Towards New Formulation of Quantum field theory: Geometric Picture for Scattering Amplitudes

Part 1

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Work with Nima Arkani-Hamed, Jacob Bourjaily, Freddy Cachazo, Alexander Goncharov, Alexander Postnikov, arxiv: 1212.5605
Work with Nima Arkani-Hamed, arxiv: 1312.2007

Motivation

- One of the most important challenges of theoretical physics: Quantum gravity.
- Method 1: Solve the problem. Most promising candidate: String theory.
- Method 2: Detour - take the inspiration from history of physics. Reformulate Quantum field theory.
- Standard formulation of Quantum field theory: space-time, path integral, Lagrangian, locality, unitarity.
- Perturbative expansion using Feynman diagrams.
- Ultimate goal: Find the reformulation of Quantum field theory where these words emerge as derived concepts from other principle.

Motivation

- This is an extremely hard problem with no guarantee of success. To have any chance we should be able to do it in the simplest set-up.
- We consider the simplest Quantum field theory: $\mathcal{N} = 4$ Super-Yang Mills theory in planar limit.
- We choose one set of objects: on-shell scattering amplitudes.
- In the process of reformulation we make a connection with active area of research in combinatorics and algebraic geometry: Positive Grassmannian $G_+(k, n)$.
- The final result is formulated using a new mathematical object – Amplituhedron which is a significant generalization of the Positive Grassmannian.

Plan of lectures

Lecture 1: Introduction to scattering amplitudes

Lecture 2: Positive Grassmannian

Lecture 3: The Amplituhedron

Very brief introduction to Scattering Amplitudes

On-shell scattering amplitudes

- Fundamental objects in any quantum field theory that describe interactions of particles.

$$\mathcal{M} \sim \langle in \,|\, out \rangle$$

- Each particle is characterized by the four-momentum p_μ and also by spin information.
- The relevant fields have spin ≤ 2, non-gravitational theories have spin $0, \frac{1}{2}, 1$. The information is captured for spin $\frac{1}{2}$ by spinor while for spin 1 by a vector. Quantum numbers: s, $m = (-s, \ldots, s)$.
- On-shell: $p_i^2 = m_i^2$, in many cases we consider $m_i = 0$.
- For massless amplitudes p_μ has three degrees of freedom and m is replaced by helicity $h = (-s, +s)$.

Kinematics

- Massless momentum p_α can be written in 2x2 matrix as

$$p_{a\dot{a}} = \sigma^{\alpha}_{a\dot{a}} p_\alpha$$

- The fact that $p^2 = 0$ is reflected in $\det p_{a\dot{a}} = 0$. Therefore $p_{a\dot{a}}$ can be written as a product of two spinors λ_a and $\tilde{\lambda}_{\dot{a}}$.

$$p_{a\dot{a}} = \lambda_a \tilde{\lambda}_{\dot{a}}$$

  where in (2,2) signature λ, $\tilde{\lambda}$ are real and independent while in (3,1) signature they are complex and conjugate.

- Scalar products

$$\langle 12 \rangle = \epsilon^{ab} \lambda_{1a} \lambda_{2b}, \qquad [12] = \epsilon^{\dot{a}\dot{b}} \tilde{\lambda}_{1\dot{a}} \tilde{\lambda}_{2\dot{b}}$$

  are related to the original scalar product $p_1 \cdot p_2$ as

$$(p_1 + p_2)^2 = 2(p_1 \cdot p_2) = \langle 12 \rangle [12]$$

Scattering amplitudes

- The amplitude $\mathcal{M}$ is a function of p_μ and spin information and is directly related to the probabilities in scattering experiment given by cross sections,

$$\sigma \sim \int d\Omega \, |\mathcal{M}|^2$$

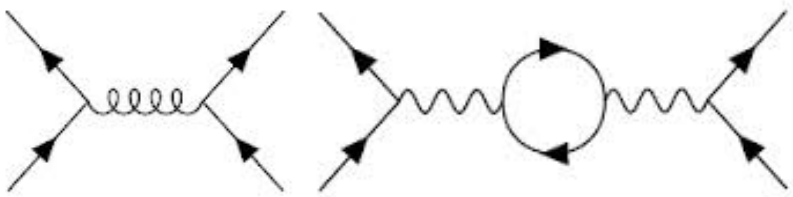
- Despite the physical observable is σ, the amplitude $\mathcal{M}$ itself satisfies many non-trivial properties from QFT.
- Studying scattering amplitudes was crucial for developing QFT in hands of Dirac, Feynman, Schwinger, Dyson and others.
- Two main approaches:
  - Analytic S-matrix program: the amplitude as a function can be fixed using symmetries and consistency constraints.
  - Feynman diagrams: expansion of the amplitude using pieces that represent physical processes with virtual particles.
- In history of physics the second approach was the clear winner, demonstrated most manifestly in development of QCD.

Feynman diagrams

- Theory is characterized by the Lagrangian $\mathcal{L}$, for example

$$\mathcal{L}_{\phi^4} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) + \lambda \phi^4$$

- Standard QFT approach: generating functional $\rightarrow$ correlation function $\rightarrow$ on-shell scattering amplitude.
- Diagrammatic interpretation: draw all graphs using fundamental vertices derived from Lagrangian, and evaluate them using certain rules.



- Perturbative expansion: tree-level (classical) amplitudes and loop corrections.

