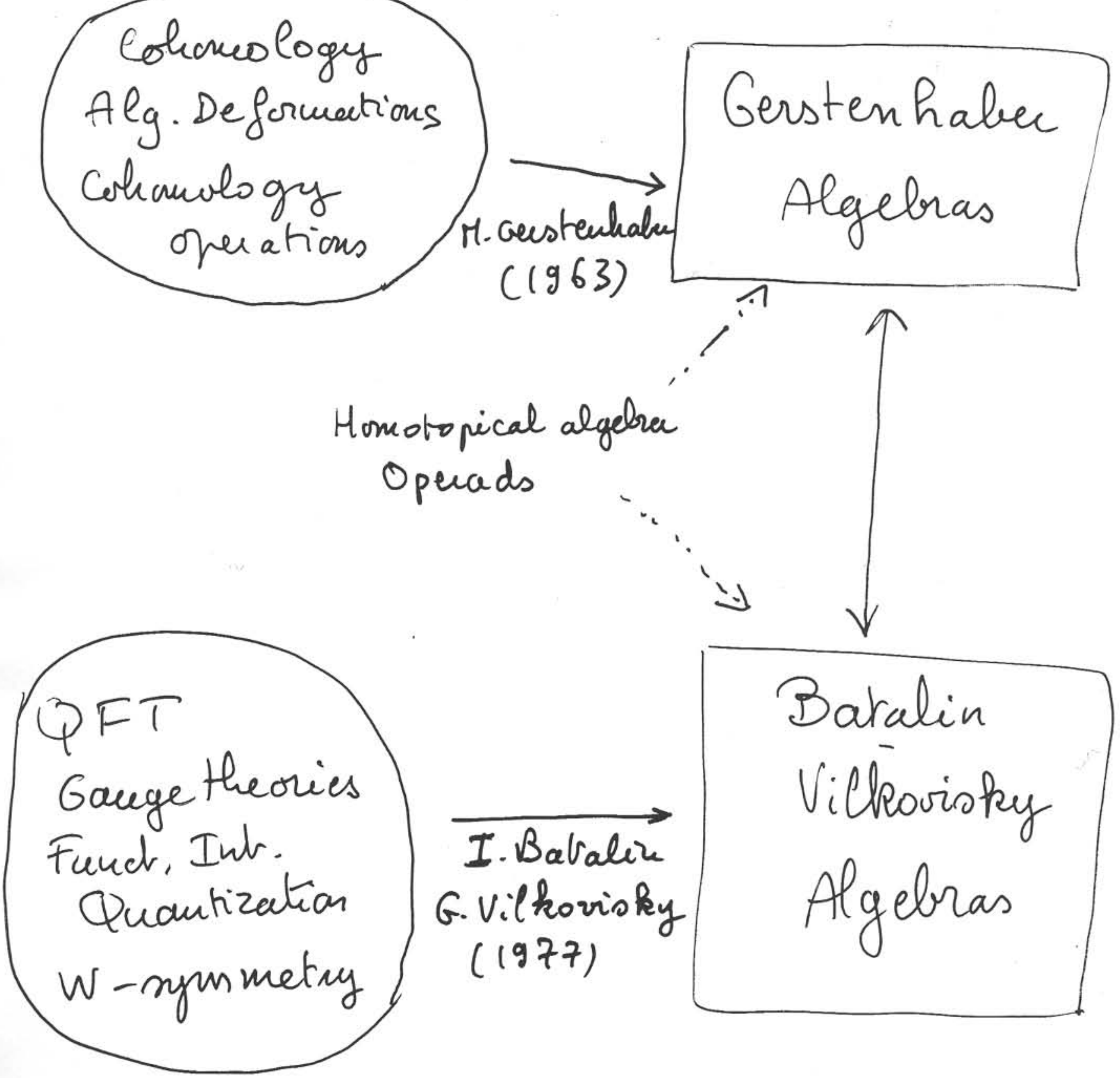
Gerstenhaber and Batalin -Vilkovisky algebras; algebraic, geometric and physical aspects.

Abstract

We shall give a survey of classical examples,together with algebraic methods to deal with those structures: graded algebra, cohomologies, cohomology operations.
The corresponding geometric structures will be described (e.g., Lie algebroids),with particular emphasis on supergeometry, ***odd supersymplectic*** *structures (or periplectic structures) and their classification. Finally,we shall explain how BV structures appeared in Quantum Field Theory, as a version of functional integral quantization.*

Claude ROGER
Université Claude Bernard (Lyon1)



Cohomology
Alg. Deformations
Cohomology operations
M. Gerstenhaber (1963)
Gerstenhaber Algebras
Homotopical algebra
Operads
QFT
Gauge theories
Fund. Int.
Quantization
W-symmetry
I. Batalin
G. Vilkovisky
(1977)
Batalin Vilkovisky Algebras

The roots .....

BIBLIOGRAPHY

Algebraic results

Voronov, A. A. ; Gerstenhaber, M.
Higher-order operations on the Hochschild complex. (Russian. Russian summary) Funktsional. Anal. i Prilozhen. 29 (1995), no. 1, 1--6, 96; translation in
Funct. Anal. Appl. 29 (1995), no. 1, 1--5

Akman, Füsun
A master identity for homotopy Gerstenhaber algebras. Comm. Math. Phys. 209 (2000), no. 1, 51--76.

Akman, Füsun
On some generalizations of Batalin-Vilkovisky algebras. J. Pure Appl. Algebra 120 (1997), no. 2, 105--141.

De Wilde, M.; Lecomte, P. B. A.
An homotopy formula for the Hochschild cohomology. Compositio Math. 96 (1995), no. 1, 99--109.

