Conformal boundaries in pseudo-Riemannian geometry I

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- This way, we get a "boundary at infinity" $\partial_\sigma(M)$ as the topological boundary of $\sigma(M)$ in N.
- Of course, some natural geometric conditions are to be put on σ so that $\partial_\sigma(M)$ reflects the geometry at infinity.

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Lemma

Let (M, g) be a complete pseudo-Riemannian n-dimensional manifold, and $\sigma : (M, g) \to (N, h)$ an isometric embedding, where (N, h) is also n-dimensional. Then the map σ is onto.

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- In Lorentzian signature, σ preserves the distribution of lightcones, namely σ preserves causality.

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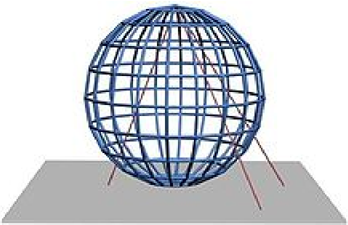
- We can replace the condition (N, h) compact by (N, h) conformally maximal.
- Observe that with this definition, $\partial_\sigma(M)$ might be very irregular.

Some basic examples

The (inverse of) the stereographic projection embeds the euclidean space $\mathbf{E}^n$ into the round sphere $\mathbf{S}^n$.



Stereographic projection of pole ν.

This yields a single point as the conformal boundary of $\mathbf{E}^n$.

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- Moreover this sphere is naturally endowed with a Riemannian conformal structure, which is the standard one : $\partial_\sigma \mathbf{H^n} \simeq \mathbf{S^{n-1}}$.
- More generally, we will see that all pseudo-Riemannian 1-connected complete spaces of constant curvature inherit a conformal boundary, by embedding them into a nice "universal" space.

Einstein's universe

- Let p and q be two integers, $p \leq q$, $p + q = n$ at least 2.
- We endow $\mathbb{R}^{p+q}$ with the quadratic form

$$q^{p+1,q+1} = -x_0^2 - x_1^2 - \ldots - x_{p-1}^2 + x_p^2 + \ldots + x_{n+1}^2.$$

- We call $C^{p+1,q+1}$ the null cone of $q^{p+1,q+1}$, and $Q^{p,q}$ the projectivization of $C^{p+1,q+1}$ in $\mathbf{RP}^n$.

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- The pair $(Q^{p,q},[g])$ is called Einstein's universe of type (p,q), denoted by $\mathbf{Ein}^{p,q}$.

Topology

- let Σ be the euclidean sphere in $\mathbb{R}^{n+2}$. Then $C^{p+1,q+1} \cap \Sigma$ is diffeomorphic to $\mathbb{S}^p \times \mathbb{S}^q$, and is a covering of order 2 of $\mathbf{Ein}^{p,q}$.
- If we lift the conformal structure of $\mathbf{Ein}^{p,q}$ to this double covering, we just get the conformal structure $[-g_{\mathbb{S}^p} \oplus g_{\mathbb{S}^q}]$ on $\mathbb{S}^p \times \mathbb{S}^q$. We adopt the notation $\widehat{\mathbf{Ein}}^{p,q}$ for this space.

Conformal group

By construction of $\mathbf{Ein}^{p,q}$, there is a conformal action of $\mathrm{PO}(p+1, q+1)$ on $\mathbf{Ein}^{p,q}$. Actually, the conformal group of $\mathbf{Ein}^{p,q}$ is exactly $\mathrm{PO}(p+1, q+1)$.

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Theorem (Liouville)

We assume $p+q \geq 3$. Let U and V be two connected open subsets of $\mathbf{Ein}^{p,q}$, and

$$\varphi : U \to V$$

a conformal transformation. Then there exists a unique $g \in \mathrm{PO}(p+1, q+1)$ coinciding with φ on U.