Feynman diagrams

- At tree-level the amplitude is a rational function with simple poles of external momenta and spin structure,

$$M_0 = \frac{N(p_i, s_i)}{p_1^2 p_2^2 p_3^2 \dots p_k^2}$$

  where the poles are of the form $p_j^2 = (\sum_k p_k)^2$.
- At loop level the amplitude is an integral over the rational function,

$$M_L = \int d^4\ell_1 \dots d^4\ell_L \frac{N(p_i, s_i, \ell_j)}{p_1^2 \dots p_k^2}$$

  where the poles now also depend on ℓ_i.
- The class of functions we get for M_L is not known in general.

Simple amplitudes

- Amplitudes are much simpler than could be predicted from Feynman diagram approach.
- Most transparent example: Park-Taylor formula (1984)
  - Original calculation: $2 \to 4$ tree-level scattering
  - Most complicated process calculated by that time.
  - Result written on 16 pages using small font.
  - Final result simplifies to one-line expression.

$$M = \frac{\langle ij \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 56 \rangle \langle 61 \rangle}$$

  - The simplicity generalizes to all "MHV" amplitudes, invisible in Feynman diagrams.
- This started a new field of research in particle physics, many new methods and approaches have been developed. The progress rapidly accelerated in last few years.

Simple amplitudes

- Feynman diagrams work in general for any theory with Lagrangian, however, the results for amplitudes are artificially complicated.
- Moreover, in many cases there are hidden symmetries for amplitudes which are invisible in Feynman diagrams and are only restored in the sum.
- Advantages from both approaches: perturbative QFT and analytic theory for S-matrix.
  - We use perturbative definition of the amplitude using Feynman diagrams and it also serves like a reference result.
  - On the other hand we can use properties of the S-matrix to constrain the result: locality, unitarity, analyticity and global symmetries.
- In our discussion we focus on the tree-level amplitudes and integrand of loop amplitudes.

Other aspects

- Integrated amplitudes: there is a recent activity in classifying functions one can get for amplitudes.
- In certain theories we have a good notion of transcendentality related to the loop order of the amplitude: symbol of the amplitude.
- Relation to multiple zeta values and motivic structures.
- In many theories there are also important non-perturbative effects not seen in the standard expansion.
- This is completely absent in the theory I am going to discuss now – $\mathcal{N} = 4$ SYM in planar limit.
- Despite it is a simple model, it is still an interesting 4-dimensional interacting theory, closed cousin of Quantum Chromodynamics (QCD).

Toy model for gauge theories

$\mathcal{N} = 4$ Super Yang-Mills theory in planar limit.

- Maximal supersymmetric version of $SU(N)$ Yang-Mills theory, definitely not realized in nature.
- Particle content: gauge fields "gluons", fermions and scalars. At tree-level: amplitudes of gluons and fermions identical to pure Yang-Mills theory. Superfield Φ,

$$\Phi = G_+ + \eta^A \Gamma_A + \frac{1}{2}\eta^A \eta^B S_{AB} + \frac{1}{6}\epsilon_{ABCD}\eta^A \eta^B \eta^C \overline{\Gamma}^D + \frac{1}{24}\epsilon_{ABCD}\eta^A \eta^B \eta^C \eta^D G_-$$

- The theory is conformal, UV finite. In planar limit (large N) hidden infinite dimensional (Yangian) symmetry which is completely invisible in any standard QFT approach.
- The theory is integrable: should have an exact solution. In AdS/CFT dual to type IIB string theory on $AdS_5 \times S_5$.

Properties of amplitudes in toy model

- The theory has $SU(N)$ symmetry group, in Feynman diagrams we get different group structures. In planar limit only single trace survives

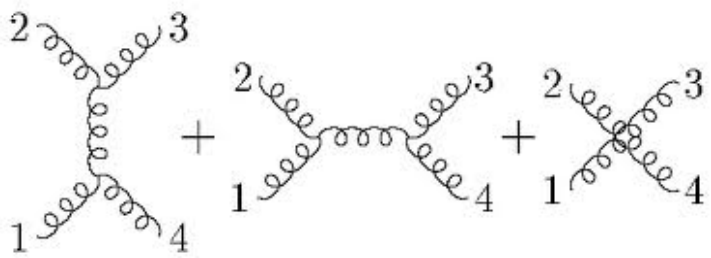
$$\mathcal{M}_{123\ldots n} = \sum_{\sigma/\pi} \mathrm{Tr}\left(T^{a_1} T^{a_2} \ldots T^{a_n}\right) M_{a_1 a_2 \ldots a_n}$$

  We consider the "color-stripped" amplitude M which is cyclic.
- New kinematical variables: n twistors Z_i, points in $\mathbb{P}^3$, and a set of Grassmann variables η_i. Natural $SL(4)$ invariants $\langle Z_1 Z_2 Z_3 Z_4 \rangle$.
- The loop momentum is off-shell and has 4 degrees of freedom, represented by a line $Z_A Z_B$ in twistor space.
- The amplitude is then a rational function of $\langle \cdots \cdot \rangle$ with homogeneity 0 in all Zs with single poles. The pole structure is dictated by locality of the amplitude:

$$\langle Z_i Z_{i+1} Z_j Z_{j+1} \rangle \quad \text{or} \quad \langle Z_A Z_B \, Z_i Z_{i+1} \rangle \quad \text{or} \quad \langle Z_A Z_B Z_C Z_D \rangle$$