Gerstenhaber, Murray
The cohomology structure of an associative ring.
Ann. of Math. (2) 78 1963 267--288.

Gerstenhaber, Murray; Schack, Samuel D.
Algebraic cohomology and deformation theory. Deformation theory of algebras and structures and applications (Il Ciocco, 1986), 11--264, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 247, Kluwer Acad. Publ., Dordrecht, 1988.

Penkava, Michael; Schwarz, Albert
On some algebraic structures arising in string theory. Perspectives in mathematical physics, 219--227, Conf. Proc. Lecture Notes Math. Phys., III, Int. Press, Cambridge, MA, 1994

Geometrical aspects

Huebschmann, Johannes
Lie-Rinehart algebras, Gerstenhaber algebras and Batalin-Vilkovisky algebras. Ann. Inst. Fourier (Grenoble) 48 (1998), no. 2, 425--440.

Kosmann-Schwarzbach, Y.; Monterde, J.
Divergence operators and odd Poisson brackets. Ann. Inst. Fourier (Grenoble) 52 (2002), no. 2, 419—456.

Kosmann-Schwarzbach, Y.
Exact Gerstenhaber algebras and Lie bialgebroids. Geometric and algebraic structures in differential equations. Acta Appl. Math. 41 (1995), no. 1-3, 153--165.

Xu, Ping
Gerstenhaber algebras and BV-algebras in Poisson geometry. Comm. Math. Phys. 200 (1999), no. 3, 545—560

Physical aspects

Schwarz, Albert
Geometry of Batalin-Vilkovisky quantization. Comm. Math. Phys. 155 (1993), no. 2, 249--260.

Losev, Andrei
From Berezin integral to Batalin-Vilkovisky formalism: a mathematical physicist's point of view
http: //www.math.columbia.edu/~woit/wordpress/?p=596

Khudaverdian, Hovhannes M.
Semidensities on odd symplectic supermanifolds. Comm. Math. Phys. 247 (2004), no. 2, 353--390.

Khudaverdian, Hovhannes M. , Voronov, Theodore Th.
Differential forms and odd symplectic geometry
arXiv:math/0606560

A. Cattaneo
On the BV Formalism
Internal notes, 1996
http://www.math.uzh.ch/index.php?technicalreports

Quantum fields and strings: a course for mathematicians. Vol. 1, 2. Material from the Special Year on Quantum Field Theory held at the Institute for Advanced Study, Princeton, NJ, 1996--1997. Edited by Pierre Deligne, Pavel Etingof, Daniel S. Freed, Lisa C. Jeffrey, David Kazhdan, John W. Morgan, David R. Morrison and Edward Witten. American Mathematical Society, Providence, RI; Institute for Advanced Study (IAS), Princeton, NJ, 1999.

Bouwknegt, Peter ; McCarthy, Jim ; Pilch, Krzysztof
The W(3) algebra.
Modules, semi-infinite cohomology and BV algebras. Lecture Notes in Physics. New Series m: Monographs, 42. Springer-Verlag, Berlin, 1996. xii+204 pp.

Chapter 1: BV-algebras and G-algebras. Generalities and main examples

A few algebraic preliminaries:

- We shall deal with graded spaces $E^* = \bigoplus_{p \in \mathbb{Z}} E^p$ ($\underline{k}$ = base field) (usually $E^p = \{0\}$ if $p \leq p_0$), and various kind of graded algebras, for which "everything" respect graduation.

For $a \in E^*$, degree of a is denoted by $|a|$.

- Shift of graduation: for E^*, let $E^*[1] = E^* \otimes_{\underline{k}} \underline{k}[1]$ so $E[1]^n = E^{n+1}$ for $n \in \mathbb{Z}$.
- Differential operators: for A^* a graded commutative algebra, 0^{th} order operator are $\mu_a : A \to A$, $\mu_a(b) = a.b$ for some $a \in A$.

$\Delta : A^* \to A^*$ will be n^{th} order operator if for any $a \in A$, one has $[\Delta, \mu_a] - \mu_{\Delta(a)}$ is a $(n-1)^{th}$ order operator (recurrence)

$\boxed{1.1}$

Gerstenhaber structures (structure-G)

A^* is a Gerstenhaber algebra if one has:

(1) $A^* \times A^* \xrightarrow{\cdot} A^*$ associative, graded commutative mult.

$$a.b = (-1)^{|a||b|} b.a$$

(2) $A^*[1] \times A^*[1] \xrightarrow{[\,]} A^*[1]$ graded Lie algebra bracket

$$[a, b] = (-1)^{|a||b|+1} [b, a] \quad \text{(graded antisymmetry)}$$

$$\sum (-1)^{(|a|-1)(|c|-1)} [a, [b, c]] = 0 \quad \text{(graded Jacobi)}$$

(3) Operations $\cdot$ and $[\,,\,]$ are compatible through Leibniz relation

$$[a, b.c] = [a, b].c + (-1)^{(|a|-1)(|b|)} b.[a.c]$$

⚠ Difficulties lie in change of graduation

(Five axioms)

<u>BV-algebras</u>: a graded space A^* is a graded algebra if:

(1) A^* is an associative graded commutative algebra.

(2) one has $\Delta : A^* \to A^*$ differential operator of order 2 and degree (-1)

(3) $\Delta^2 = 0$

(Four axioms)

Explicitly, Δ of order 2 means:

$$\Delta(abc) = \Delta(ab)c + (-1)^{|a|} a\Delta(bc) + (-1)^{|b|(|a|+1)} b\,\Delta(ac) -$$
$$- \Delta(a)bc - (-1)^{|a|} a\,\Delta(b)c - (-1)^{|a|+|b|} ab\,\Delta(c).$$

for every a, b, c.

