Introduction to Generalized complex structures

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Aim: Deformation theorem

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- First lecture: linear algebra.

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- Derived brackets and integrability.

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- Derived brackets and integrability.
- Statement and proof of the theorem.

References

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- M. Gualtieri. Generalized complex geometry. Ph.D. thesis 2004.

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- M. Gualtieri. Generalized complex geometry. Ph.D. thesis 2004.
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- Generalized CR (D. Iglesias and A. Wade, Contact manifolds and generalized complex structures, J. Geom. Phys. 53 (2005), 249-258), gen. F structures...

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- Lie algebra: $\mathfrak{so}(n,n)$ = skew symmetric matrices = $\wedge^2(T \oplus T^*)$.

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$$\mathcal{J}^2 = -Id \qquad \langle \mathcal{J}v, \mathcal{J}w \rangle = \langle v, w \rangle.$$

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Aside 1: $L < V \oplus V^*(\otimes\mathbb{C})$ Lagrangian is called a (complex) Dirac structure.

Aside 2: $L_1, L_2 < V \oplus V^*(\otimes\mathbb{C})$ Dirac structures satisfying $L_1 \cap L_2 = \{0\}$ is a Lie bialgebroid (well not really).

Gcss: pure forms

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$$\mathcal{J} \text{ gcs } \iff K < \wedge^{\bullet} V_{\mathbb{C}}^{*} \text{ a line}$$

generated by a pure form:

$$\rho = e^{B+i\omega} \wedge \Omega,$$

where

- B and ω are real 2-forms;
- $\Omega = \theta_1 \wedge \theta_2 \wedge \cdots \wedge \theta_k$, with θ_i 1-forms;
- $\Omega \wedge \overline{\Omega} \wedge \omega^{n-k} \neq 0$.

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- The integer $k = \deg(\Omega)$ is the *type of* $\mathcal{J}$;
- **Exercise**: show that type of $\mathcal{J}$ = codim$(\pi_V(\mathcal{J}V^*))$.

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- $L < V_{\mathbb{C}} \oplus V_{\mathbb{C}}^*$ maximal isotropic such that $L \cap \overline{L} = \{0\}$,

- Line $K < \wedge V_{\mathbb{C}}^*$ generated by a pure form ρ satisfying

$$(\rho, \overline{\rho}) \neq 0.$$

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- then $\mathcal{J} \cdot \phi = ik\phi$ for $\phi \in U^k$,
- for a complex structure $U^k = \oplus_{p-q=k} \wedge^{p,q} V^*$.

Gcss on manifolds

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- $L < T_{\mathbb{C}}M \oplus T^*_{\mathbb{C}}M$ Lagrangian with $L \cap \overline{L} = \{0\}$,
- $K < \wedge^{\bullet} T^*_{\mathbb{C}}M$, the canonical bundle generated by a pure form,
- A decomposition of the space of forms $\wedge^{\bullet} T^*_{\mathbb{C}}M = \oplus_{k=-n}^{n} U^k$.

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But where does this come from?

Derived brackets

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Given a differential graded Lie algebra $(\mathcal{A}^\bullet, [\cdot,\cdot], D)$ we define the *derived bracket*:

$$[a,b]_D = (-1)^{n+|a|+1}[Da, b],$$

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The derived bracket has degree $n+1$, is not skew, but satisfies Leibniz/Jacobi:

$$[a,[b,c]_D]_D = [[a,b]_D,c]_D + (-1)^{(n+1+a)(n+1+b)}[b,[a,c]_D]_D.$$

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Example: If $(\mathcal{A}, [\cdot,\cdot])$ is a graded Lie algebra and $d \in \mathcal{A}$ is an odd element with $[d,d] = 0$, then $D = [d,\cdot]$ makes $\mathcal{A}$ into a DGLA.

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Example: Let $\mathcal{A}$ be the the of all differential operators on differential forms on a manifold and $[\cdot,\cdot]$ the graded commutator and $D = [d, \cdot]$ be the differential.

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- $\Gamma(T^*) \Longrightarrow$ trivial bracket,
- $\Gamma(Clif(T \oplus T^*)) \Longrightarrow$ not closed under the derived bracket.

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An *integrable Dirac structure* is a Lagrangian subspace $L < TM \oplus T^*M$ for which (1) holds.

A *Lie bialgebroid* is a pair of (integrable) Dirac structures L_1, L_2 for which $L_1 + L_2 = TM \oplus T^*M$

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- If L is a Dirac structure, $\Gamma(\wedge^\bullet L) \subset \mathcal{A} \Longrightarrow$ Schouten type bracket of bracket of multisections.