Properties of amplitudes in toy model

- All amplitudes are labeled by three numbers n, k, L where a k is a k-charge of $SU(4)$ symmetry of the amplitude. It has physical interpretation in terms of helicities of component gluonic amplitudes (number of $-$ helicity gluons). In fact we better use the label $k \equiv k' = k - 2$.
- Feynman diagram approach is extremely inefficient. For example, $n = 4$, $k = 0$:

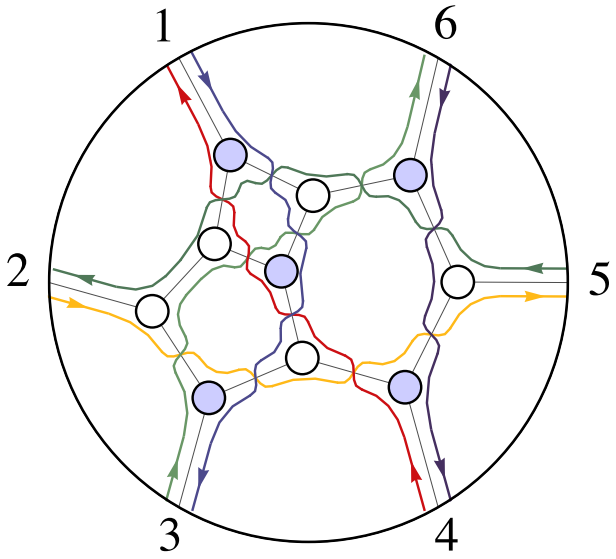


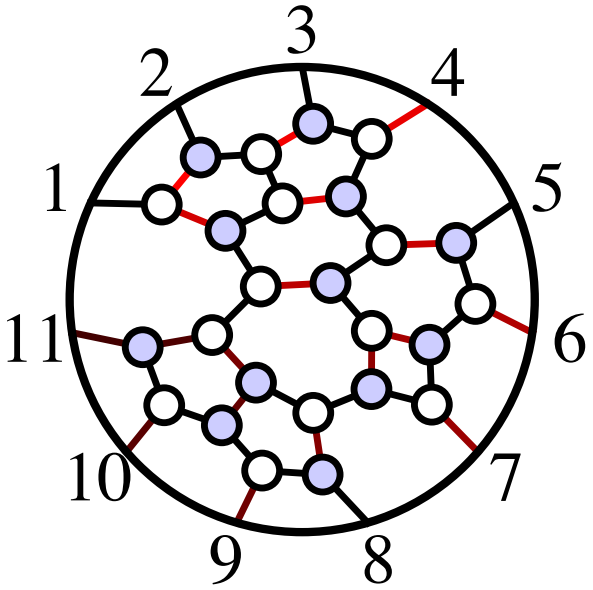
0	1	2	3	4	5	6	7
3	940	47.380	4×10^6	6×10^8	10^{11}	10^{13}	10^{15}

Overview of the program

- Our ultimate goal: to find a geometric formulation of the scattering amplitude as a single object.
- This formulation should make all properties of the amplitude manifest.
- It better does not use any physical concepts which should emerge as derived properties from the geometry.
- We will proceed in two steps:
  - Step 1: We find a new basis of objects which serve as building blocks for the amplitude. It will be an alternative to Feynman diagrams with very different properties. They will have a direct connection to Positive Grassmannian.
  - Step 2: Inspired by that we find a unique object which represents the full scattering amplitude - Amplituhedron - a natural generalization of Positive Grassmannian. The problem of calculating amplitudes is then reduced to the triangulation.
- The final picture involves new mathematical structures which should be understood more rigorously.

Scattering Amplitudes and Positive Grassmannian

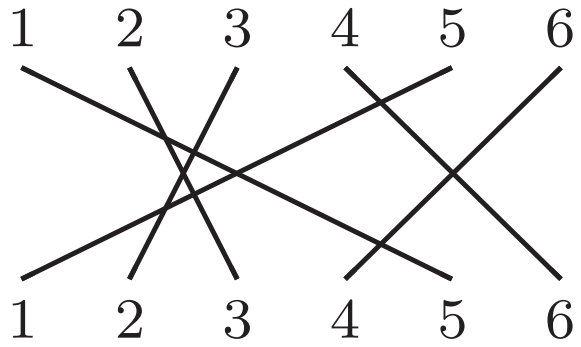




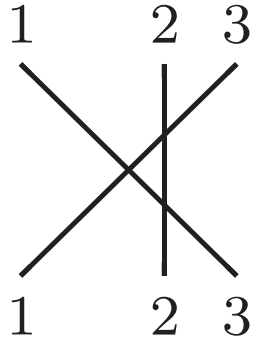
Permutations

Permutations

- Standard permutation: $(1, 2, \ldots n) \to (\sigma(1), \sigma(2), \ldots \sigma(n))$.

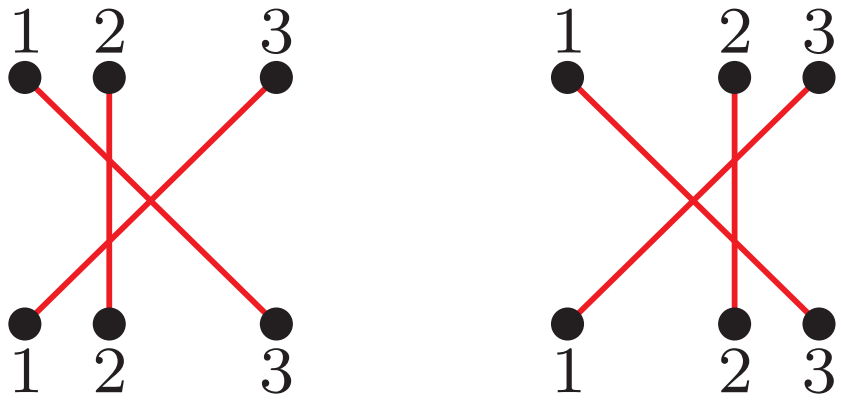


- Scattering process in $1 + 1$ dimensions.
- Most trivial example: $(1, 2, 3) \to (3, 2, 1)$.