<u>A BV-algebra is an algebra-G</u>

Define the graded Lie bracket as the obstruction that Δ should be a derivation

(1) $[a, b] = (-1)^{|a|}\left(\Delta(ab) - \Delta(a)b - (-1)^{|a|} a\,\Delta(b)\right)$

then $(\cdot, [\,,\,])$ define an algebra-G structure on A^*.

Moreover Δ is then a <u>graded derivation</u> of $[\,,\,]$

(2) $\Delta([a, b]) = [\Delta(a), b] + (-1)^{|a|-1}[a, \Delta(b)]$

<u>Rem</u>: for an algebra-G, one may have (1) for some Δ (*) which doesn't necessarily satisfy $\Delta^2 = 0$ (<u>coboundary alg.-G</u>) We shall see later that Δ^2 derives. $\Rightarrow \Delta$ derives $[\,,\,]$

(<u>Exercise</u>: let A^* an algebra-G, s.t. $A^* = \{0\}$ for $* < 0$ show that the axioms imply A^0 associative commutative algebra, A^1 is a Lie algebra and morphism $A^1 \to \mathrm{Der}(A^0)$)

(*) One says that Δ <u>generates</u> bracket $[\,,\,]$.

1.2 Basic examples of structures - G

(1) Schouten bracket : Let X differentiable manifold $\tau_X \to X$ tangent bundle, $\Lambda_* \tau_X \to X$ associated exterior algebra bundle, let $\Omega_*(X) = \Gamma(X, \Lambda_* \tau_X)$ space of sections = space of antisym. contravariant tensor fields.
$(\Omega_*(X), \wedge, [\ ])$ is an algebra G for exterior product $\wedge$ of tensor fields, $[\,,\,]$ being the Schouten bracket
(can be defined as the unique graded prolongation of Lie bracket)

(2) If $\mathcal{G}$ is a Lie algebra, then $\Lambda_*(\mathcal{G})$ is an algebra - G.
for exterior product + natural extension of bracket.
Easily obtained from (1) : $\Lambda_*(\mathcal{G}) = \mathrm{Inv}_G \Omega_*(G)$

(3) Algebraization of (1) (2) :
A comm. assoc M A-mod. $\mathcal{P}^n(A, M) \subset \mathrm{Hom}(\overset{n}{\otimes} A, M)$
antisymmetric, multiderivation mappings.
$\mathcal{P}^n(A) = \mathcal{P}^n(A, A)$ $\qquad \mathcal{P}^*(A) = \bigoplus_{n=0}^{+\infty} \mathcal{P}^n(A)$.
Then $(\mathcal{P}^*(A), \cdot, [\ ]_S)$ is an algebra - G
for $\cdot$ cup product of cochains

$$(C_1 \cdot C_2)(x_1, \ldots, x_{m+n}) = (-1)^{mn} \sum_{\sigma \in \mathfrak{S}_{m+n}} \varepsilon(\sigma)\, C_1(x_{\sigma(1)}, \ldots x_{\sigma(m)}) C_2(x_{\sigma(m+1)}, \ldots, x_{\sigma(m+n)})$$

$[\,,\,]_S$ generalized Schouten bracket being defined as the unique prolongation of $[a, b]_S = 0$ if $|a| = |b| = 0$

$$[a, b]_S = a(b) \text{ if } |a| = 1,\ |b| = 0$$

If $A = C^\infty(X)$, then one recovers (1)

Basic examples (cont.)

(4) Geometric generalization: Lie algebroïds (contains all previous examples)

. A Lie algebroïd on X is a vector bundle $A \to X$ with a bundle map $a: A \to \tau_X$ such that

. $\Gamma(A)$ is equipped with a Lie bracket

. $a : \Gamma(A) \to \Gamma(X, \tau_X) = \mathrm{Vect}(X)$ is a Lie morphism

. $[\xi, f\eta] = f[\xi,\eta] + (L_{a(\xi)}f)\eta$ For $\xi, \eta \in \Gamma(A)$, $f \in C^\infty(X)$

. For any vector bundle $A \to X$, $\mathcal{A} = \bigoplus_{0 \leq k \leq n} \Gamma(\Lambda_k A)$ is an associative graded-commutative algebra for exterior product $\wedge$, then one has

$$\boxed{A \text{ is a Lie algebroïd} \iff \mathcal{A} \text{ is an algebra-}G}$$

Proof: See [YKS] or [Xu].

[Exercise: A is a Lie algebroïd, there exists a differential d such that $(\Gamma(\Lambda^\cdot(A')), d)$ is a differential commutative graded Lie algebra. "DGA, dual to alg-G" in some sense]

Examples: (1) $A = \tau_X$, $a = Id$ (2) $A = \mathfrak{g}$, $X = \{*\}$
(one recovers (1) and (2) above)

(3) A tgt bundle to a foliation, a natural inclusion.

