Introduction to Generalized complex structures

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Last lecture: gcss

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- A gcss decomposes $\wedge^\bullet T^*_{\mathbb{C}}$ into eigenspaces of $\mathcal{J}$.

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$$(e^{-a} v)\rho = e^{-a}(v \cdot e^a(\rho)) \qquad v \in T \oplus T^*, \rho \in \wedge^\bullet T^*.$$

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But where does this come from?

Derived brackets

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Given a differential graded Lie algebra $(\mathcal{A}^\bullet, [\cdot,\cdot], D)$ we define the *derived bracket*:

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The derived bracket has degree $n+1$, is not skew, but satisfies Leibniz/Jacobi:

$$[a,[b,c]_D]_D = [[a,b]_D,c]_D + (-1)^{(n+1+a)(n+1+b)}[b,[a,c]_D]_D.$$

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Example: If $(\mathcal{A}, [\cdot,\cdot])$ is a graded Lie algebra and $d \in \mathcal{A}$ is an odd element with $[d,d] = 0$, then $D = [d,\cdot]$ makes $\mathcal{A}$ into a DGLA.

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Then we can form the derived bracket:

$$[a,b]_d = [[a,d],b].$$

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A *Lie bialgebroid* is a pair of (integrable) Dirac structures L_1, L_2 for which $L_1 + L_2 = TM \oplus T^*M$

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d_L satisfies Leibniz.

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If $\mathcal{J}$ is a gcs, $\overline{L} \cong L^*$ is endowed with a bracket and a differential.

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- $\omega^{n-k} \wedge \Omega \wedge \overline{\Omega} \neq 0$

- for every nonvanishing local section ρ of K there is $v \in T \oplus T^*$ such that

$$d\rho = v \cdot \rho.$$

Examples

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The operators ∂ and $\overline{\partial}$

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Theorem (Gualtieri): $\mathcal{J}$ is integrable iff

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Define ∂ and $\overline{\partial}$ as the projections of d, so

$$\partial : \Gamma(U^k) \to \Gamma(U^{k+1}) \qquad \overline{\partial} : \Gamma(U^k) \to \Gamma(U^{k-1})$$

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$$\overline{\partial}(\alpha\cdot\varphi) = d_L(\alpha)\cdot\varphi + (-1)^{|\alpha|}\alpha\cdot\overline{\partial}\varphi \qquad \forall\alpha\in\Gamma(\wedge^\bullet\overline{L}), \varphi\in\Omega^\bullet(M;\mathbb{C}).$$

Small deformations

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Small deformations are given by $\epsilon \in \Gamma(\wedge^2 \overline{L})$.

Theorem (Liu–Weinstein–Xu, Gualtieri): The deformed gcs is integrable iff

$$d_L \epsilon + \frac{1}{2}[\epsilon, \epsilon] = 0$$