Permutations

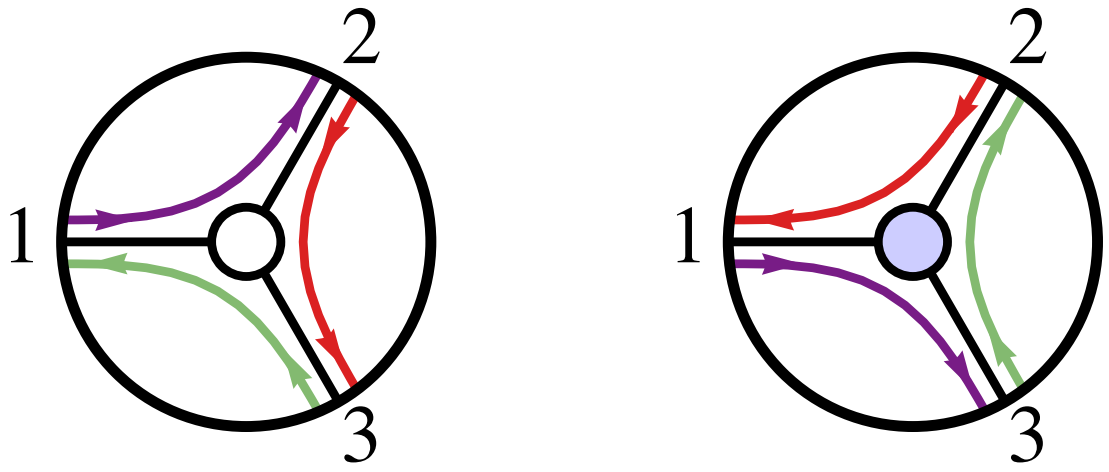
- The picture is not unique: Yang-Baxter move



- Unfortunately, this can not be applied to $3+1$ dimensions
  - No particle creation/destruction.
  - Fundamental 4pt interactions.
- We need fundamental 3pt vertices. Is there a way how to represent a permutation with a diagram which has only 3pt vertices?
- It is not possible to do it with a single 3pt vertex.

Permutations

- Fundamental 3pt vertices:

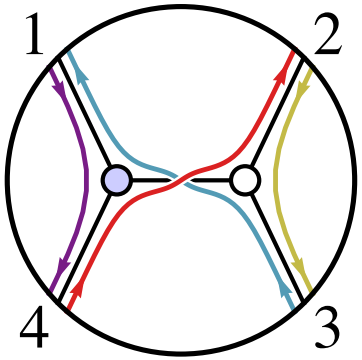


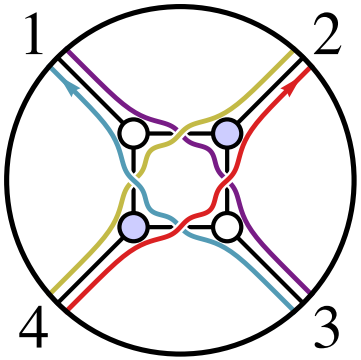
  represent permutations $(1, 2, 3) \to (2, 3, 1)$ and $(1, 2, 3) \to (3, 2, 1)$.

- Left-Right paths in the graph: left on white vertex, right on black vertex.

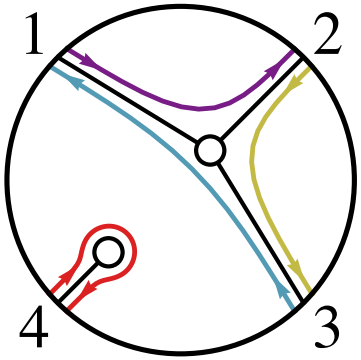
Permutations

- Build a 4pt diagram:



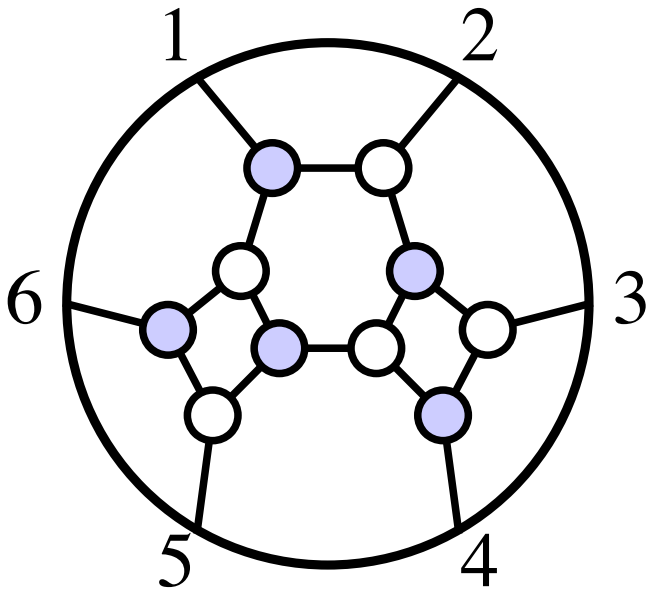


- Permutations: $(1,2,3,4) \to (4,3,1,2)$, resp. $(1,2,3,4) \to (3,4,1,2)$.
- In case $k \to k$ we draw the lollipop, for $(1,2,3,4) \to (2,3,1,4)$



Permutations

- We can build a diagram and find a permutation.



