Conformal boundaries in pseudo-Riemannian geometry II

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- This raises two natural questions :
  1. The first one is the existence of such embeddings
  2. The second one is the independence of the boundary from the conformal embedding.

About the existence issue

Question (existence issue)

Let (M, g) be a noncompact pseudo-Riemannian manifold. Does there exist any pseudo-Riemannian manifold (N, h) of type (p, q), together with a conformal embedding

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- When such embedding σ as above do not exist, we say that (M, g) is **conformally maximal**.
- The notion of maximality can be formulated for any class of geometric structure, and can be seen as a weak notion of completeness.

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- In other words, we ask wether the conformal boundary, which was defined as an extrinsic notion, is actually intrinsic.

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3. *The Riemannian product $\mathbf{H}^m \times \mathbf{S}^k$, with $m \geq 1$, $k \geq 1$, $m + k \geq 3$ (with the convention that $\mathbf{H}^1$ is the 1-dimensional Euclidean space and $\mathbf{S}^1$ is the circle endowed with the homogeneous Riemannian metric of total length 2π).*

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- Except for the real hyperbolic space, all symmetric spaces of noncompact type are conformally maximal.
- Subtle phenomena : $\mathbf{E}^1 \times \mathbf{S}^{n-1}$ is not conformally maximal, while $\mathbf{E}^1 \times \mathbf{RP}^{n-1}$ is.

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- The first point fails in dimension 2.

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- Illustration in the case of the Euclidean space.

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- Following Felix Klein : "*A geometry is the set of properties invariant under the (transitive) action of a (Lie) group*". So what is the geometry preserved by $\mathrm{PO}(p+1, q+1)$ on $\mathbf{Ein}^{p,q}$?

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- "The natural conformal class of type (p, q) on $\mathbf{Ein}^{p,q}$" is a good answer.
- But there is another answer, which turns out to be equally good.

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- Consider the lie group $G = \mathrm{PO}(p+1, q+1)$ as a P-principal fiber bundle over $\mathbf{Ein}^{p,q}$.

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- Denote by ω^G the left-invariant Maurer-Cartan form on $\mathrm{PO}(p+1, q+1)$.
- Fact : any $\varphi : \mathrm{PO}(p+1, q+1) \to \mathrm{PO}(p+1, q+1)$ which is a bundle automorphism such that $\varphi^*\omega^G = \omega^G$ corresponds to the left translation by some element of $\mathrm{PO}(p+1, q+1)$.

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- The geometric data consisting in the P-principal fiber bundle

$$PO(p+1, q+1) \to \mathbf{Ein}^{p,q}$$

together with the form ω^G can be considered as equivalent to the conformal class on $\mathbf{Ein}^{p,q}$.

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Theorem (E. Cartan)

Let $(M,[g])$ be a pseudo-Riemannian manifold of type (p,q) with $p+q \geq 3$. Then $(M,[g])$ defines in a canonical way a P-principal bundle $\hat{M} \to M$, as well as a 1-form $\omega : T\hat{M} \to \mathfrak{o}(p+1,q+1)$ with the following properties :

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- The unicity of ω is ensured by suitable normalization conditions on its curvature
- If $(M, [g])$ and $(N, [h])$ are two conformal structures of dimension $n \geq 3$, and if $\varphi : M \to N$ is a conformal map, then φ lifts to $\hat{\varphi} : \hat{M} \to \hat{N}$ a bundle morphism satisfying $\hat{\varphi}^*\omega^N = \omega^M$

Cauchy boundary of a conformal structure

Thanks to Cartan theorem, we are going to associate to **any** conformal of class $[g]$ of pseudo-Riemannian metrics on a manifold M ($\dim M \geq 3$) an abstract boundary. The construction we will present here is due to B. Schmidt, and is called "*b*-boundary construction".

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- The boundary will be a metric space (well-defined up to bi-Lipschitz equivalence).
- It will carry a continuous action of the group $P = (\mathbb{R}_+^* \times \mathrm{O}(p,q)) \ltimes \mathbb{R}^n$, and an isometric action of the conformal group $\mathrm{Conf}(M,g)$.

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- In certain cases, the topological boundary of any conformal embedding of (M, g) into a compact manifold will be the same as this abstract boundary. Hence the topological boundary will be independant of the embedding under consideration.