(4) (P, Λ) Poisson manifold

$A = T^*P$ and $a: T^*P \to TP$ the "musical" morphism associated to Λ $\Lambda^\#(\alpha)(\beta) = \Lambda(\alpha,\beta)$, α, β being 1-forms

$[\alpha, \beta] = -d(\Lambda(\alpha,\beta)) + L_{\Lambda^\#(\alpha)}\beta - L_{\Lambda^\#(\beta)}\alpha$. (Koszul)

if Λ symplectic then a is an isomorphism

(Notation: $\Lambda^\#(\alpha)$ or $i_\alpha\Lambda$ and $\Lambda(\alpha,\beta)$ or $\langle \Lambda, \alpha\wedge\beta\rangle$)

$\boxed{1.3}$ Above examples are BV-algebras

---

(1) Particular case of (3) above. Let $A_n = \underline{k}[x_1, \dots, x_n]$

$$\mathrm{Der}(A_n) = \left\{ \sum_{i=1}^{n} p_i \theta_i \;\middle|\; p_i \in A_n \quad \theta_i = \frac{\partial}{\partial x_i} \right\}$$

Then $\mathcal{P}^*(A_n) = \Lambda^*_{A_n}(\mathrm{Der}(A_n)) = \Lambda^{\cdot}(\theta_1, \dots, \theta_n) \otimes \underline{k}[x_1, \dots, x_n]$

$\phi \in \mathcal{P}^*(A_n) \qquad \phi = \phi^I \theta_I \qquad \theta_I = \theta_{i_1 \dots i_m}$ for $I = \{i_1, \dots, i_m\}$

$$\phi^I = \phi^{i_1 \dots i_m} = (-1)^{\binom{m}{2}} \frac{\phi(x_{i_1}, \dots, x_{i_m})}{m!}$$

$$\boxed{[\phi, \psi] = \sum_{k=1}^{m} (-1)^{m+k} \phi^{i_1 \dots i_m} \frac{\partial}{\partial x_{i_k}} \psi^{j_1 \dots j_l} \theta_{I_k} \theta_J - (-1)^{(m-1)(l-1)} \sum_{k=1}^{l} \psi^{j_1 \dots j_l} \frac{\partial}{\partial x_{j_k}} \phi^{i_1 \dots i_m} \theta_{J_k} \theta_I}$$

$I = \{i_1 \dots i_m\} \subset \{1 \dots n\} \qquad I_k = \{i_1 \dots i_{k-1}, i_{k+1}, \dots, i_m\}$
$J = \{j_1 \dots j_l\} \subset \{1 \dots n\} \qquad J_k = \{j_1 \dots j_{k-1}, j_{k+1}, \dots, j_l\}$

Explicit formula for Schouten bracket for $\phi \in \mathcal{P}^m(A_n), \psi \in \mathcal{P}^l(A_n)$

<u>Rmk</u>: One has $i(x) : \mathcal{P}^k(A_n) \to \mathcal{P}^{k-1}(A_n) \quad i(x)\phi = [\phi, x]$
for $x \in A_n = \mathcal{P}^0(A_n)$; for x^k $k = 1 \dots n$, then $i(x_k) = \dfrac{\partial}{\partial \theta_k}$

<u>Thm</u>. Let $\Delta = -\dfrac{\partial}{\partial x_i} \dfrac{\partial}{\partial \theta_i}$ (Einstein convention)
then $\Delta^2 = 0$, Δ is a diff. operator of order 2. and degree (-1)
which generate Schouten bracket on $\mathcal{P}^*(A_n)$.

<u>Proof</u>: Exercise.

<u>Rmk</u>: There is a curious analogy (not only because of notations!) between Δ and a laplacian. It will be clear in next chapter, in the context of supergeometry.

(2) $(\Omega_*(X), \wedge, [\,,\,])$ is a BV-algebra with De Rham codifferential on contravariant tensor fields, provided X is orientable. Take $\omega \in \Omega^n(X)$ a volume form, it defines musical isomorphisms ($\nearrow$ and $\searrow$ for indices), which transfer De Rham d to codifferential δ:

$$\begin{tikzcd} \Omega^p \arrow[r, "d"] \arrow[d, "\#"] & \Omega^{p+1} \arrow[d, "\#"] \\ \Omega_{n-p} \arrow[r, "\delta"] & \Omega_{n-p-1} \end{tikzcd}$$

δ is of order 2, and degree -1, $\delta^2 = 0$ is obvious

$(\Omega_*(X), \wedge, \delta)$ a BV algebra is easily checked

<u>Rem</u>: Non-unicity since it depends on the choice of ω

δ can be changed into $\delta' = \delta + i(d\varphi)$

(3) Same construction works for $\Lambda.(\mathfrak{g})$

$(\Lambda.(\mathfrak{g}), \wedge, \delta)$ is a BV-algebra, where δ is the differential of the homological complex (Chevalley-Eilenberg) of the Lie algebra $\mathfrak{g}$ with scalar coefficients

$$\delta(x_1 \wedge \cdots \wedge x_p) = \sum_{1 \leq i < j \leq p} (-1)^{i+j} [x_i, x_j] \wedge x_1 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge \hat{x}_j \wedge \cdots \wedge x_p.$$

(4) For (P, Λ) a Poisson manifold; $(\Omega^*(P), \wedge, [\ ])$ the algebra-G structure associated to it on $\Omega^*(P)$ set $d_\Lambda = [i(\Lambda), d]$ the commutator between exterior diff. and inner product by Λ

$(\Omega^*(P), \wedge, d_\Lambda)$ is a BV-algebra (associated with the structure-G)

Then the cohomology of $(\Omega^*(P), d_\Lambda)$ is the <u>Poisson homology</u> of (P, Λ) (Brylinski).

(5) For the general case of Lie algebroïds, the problem has been geometrized by Ping-Xu [ ].

For an algebroïd A, he defines the notion of A-connection, generalizing directly linear connections:

$$\Gamma(A)\times\Gamma(E)\xrightarrow{\nabla}\Gamma(E) + \text{standard axioms.}$$

To an A-connection on $\Lambda^n A$, he associates covariant derivatives $D_\nabla : \Gamma(\Lambda^k A)\longrightarrow\Gamma(\Lambda^{k-1}A)$.