- The permutation is $(1, 2, 3, 4, 5, 6) \to (5, 4, 6, 1, 2, 3)$.
- Every permutation can be represented like this!

Permutations

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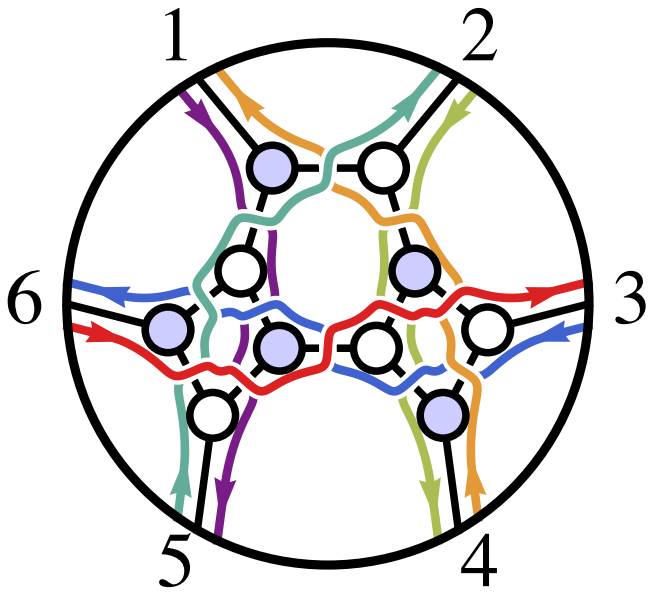

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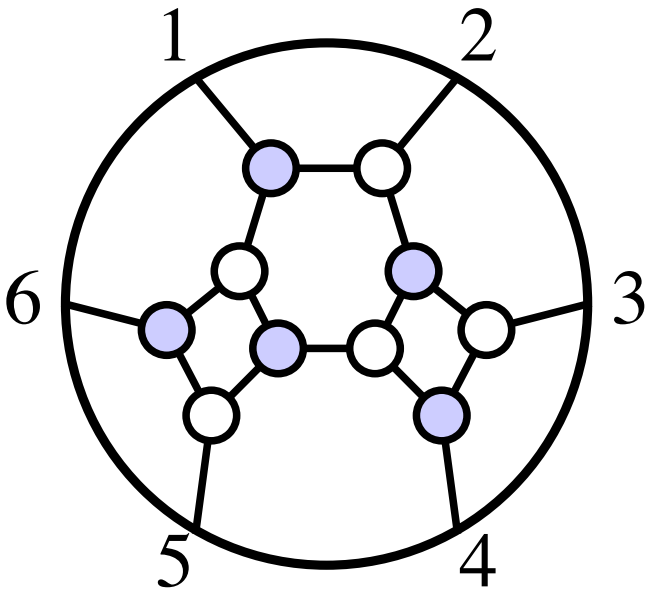
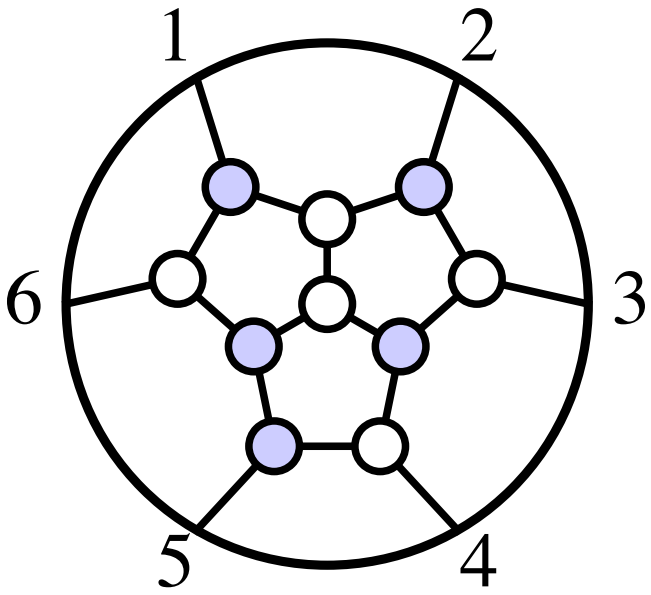
Permutations

- There exists a different diagram that gives the same permutation





- The map diagrams $\leftrightarrow$ permutations is not unique!
- Reduced graphs: minimal number of faces (loops) - they represent permutations.