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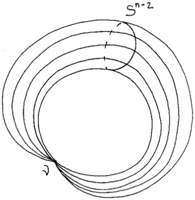
- In $\mathbf{Ein}^{p,q}$, all lightlike geodesics are obtained as projections $\pi(P)$, where P is a null 2-plane in $\mathbb{R}^{p+1,q+1}$.
- The set of all lightlike geodesics through a same point is called a lightcone.
- If $x = \pi(\tilde{x})$ is a point of $\mathbf{Ein}^{p,q}$, the lightcone $C(x)$ is the set

$$\pi(\tilde{x}^{\perp} \cap C^{p+1,q+1}).$$

$$C(x) \setminus \{x\} \simeq \mathbb{R} \times \mathbf{Ein}^{p-1,q-1}.$$

A Lorentzian lightcone

Here is a lightcone in the lorentzian Einstein's universe $\mathbf{Ein}^{1,n-1}$.



Conformal compactification of Minkowski space

We call Minkowski space of type (p, q), denoted $\mathbf{Min^{p,q}}$, the manifold $\mathbb{R}^{p+q}$ endowed with the flat metric

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Proposition

There exists a conformal embedding

$$\sigma : \mathbf{Min^{p,q}} \to \mathbf{Ein^{p,q}}.$$

The image $\sigma(\mathbf{Min^{p,q}})$ is a dense open subset of $\mathbf{Ein}^{p,q}$, the boundary of which is a lightcone in $\mathbf{Ein}^{p,q}$.

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- In any nonriemannian signature, the boundary of Minkowski space in $\mathbf{Ein}^{p,q}$ is never smooth.

- Instead of $q^{p+1,q+1}$, we work with

$$\tilde{q}(x) = 2x_0x_{n+1} - x_1^2 - \ldots - x_p^2 + x_{p+1}^2 + \ldots + x_n^2.$$

- We will identify $\mathbf{Min^{p,q}}$ with $\mathrm{Span}(e_1, \ldots, e_n)$.
- Define $\sigma : \mathbf{Min^{p,q}} \to \mathbf{Ein^{p,q}}$ by

$$x \mapsto [-\frac{1}{2}\tilde{q}(x) : x : 1].$$

- The map σ is conformal and its image is everything but $\pi(e_0^\perp \cap C^{p+1,q+1})$.

De Sitter space

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- De Sitter space is diffeomorphic to $\mathbb{R} \times \mathbb{S}^{n-1}$, and is complete of constant sectional curvature $+1$.
- The map $\sigma : x = (x_1, \ldots, x_{n+1}) \mapsto [1, x_1 \ldots x_{n+1}]$ maps $\mathbf{dS^n}$ conformally into $\mathbf{Ein}^{1,n-1}$.
- The boundary $\partial_\sigma(\mathbf{dS^n})$ is a $(n-1)-$Riemannian sphere.

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- Anti-de Sitter space is diffeomorphic to $\mathbb{S}^1 \times \mathbb{R}^{n-1}$, and is complete of constant sectional curvature -1.
- The map $\sigma : x = (x_0, \ldots, x_n) \mapsto [x_0 \ldots x_n, 1]$ maps $\mathbf{AdS^n}$ conformally into $\mathbf{Ein}^{1,n-1}$.
- The boundary $\partial_\sigma(\mathbf{AdS^n})$ is a codimension 1 Einstein's sub-universe $\mathbf{Ein}^{1,n-2}$.

Other spaces

As an exercise, and using maps built in the same way as those we just studied, you can check that the spaces below can be embedded conformally either in $\mathbf{S}^n$, or in $\mathbf{Ein}^{1,n-1}$, or in $\widehat{\mathbf{Ein}}^{1,n-1}$, and determine the conformal boundary obtained in this way.

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- The Lorentzian product $\mathbf{dS^{n-1}} \times \mathbf{E^1}$.
- The Lorentzian product $(-\mathbf{E}^1) \times \mathbf{H^{n-1}}$.