Then: (1) D_∇ generates the Gerstenhaber bracket on $\Gamma(\Lambda^\cdot A)$

(2) $D_\nabla^2 = -i(R) \quad R\in\Gamma(\Lambda^2 A^*\otimes \mathrm{End}(\Lambda^n A))$

R being the curvature of ∇.

So: coboundaries for algebra-G on $\mathcal{A}=\bigoplus_{k=0}^{n}\Gamma(\Lambda^k A)$

$$\Updownarrow$$

A-connections on $\Lambda^n A$ (determinant bundle)

BV-structures associated to algebra G on $\mathcal{A}=\bigoplus_{k=0}^{n}\Gamma(\Lambda^k A)$

$$\Updownarrow$$

flat A-connections on $\Lambda^n A$

In the C^∞-context, one has no obstruction to existence of connections so every algebra-G is a coboundary; as long as determinant bundle $\Lambda^n A$ is trivial, it admits flat connections as well so one gets BV-structures of each algebra-G.

1.4 Algebraic computations through graded Lie algebras (G.L.A) Relations with Hochschild cohomology and Chevalley-Eilenberg cohomology.

- Let L^* be a G.L.A ($L^k = 0$ if $k \le k_0$, usually $k_0 = -2$)

An element with vanishing square is a $c \in L^1$, s. that $[c,c]=0$.
If $\partial_c(x) = [c, x]$ then (L^*, ∂_c) is a cohomological complex

Exercise: its cohomology $H_c(L^*)$ is a G.L.A for induced bracket.

- Deformation theory: let c be square vanishing, a deformation of c will be a $c+\gamma$, also square-vanishing.

One deduces from $[c+\gamma, c+\gamma]=0$, Maurer-Cartan equation $\boxed{\partial_c \gamma + \frac{[\gamma,\gamma]}{2} = 0}$

One obtains cohomological classification of deformations through $H^1_c(L^*)$, and obstructions, using square map $Sq : H^1_c(L^*) \longrightarrow H^2_c(L^*)$.

- Fundamental examples:

(1) E vector space $\quad M^*(E) = \bigoplus_{p=-1}^{+\infty} M^p(E)$ where

$M^p(E) = C^{p+1}(E,E)$ $(p+1)$-linear mappings from E into E

$c \in M^p(E)$, $|c| = p$

$c_a \in M^a(E)$, $c_b \in M^b(E)$ $\quad i(c_a).c_b \in M^{a+b}(E)$

$$i(c_a)c_b(x_0, .., x_{a+b}) = \sum_{k=0}^{b} (-1)^{ak} c_b(x_0, .., x_{k-1}, c_a(x_k, .. x_{k+a}), x_{k+a+1}, .., x_{a+b})$$

Then $[c_a, c_b] = i(c_a).c_b - (-1)^{ab} i(c_b).c_a$ defines a GLA structure on $M^*(E)$ (Gerstenhaber 1963 - implicit)

$[c,c]=0 \iff c$ is associative

$H_c(M^*(E))$ is Hochschild cohomology of associative algebra structure defined by c on E.

(2) E vector space $A^*(E) = \bigoplus_{p=-1}^{+\infty} A^p(E)$

$A^p(E) = \mathrm{Alt}^{p+1}(E,E) = \Lambda^{p+1}(E')\otimes E$ if finite dimensional.

G.L.A bracket (Richardson - Nijenhuis) deduced from above by antisymmetrisation, explicitly:

$[\alpha\otimes X, \beta\otimes Y] = \alpha\wedge i(X)\beta\otimes Y - (-1)^{ab}\beta\wedge i(Y)\alpha\otimes X \quad |\alpha|=a, |\beta|=b$

$[c,c]=0 \iff c$ satisfies Jacobi identity.

$H_c(A^*(E))$ is then adjoint Lie algebra cohomology for the Lie algebra structure on E defined by c.

[Rem: Supergeometric interpretation (cf chap 2) as $A^*(E) = \mathrm{Der}(\Lambda^* E)$

so $A^*(E) = \mathrm{Vect}(0|n)$ if $n = \dim E$]

If E is a graded space, then $A^*(E)$ and $M^*(E)$ are <u>bigraded</u> L A.

Applications to Gerstenhaber and BV-structures

(i) Let E be the graded vector space underlying a structure G,

μ = associative multiplication $\quad \mu\in M^1(E)_0$

c = Lie bracket (graded) $\quad c\in A^1(E)_{-1}\subset M^1(E)_{-1}$

If $\Delta : E\to E$ defines a BV structure associated with (μ, c) one has $\Delta\in M^0(E)_{-1}$, and $[\Delta,\mu] = c$ (Check it!)

So c is the coboundary of Δ in the Hochschild cohomology (graded) for the associative structure on E defined by μ.

(ii) Deduce from the Leibniz property of bracket c, that

$[c,\mu]=0$, so c is a 2 cocycle in Hoschild cohomology (what about the converse?)

(iii) Deduce from (i)(ii) a cohomological interpretation of existence and classification of BV structures associated with a given structure-G.

\\|

(iv) Let Δ be a coboundary for c, $[\Delta,\mu]=c$

Prove that $[\Delta^2,\mu]=[\Delta,c]$ (up to sign)

So Δ is a derivation of c iff Δ^2 is a derivation of μ (in particular if $\Delta^2=0$!)

(v) Suppose now $c=[\Delta,\mu]$ without assuming c is a Lie algebra structure. Compute $Jac(c)=[c,c]\in A^2(E)_{-2}$ and prove $Jac(c)=0 \Leftrightarrow \Delta^2$ is an operator of order 2.