Permutations

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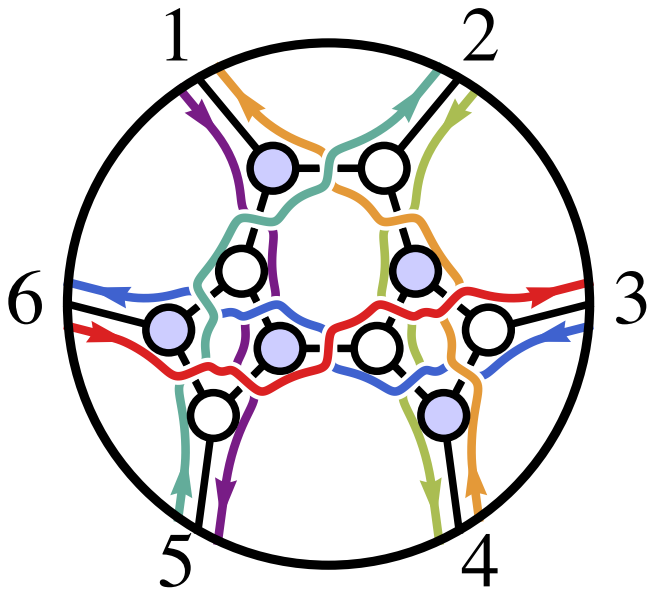

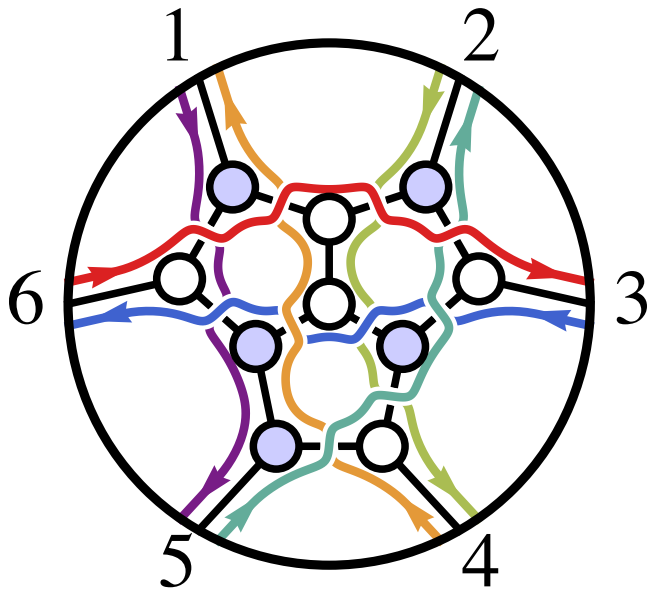

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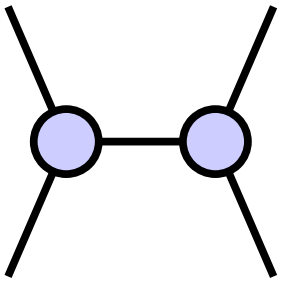
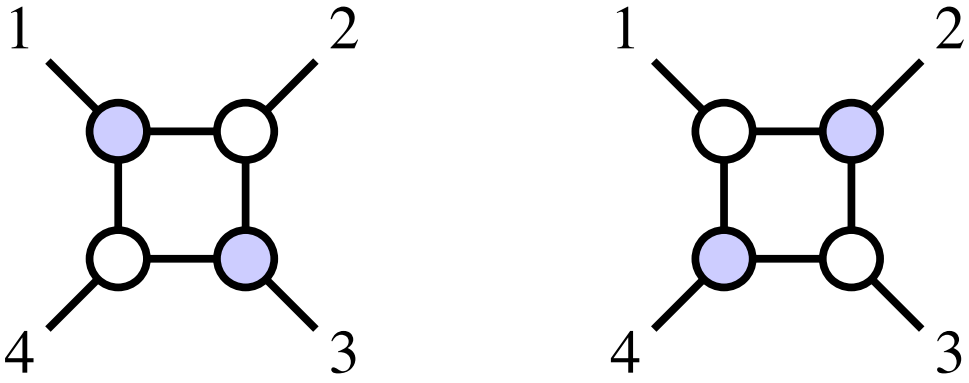
Permutations

- There are two identity moves:
  - merge-expand of black (or white) vertices



  - square move



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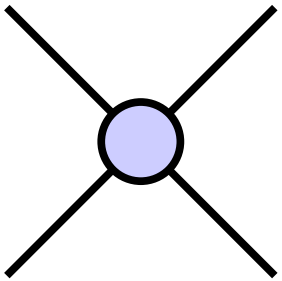

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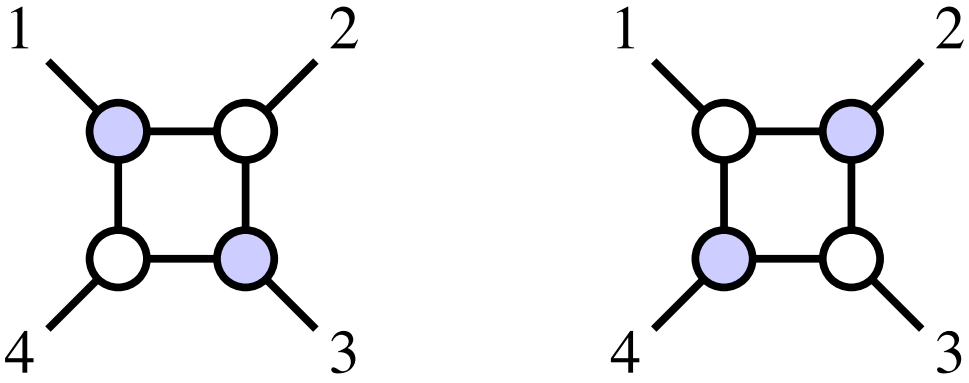

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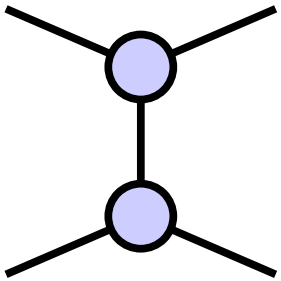

