0 & B & C & 0 \\ 0 & 0 & 0 & A \end{pmatrix}$$

where

$$A(\Theta) = \frac{qx_2}{x_1} - \frac{x_1}{qx_2},\ B(\Theta) = \frac{x_1}{x_2} - \frac{x_2}{x_1},\ C(\Theta) = q - \frac{1}{q}$$

and

$$\begin{aligned} \rho(\Theta) &= \frac{\Gamma(1+z)\Gamma(1-\gamma-z)}{2\pi i} \prod_{1}^{\infty} R_k(\Theta) R_k(i\pi - \Theta) \\ R_k(\Theta) &= \frac{\Gamma(2k\gamma + z)\Gamma(1 + 2k\gamma + z)}{\Gamma((2k+1)\gamma + z)\Gamma(1 + (2k+1)\gamma + z)},\ z = i\gamma/\pi. \end{aligned}$$

The Zamolodchikov S-matrix depends on the rapidity variables θ and the bulk coupling β via

$$x = e^{\gamma\theta},\ q = e^{i\pi\gamma},\ \gamma = \frac{8\pi}{\beta^2} - 1,$$

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$$f(q,x) = \frac{e^{i\pi(1+\gamma)/4}}{1+ie^{\gamma(\theta-\eta)}}\,\frac{r(x)}{\bar{r}(x)},$$

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Remarks ($\theta > 0$): it is tempting to suppose η (possibly renormalized) is the same parameter as in the classical model.

- $\eta < 0$ - the off-diagonal entries dominate;
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- $\eta > \theta > 0$ - the diagonal entries dominate;

• These are the same features we saw in the classical soliton-shock scattering.

• $\theta = \eta$ is not special but there is a simple pole nearby at

$$\theta = \eta - \frac{i\pi}{2\gamma} \to \eta,\ \beta \to 0$$

$$T_{a\alpha}^{b\beta}(\theta) = f(q,x)\begin{pmatrix} Q^{\alpha}\,\delta_{\alpha}^{\beta} & q^{-1/2}\,e^{\gamma(\theta-\eta)}\,\delta_{\alpha}^{\beta-2} \\ q^{-1/2}\,e^{\gamma(\theta-\eta)}\,\delta_{\alpha}^{\beta+2} & Q^{-\alpha}\,\delta_{\alpha}^{\beta} \end{pmatrix}$$

Remarks ($\theta > 0$): it is tempting to suppose η (possibly renormalized) is the same parameter as in the classical model.

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$$E = m_s \cosh\theta = m_s(\cosh\eta\cos(\pi/2\gamma) - i\sinh\eta\sin(\pi/2\gamma))$$

and a 'width' proportional to $\sin(\pi/2\gamma)$.

Using this pole and a bootstrap to define ${}^{odd}T$ leads to a non-unitary transmission matrix - interpret as the instability corresponding to the classical feature noted at $\theta = \eta$.

- The Zamolodchikov S-matrix has 'breather' poles corresponding to soliton-anti-soliton bound states at

$$\Theta = i\pi(1 - n/\gamma), \; n = 1, 2, ..., n_{\max};$$

use the bootstrap to define the transmission factors for breathers and find for the lightest breather:

$$T(\theta) = -i\frac{\sinh\left(\frac{\theta-\eta}{2} - \frac{i\pi}{4}\right)}{\sinh\left(\frac{\theta-\eta}{2} + \frac{i\pi}{4}\right)}$$

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- This is also amenable to perturbative calculation and it works out (with a renormalised η) - See Bajnok and Simon, 2007.

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Consider the x-axis with a shock located at x_0 and asymptotic values of the fields

$$\cdots \quad u = 2a\pi/\beta \qquad\qquad x_0 \qquad\qquad v = 2b\pi/\beta \quad \cdots$$

Compare $(0,0)$ and (a,b) in functional integral representations:

$$u \to u - 2a\pi/\beta,\ v \to v - 2b\pi/\beta,\ A \to A + \delta A$$

with

$$\delta A = \frac{\pi}{\beta}\int_{-\infty}^{\infty} dt(av_t - bu_t) = \frac{\pi}{\beta}(a\delta v - b\delta u)_{x_0}$$

Soliton: $(a,b) \to (a-1, b-1)$, so $\delta u = \delta v = -2\pi/\beta$

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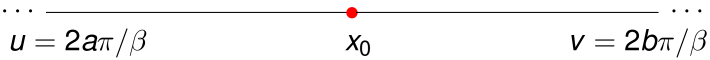

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Further questions....

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- Other field theories - shocks can be constructed within the $a_r^{(1)}$ affine Toda field theories (Bowcock, EC, Zambon, 2004) and there are several types of transmission matrices, though only partially analysed (EC, Zambon, 2007).

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- Might they be technologically useful? To control solitons? EC, Zambon 2004
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