[For more details about this kind of GLA computations, cf [Penkava-Schwarz] or [Akman]]

<u>Some more results about Hochschild cochains:</u>

. E vector space (not necessarily graded) $\mu\in M^1(E)$ assoc. mult

$(E,\mu)=A$ associative algebra, then $M^p(E)=C^{p+1}(A,A)$ the space of Hochschild cochains.

Then Gerstenhaber bracket defines GLA bracket:

(1) $C^*(A,A)[1]\times C^*(A,A)[1]\xrightarrow{[\,,\,]} C^*(A,A)[1]$

One has moreover the naturally defined cup-product

(2) $a\in C^k(A,A)$, $b\in C^l(A,A)$ $\quad a.b\in C^{k+l}(A,A)$ defined as:

$$(a.b)(x_1,\dots,x_{k+l})=(-1)^{kl}a(x_1,\dots,x_k).b(x_{k+1},\dots,x_{k+l})$$

(1).(2) doesn't define a structure-G on $C^*(A,A)$ (for example not graded commutative!). But on cohomology:

<u>$HH^\cdot(A,A)$ is an algebra-G</u> (Gerstenhaber 1963)

(for <u>any</u> associative algebra A)

This result is a generalisation of $(\Omega_*(X), \wedge, [\,])$:

<u>Hochschild-Kostant-Rosenberg thm:</u>

If A is a smooth commutative k-algebra, then one has an isomorphism

$$\wedge^*_A(\mathrm{Der}_{\underline{k}}(A)) \longrightarrow HH^\cdot(A,A)$$

So, if $A = C^\infty(X)$, $\mathrm{Der}_{\underline{k}}(A) = \mathrm{Vect}(X)$

$$\wedge^\cdot_A(\mathrm{Der}_{\underline{k}}(A)) = \Omega_*(X) = HH^\cdot(C^\infty(X), C^\infty(X))$$

[For a direct proof, valid for $A = C^\infty(X)$, see [De-Wilde-Lecomte]; For $A = \underline{k}[T_1, .., T_n]$ the theorem is readily proved using Koszul complex]

<u>Rmk</u>: homological version $HH_\cdot(A,A) \xrightarrow{\sim} \Omega^*_{\underline{k}}(A)$ (Kähler diff.)

- In fact, one can consider "structures up to homotopy" with analogous tools (GLA, Maurer-Cartan equation...) and $C^*(A,A)$ admits a structure of <u>Gerstenhaber algebra up to-homotopy</u>, using constructions of "braces" ([Akman], [Gerstenhaber-Voronov]...) + results of formality (Deligne's cj.)

- The right tool to handle with all those very complicated constructions is the theory of <u>operads</u> (BV-operad has been made explicit very recently, cf Vallette et alii...)

Chapter 2: BV-structures and supergeometry

.1 Short sketch of supergeometry:

- Superspace $\mathbb{R}^{p|q}$ with superfunctions $C^\infty(\mathbb{R}^{p|q}) = C^\infty(\mathbb{R}^p) \otimes \Lambda^*(\mathbb{R}^q)$
  parity: generators of exterior algebra are odd
- Superdomain $\mathcal{U} \subset \mathbb{R}^{p|q}$ where $C^\infty(\mathcal{U}) = C^\infty(U) \otimes \Lambda^*(\mathbb{R}^q)$
  where $U \subset \mathbb{R}^p$ open set $\quad \mathrm{Dim}(\mathcal{U}) = p|q$
  Algebra of superfunctions = assoc. graded commutative algebras.
- Supermanifold = ringed space (Grothendieck) = space with a sheaf of algebras of superfunctions.
  $\mathcal{X} = (X, \underline{O}_{\mathcal{X}}) \longrightarrow$ underlying X is a diff. manifold dim n
  covering U of X with $\quad \underline{O}_{\mathcal{X}}(U) = C^\infty(U) \otimes \Lambda^*(\mathbb{R}^m)$
  then $\dim \mathcal{X} = n|m$
- Typical example: $E \longrightarrow X$ vector bundle of rank m
  $\underline{O}_{\mathcal{X}} = \underline{\Gamma}(\Lambda^* E)$. [in some sense, the only one in the C^∞ category]
  (Batchelor)
- Functions on a supermanifold form a graded vector space:
  $$\underline{O}_{\mathcal{X}}(X) = \underline{O}_{\mathcal{X}}(X)^{\text{even}} \oplus \underline{O}_{\mathcal{X}}(X)^{\text{odd}}$$
  $$C^\infty(\mathcal{U}) = \left(C^\infty(U) \otimes \Lambda^{\text{even}}(\mathbb{R}^q)\right) \oplus \left(C^\infty(U) \otimes \Lambda^{\text{odd}}(\mathbb{R}^q)\right)$$
- Functor Π changes parity
  ΠTX : tangent bundle on X with fibres made odd.
  ΠT^*X : cotangent " " " " ".
  $\Pi TX = (X, \underline{\Omega}^*)$ sheaf of differential forms
  $\Pi T^*X = (X, \underline{\Omega}_*)$ sheaf of antisym. contravariant tensor fields.

- Basic notions of differential geometry extend to the supercase (some specific difficulties with the volume form and integration, see below). One has in particular frame bundles, and various notions of G-structures for some supergroups contained in $GL(n|m)$.

$GL(n|m)$ = group of even graded linear automorphisms of superspace $\mathbb{R}^{n|m}$ (described through block matrices)

$\mathfrak{gl}(n|m)$ the corresponding Lie superalgebra.

2.2 . <u>Supersymplectic form</u> : $\omega \in \Omega^2(\mathfrak{X})$ is a supersymplectic form if closed and non degenerate

$\omega(x) : T_x\mathfrak{X} \times T_x\mathfrak{X} \to \mathbb{R}$ super antisymmetric.