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Let us consider $(M, [g])$ a conformal structure of type (p, q), $p + q \geq 3$, and call $(\hat{M}, \omega^N)$ the associated normal Cartan bundle, endowed with the normal Cartan connection. We fix **once for all** a basis $X_1, \ldots, X_n$ of the Lie algebra $\mathfrak{g} = \mathfrak{o}(p+1, q+1)$.

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- Calling $\tilde{X}_i$ the ω^M-constant vector field on $\hat{M}$ corresponding to X_i, we get a global framing $\mathcal{R}$ on $\hat{M}$ defined at each $\hat{x} \in \hat{M}$ by

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- We call ρ^M the Riemannian metric making $\mathcal{R}$ orthonormal.

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- If $\hat{x}$ and $\hat{y}$ are in a same connected component of $\hat{M}$, call $\Gamma(\hat{x}, \hat{y})$ the set of piecewise C^1 paths, parametrized by $[0, 1]$, joining $\hat{x}$ to $\hat{y}$ and defined

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- We get thus a distance d_M on $\hat{M}$ defined by $d_M(\hat{x},\hat{y}) = \frac{\delta_M(\hat{x},\hat{y})}{1+\delta_M(\hat{x},\hat{y})}$ if $\hat{x}$ and $\hat{y}$ are in a same connected component, and $\delta_M(\hat{x},\hat{y}) = 2$ otherwise.

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- Let us call $\hat{M}_c$ the Cauchy completion of the metric space $(\hat{M}, d_M)$. We still call d_M the extension of d_M to $\hat{M}_c$. The Cauchy boundary of the conformal structure is $\partial_c\hat{M} = \hat{M}_c \setminus \hat{M}$.

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- Another choice of basis leads to a distance d'_M on $\hat{M}$ which is bi-Lipschitz equivalent to d_M. Hence the bi-Lipschitz equivalence class of the metric space $(\partial_c \hat{M}, d_M)$ is an invariant of the conformal structure $(M, [g])$.

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- Let p be an element of P. It does not act isometrically for ρ^M. Still, we have the relation $(R_p)^*\omega^M = \mathrm{Ad}\, p^{-1}.\omega^M$. Thus for each $\hat{x} \in \hat{M}$, the matrix of the differential of R_p at $\hat{x}$, read in the frames $\mathcal{R}(\hat{x})$ and $\mathcal{R}(\hat{x}.p)$ is the matrix of $\mathrm{Ad}\, p^{-1}$ in the basis $(X_1, \ldots, X_n)$.

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- Let p be an element of P. It does not act isometrically for ρ^M. Still, we have the relation $(R_p)^* \omega^M = \mathrm{Ad}\, p^{-1}.\omega^M$. Thus for each $\hat{x} \in \hat{M}$, the matrix of the differential of R_p at $\hat{x}$, read in the frames $\mathcal{R}(\hat{x})$ and $\mathcal{R}(\hat{x}.p)$ is the matrix of $\mathrm{Ad}\, p^{-1}$ in the basis $(X_1, \dots, X_n)$. In particular, it does not depend on the point, and there exist positive constants D_p and C_p so that :

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- The conclusion is that R_p is Lipschitz on $(\hat{M}, d_M)$, hence induces a Lipschitz map of $(\hat{M}_c, d_M)$.

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- What one would rather call a boundary for M is the quotient space $\partial_c M = \partial_c \hat{M}/P$.
- But this is generally a bad idea to consider this quotient, since we lose all the interesting structures (metric space+ group action).

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- It follows that the Cauchy boundary of $\mathbf{Ein}^{p,q}$ is the empty set.

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- Because $(\mathrm{PO}(p+1, q+1), d_G)$ is complete, the Cauchy boundary of $(\hat{\Omega}, d_G)$ coincide with the topological boundary $\partial\hat{\Omega}$.

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The problem is then to know when the Cauchy sequences of $(!home, d_G)$ are also Cauchy sequences in $(!home, d_\Omega)$, in which case we will conclude $\partial_c \hat{\Omega} = \partial \hat{\Omega}$.

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If $M = \Gamma\backslash\Omega$ is endowed with the conformal structure inherited from $\mathbf{Ein}^{p,q}$, then $\partial_c\hat{M}$ is homeomorphic to $\Gamma\backslash\partial\hat{\Omega}$, and the P-action is the right action of P on $\Gamma\backslash\partial\hat{\Omega}$.