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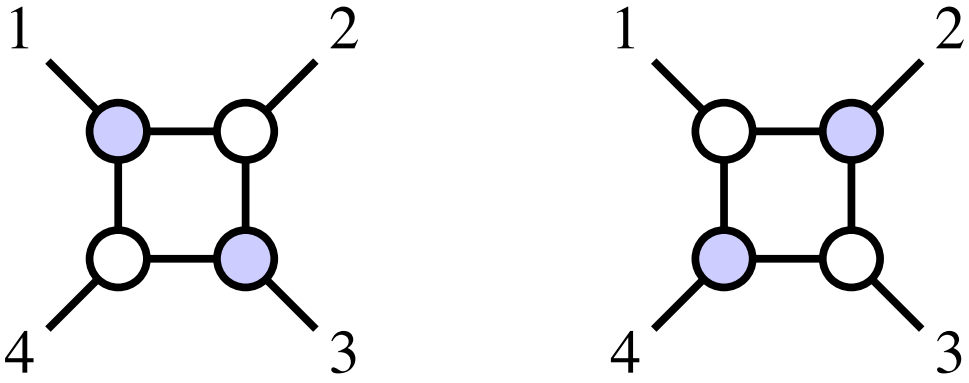
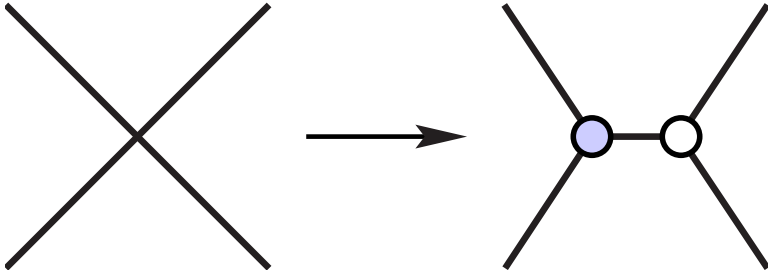
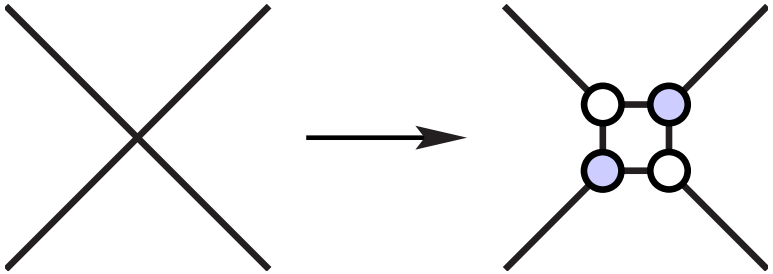