($T_x\mathfrak{X} = T_x\mathfrak{X}^{even} \oplus T_x\mathfrak{X}^{odd}$ and $T_x\mathfrak{X}^{even} = T_xX$ underlying manifold)

ω has its own parity ω even $\to$ <u>orthosymplectic structures</u>

ω odd $\to$ odd supersymplectic or <u>periplectic structures</u> (Leites)

exists only if $m=n$

Subgroup $P(n) \subset GL(n|n)$ transformations leaving canonical periplectic form on $\mathbb{R}^{n|n}$ invariant.

. (Super) Darboux theorem : let $\mathfrak{X}$ be a supermanifold with a periplectic form $\omega \in \Omega^2_{odd}(\mathfrak{X})$, then there exists at every point a chart $\mathcal{U} \subset \mathfrak{X}$ with coordinates $(x_1, \dots, x_n, \theta_1, \dots, \theta_n)$ such that $\omega_{|\mathcal{U}} = \sum_{i=1}^{n} dx_i \wedge d\theta_i$

One has the usual formalism, Hamiltonian (Leitesian), and (odd) Poisson bracket

For $f,g \in C^\infty(\mathcal{X})$, $\{f,g\} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial \theta_i} + (-1)^{|f|}\frac{\partial f}{\partial \theta_i}\frac{\partial g}{\partial x_i}$

(Buttin bracket)

. Example : ΠT^*X with Liouville form made odd is an odd symplectic supermanifold.
on $C^\infty(\Pi T^*X) = \Omega_* X$ Schouten bracket and odd Poisson bracket coincide

. This example is in fact the only one:

Thm (Schwarz) Let $\mathcal{X}$ be an $(n|n)$ dimensional manifold with an odd symplectic form. Then $\mathcal{X}$ is equivalent to ΠT^*X (for $\mathcal{X} = (X, \mathcal{O}_\mathcal{X})$)

We shall now deal with determinants and volume forms.

2.3 The Berezinian : [difficulty for determinant since no $\Lambda^{max} E$ for a superspace E] the supertrace is naturally defined by $sTr \begin{bmatrix} A & B \\ C & D \end{bmatrix} = Tr(A) - Tr(D)$ [Block dec. from $E = E^{even} \oplus E^{odd}$]

One extends the formula $Ber(\exp M) = \exp(sTr M)$

Explicitly $Ber\, M = Det(A - BD^{-1}C)\, Det(D)^{-1}$

$Ber : GL(n|m) \longrightarrow GL(1|0)$ is a group homomorphism whose kernel is $SL(n|m)$

One can define $SP(n) = P(n) \cap SL(n|n)$

or equivalently $SP(n) = \{M \in P(n) \mid Ber(M) = 1\}$

Direct computation shows that if $\begin{bmatrix} A & B \\ C & D \end{bmatrix} \in P(n)$

Then $Ber\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) = Det(A)^2$

2.4 Berezin integral:

Extend to the supercase the principle $\int_X (L_\xi f) d\mu = 0$

One must have $\int_{X_{super}} \frac{\partial f}{\partial \theta} \mathcal{D}\theta = 0$ for Berezin mesure $\mathcal{D}\theta$

. In "purely super" content one then has

$\int_{\mathbb{R}^{0|1}} (a + b\theta) \mathcal{D}\theta = b$ .... $\int_{\mathbb{R}^{0|n}} \sum c_I \theta^I \mathcal{D}(\theta_1 \cdots \theta_n) = C_{\{1,\dots,n\}}$

. on ΠTX one has the canonical Berezin mesure

for a chart $U \subset X$, one has $\mathcal{D}(x,\theta)_{|U} = \prod_{i=1}^n dx^i \prod_{i=1}^n \mathcal{D}\theta^i$

[Warning: This is not supervolume form.
if $x_i \to \lambda x_i, \theta_i \to \lambda \theta_i$, then $dx_i = \lambda dx_i$ but $\mathcal{D}\theta_i = \lambda^{-1} \mathcal{D}\theta_i$

Rmk: There is no canonical volume form on periplectic mfd!

. On any supermanifold X, one has the sheaf of densities $Ber(\Omega^1_X)$, seen as linear forms on (super) functions. One deduces the notion of integral forms which can be integrated on submanifolds.

. Thm (Khudaverdian) Integral forms on a supermanifold X can be identified with half densities on the odd symplectic supermanifold ΠT^*X.

. Change of variables for integrals on superdomains:

$$\int_{\Phi(\mathcal{U})} f(y,\psi) \mathcal{D}(y,\psi) = \int_{\mathcal{U}} f(\Phi(x,\theta)) \left| Ber(T\Phi_{x,\theta}) \right| \mathcal{D}(x,\theta)$$

From the previous formula, one deduces:

Elements $\sigma = S(x,\theta)\,[dx^1 \dots dx^n]$ transforms

as $\Phi^*(\sigma) = \sigma \operatorname{Det}(T\Phi_{11}) = \sigma \sqrt{\operatorname{Ber}(T\Phi)}$

if Φ is an odd symplectomorphism and

$$T\Phi = \left[\begin{array}{c|c} T\Phi_{11} & \\ \hline & \end{array}\right]$$

So σ can be written as $\sigma = S(x,\theta)\sqrt{\mathcal{D}(x,\theta)}$ and identified with half densities

2.5 BV operator ("odd Laplacian")

X supermanifold $\longrightarrow \Pi T^*X$ odd symplectic

$\longrightarrow C^\infty(\Pi T^*X) = \Omega_*(X)$ is an algebra-G

(Gerstenhaber bracket = Buttin bracket in that case)

One can construct a BV operator which generates this bracket, but non canonically.

