Introduction to Generalized complex structures

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Last lecture: Derived bracket and integrability

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- Induced an exterior derivative d_L on sections of $\wedge^\bullet L^*$ for a Dirac structure L,

- Induced operators ∂ and $\overline{\partial}$ on $\Omega^\bullet(M)$ from a gcs (a Lie bialgebroid).

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- $\overline{L} = T^{1,0} \oplus T^{*}0, 1$ and d_L is also $\overline{\partial}$ ($T^{1,0}$ is a hol bundle).

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$$\overline{\partial}(\alpha \cdot \varphi) = (d_L\alpha) \cdot \varphi + (-1)^{|\alpha|}\alpha \cdot \overline{\partial}\varphi$$

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Theorem (Liu–Weinstein–Xu): The deformed gcs is integrable iff

$$d_L \epsilon + \frac{1}{2}[\epsilon, \epsilon] = 0$$

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Example 2: Transformation by $\pi \in \Gamma(\wedge^2 T)$. Integrability holds if $[\pi, \pi] = 0$.

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Elements in $\wedge^{2,0} T$ give rise to deformations into other types of gcs.

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Example: In $\mathbb{C}^2$, take $\epsilon = \partial_{z_1}\partial_{z_2}$ then the deformed structure is

$$e^{\epsilon} \cdot dz_1 \wedge dz_2 = (1 + \partial_{z_1}\partial_{z_2}) \cdot dz_1 dz_2 = 1 + dz_1 \wedge dz_2 = e^{dz_1 \wedge dz_2}$$

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Then the type change happens along the zeros of f.

Symmetries

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If $\varphi : M \to M$ a diffeo and $B \in \Omega^2_{cl}(M)$, the map

$$Y + \eta \mapsto \varphi_* Y + (\varphi^*)^{-1}\eta - i_{\varphi_* Y} B$$

is a symmetry.

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If $X \in \Gamma(T)$ and $B \in \Omega^2_{cl}(M)$, the map

$$Y + \eta \mapsto [X, Y] + \mathcal{L}_X \eta - i_Y B$$

is an infinitesimal symmetry.

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Structures are *equivalent* if they can be related via inner symmetries.

$$0 \to \Omega^1_{cl} \to \Gamma(T \oplus T^*) \to Sym(T \oplus T^*) \to H^2(M) \to 0.$$

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The tangent space of the moduli space is

$$\frac{\ker d_L}{\{\text{inf inner symmetries}\}}.$$

Which is the same as

$$\frac{\ker d_L}{\mathrm{Im} d_L} = H^2_{d_L}.$$

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Theorem (Gualtieri): Let $(M, \mathcal{J})$ be a compact gcm. Then there exists an open neighbourhood $U \subset H^2_{d_L}$ containing zero, a smooth family $\tilde{\mathcal{M}} = \{\epsilon_u : u \in U, \epsilon_0 = 0\}$ of generalized almost complex deformations of $\mathcal{J}$, and an analytic obstruction map $\Phi : U \to H^3_{d_L}$ with $\Phi(0) = 0$ and $d\Phi(0) = 0$, such that the deformations in the sub-family $\mathcal{M} = \{\epsilon_z : z \in \Phi^{-1}(0)\}$ are precisely the integrable ones. Furthermore, any sufficiently small deformation ϵ of $\mathcal{J}$ is equivalent to at least one member of the family $\mathcal{M}$.