$$\omega_{|U} = \sum_{i=1}^{n} dx_i \wedge d\theta_i \quad \text{then } \Delta = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\frac{\partial}{\partial \theta_i}$$

(or deduce from De Rham diff. using "odd Fourier transform")

But: (1) Δ acts canonically on half densities:

$$\sigma = S(x,\theta)\sqrt{\mathcal{D}(x,\theta)} \longrightarrow \Delta\sigma = \sum_{i=1}^{n} \frac{\partial^2 \sigma}{\partial x_i \partial \theta_i}\sqrt{\mathcal{D}(x,\theta)}$$

(invariant under odd symplectomorphisms)

(2) if one changes the volume form by a factor ρ

then $\Delta \longrightarrow \Delta_\rho = \Delta + \frac{1}{2}\{\operatorname{Log}\rho, \cdot\}$

(cf cohomological interpretation)

6 An integration formula in supergeometry:

Some relations (e.g. group actions) in physical applications, are sometimes valid only on the set of solutions of the equation ("<u>on shell</u>").

For example, compute $\int_{\{f^{-1}(0)\}} \omega = \int_X \delta_f \wedge \omega$

where $[\delta_f] \in H^1(X)$ and $[f^{-1}(0)] \in H_{n-1}(X)$ are Poincaré-dual.

δ_f is a current which can be regularized as a 1 form

$\delta_f^{(m)} = \frac{1}{\sqrt{\pi}} \exp(-m^2 f^2)\, m\, df$ and as $m \to +\infty$

$$\int_{\{f^{-1}(0)\}} \omega = \lim_{m \to +\infty} \int_X \delta_f^{(m)} \wedge \omega \quad (\text{"concentrates" on } f^{-1}(0))$$

One can express this integral in supergeometric domain

Let θ' be an odd variable, then:

$$\delta_f^{(m)} = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}^{0|1}} \exp(-m^2 f^2 + \theta' m\, df)\, \mathcal{D}\theta'$$

Let then ℓ' be an auxiliary even variable (Lagrange multiplier!)

$$\delta_f^{(m)} = \frac{1}{2\pi} \int_{\mathbb{R}^{1|1}} \exp\left(i\ell' m f + \theta' m\, df - \frac{\ell'^2}{4}\right) \mathcal{D}\theta'\, d\ell' \quad (\text{Fourier transform!})$$

Chg of variables $\ell = \frac{m\ell'}{2\pi}$, $\theta = m\theta'$ yields

$\longrightarrow$

$$\delta_f^{(m)} = \int_{\mathbb{R}^{1|1}} \exp\left(2i\pi \ell f + \theta df - \left(\frac{\ell\pi}{m}\right)^2\right) \mathcal{D}\theta d\ell$$

So when $m \to \infty$ $\quad \delta_f = \int_{\mathbb{R}^{1|1}} \exp(2i\pi\ell f + \theta df)\, \mathcal{D}(\ell,\theta)$

Now let $\tilde{X}$ the supermanifold obtained from X with an odd variable θ added: (For $U \subset X$ $\quad \mathcal{O}_{\tilde{X}}(U) = C^\infty(U) \otimes \Lambda^{\cdot}(\theta)$)

$\Pi T\tilde{X}$ its tangent space $\left(\Pi T\tilde{X} \cong \Pi TX \times \mathbb{R}^{1|1}\right)$

$$2i\pi\ell f + \theta df = 2i\pi\ell f + \theta \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\psi_i$$

$$= \left[2i\pi\ell \frac{\partial}{\partial\theta} + \sum_{i=1}^{n} \psi_i \frac{\partial}{\partial x_i}\right](\theta f)$$

$$= \tilde{d}(\theta f)$$

$\tilde{d}$ denotes exterior derivative of (super) functions on $\Pi T\tilde{X}$

(x_i even $\to \psi_i$ odd and θ odd $\to (2i\pi)\ell$ even)

One obtains finally:

$$\boxed{\int_{\{f^{-1}(0)\}} \omega = \int_{\Pi T\tilde{X}} \omega \exp(\tilde{d}(\theta f))\, \mathcal{D}(x_i,\psi_i,\ell,\theta)}$$

<u>Rmk</u>: ω is just a function on $\Pi T\tilde{X}$!

Old idea of Lagrange multipliers: the constraints enter the Lagrangian as supplementary variables, but here as <u>odd</u> variables ("twisted fermions")

2.7 About symplectomorphisms of ΠT^*X:

Schwarz' results says that all periplectic manifold are equivalent to some ΠT^*X, but non canonically, i.e. up to some symplectomorphism.

So what about them? They have been studied by Schwarz and Khudaverdian

(I) $\mathrm{Aut}(T^*X) \hookrightarrow \mathrm{Sympl}(\Pi T^*X, \omega)$

autom. of the vector bundle — obvious inclusion

The above inclusion is a homotopy equivalence (Schwarz)

(II) Khudaverdian distinguishes 3 kinds of symplectomaphisms

(a) <u>"ponctual"</u> $\mathrm{Diff}(X) \hookrightarrow \mathrm{Sympl}(\Pi T^*X, \omega)$

(b) <u>"special"</u> $\Omega^1(X) \hookrightarrow \mathrm{Sympl}(\Pi T^*X, \omega)$

$\alpha = \sum_{i=1}^{n} \alpha_i dx_i$ gives $(x, \theta_i) \to (x, \theta_i + \alpha_i)$ $\quad i = 1 \dots n$

(c) <u>"adjusted"</u> more mysterious, they mix odd and even variables

(a)(b)(c) taken together generate the whole group of symplectomorphisms.

All this will be used in next chapter for Lagrangians and BV-quantization .....