Permutations

- Go back to the Yang-Baxter move. We expand



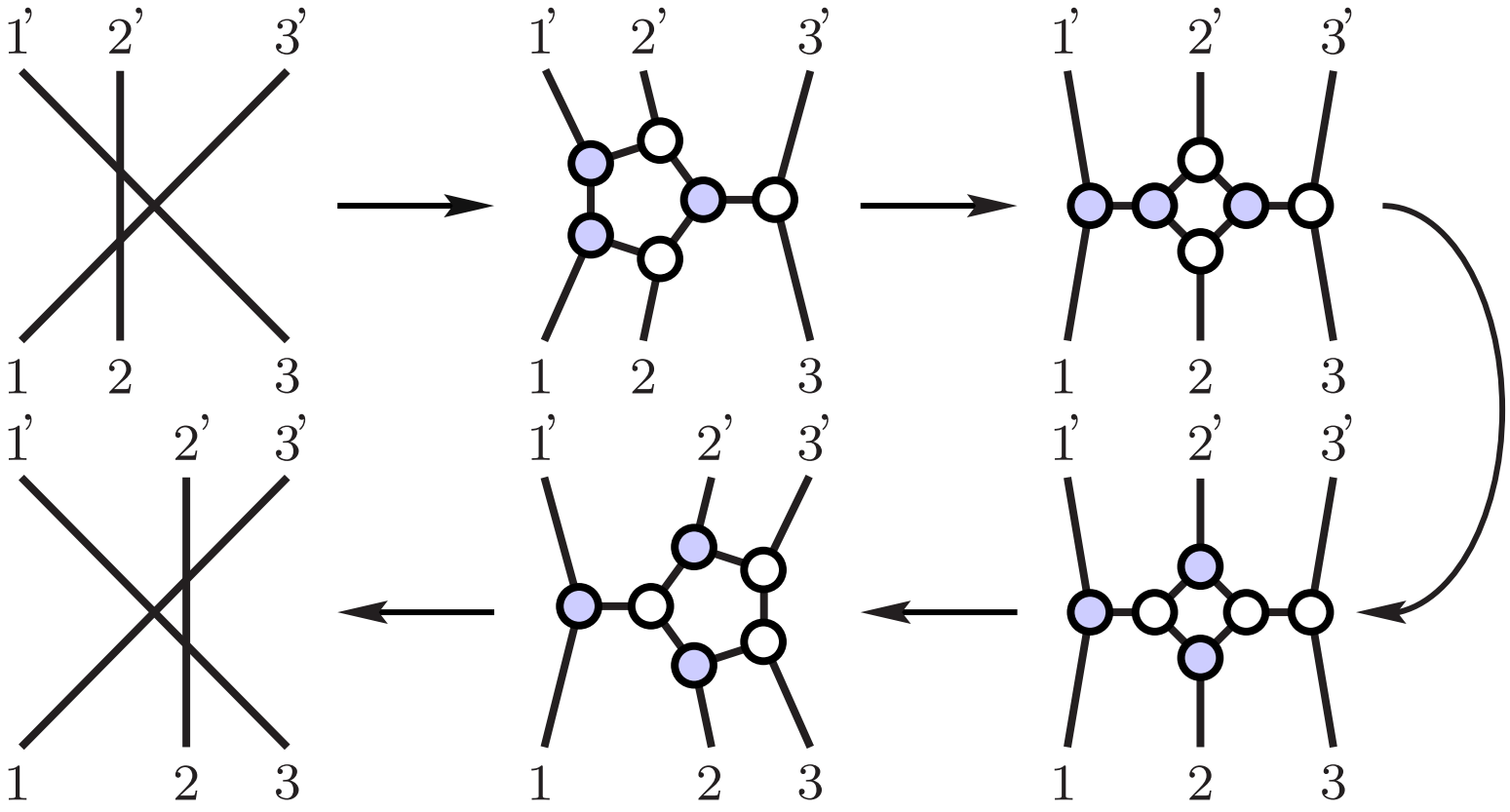
- We could also use the substitution



and prove the same identity.

Permutations

- Then we get



Old diagrams are included as a subset of new diagrams.

Permutations

- We will use affine permutation:

$$k \to \sigma(k)$$

where

$$k + n \geq \sigma(k) \geq k$$

and $\sigma(k)$ mod k is a permutation.

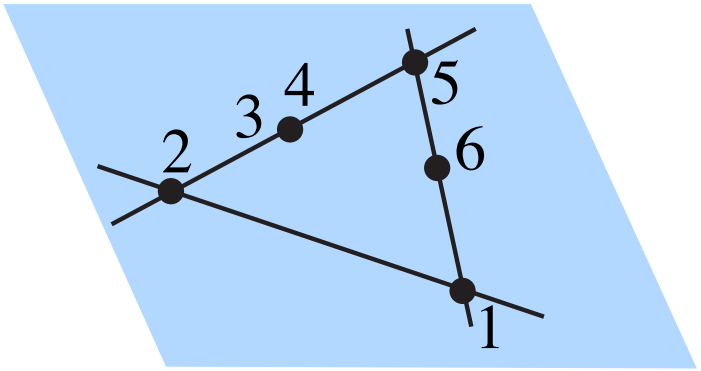
$$1 \to 3$$
$$2 \to 4$$
$$3 \to 1 + 4 = 5$$
$$4 \to 2 + 4 = 6$$

Positive Grassmannian

Configuration of vectors

- Permutations $\leftrightarrow$ Configuration of vectors with consecutive linear dependencies.
- Configuration of n pt in $\mathbb{P}^{k-1}$

$$k \to \sigma(k) \qquad \text{means that} \qquad k \subset \mathrm{span}(k{+}1, \ldots \sigma(k))$$



$1 \subset (2, 34, 5, 6) \to \sigma(1) = 6, \qquad 2 \subset (34, 5) \to \sigma(2) = 5,$
$3 \subset (4) \to \sigma(3) = 4, \qquad 4 \subset (5, 6, 1, 2) \to \sigma(4) = 2,$
$5 \subset (6, 1) \to \sigma(5) = 1, \qquad 6 \subset (1, 2, 3) \to \sigma(6) = 3.$

- The permutation is $(1, 2, 3, 4, 5, 6) \to (6, 5, 4, 8, 7, 9)$.

The Positive Grassmannian

- Grassmannian $G(k,n)$: space of k-dimensional planes in n dimensions, represented by $k \times n$ matrix modulo $GL(k)$,

$$C = \begin{pmatrix} * & * & * & \dots & * & * \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ * & * & * & \dots & * & * \end{pmatrix} = \begin{pmatrix} v_1 \\ \vdots \\ v_k \end{pmatrix} = \begin{pmatrix} c_1 & c_2 & \dots & c_n \end{pmatrix}$$

- We can think about it as collection of k vectors $v_1, \dots, v_k$ in n dimensions which specify the plane.
- We consider a positive part of $G(k,n)$ which is a space with boundaries.

The Positive Grassmannian

- Positive part:

$$C = [c_1\, c_2\, \ldots c_n]$$

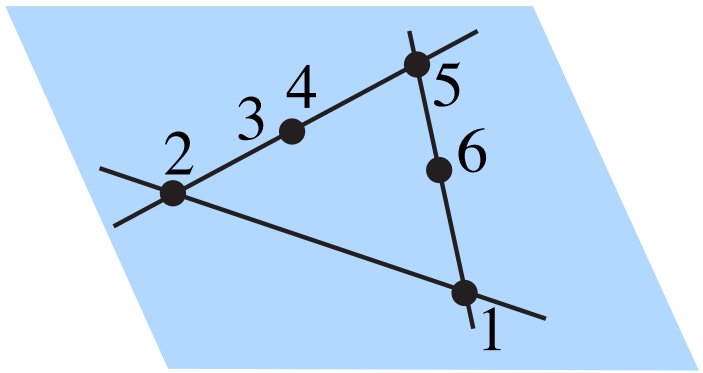
  All minors

$$(c_{i_1} \ldots c_{i_k}) > 0 \quad \text{for} \quad i_1 < i_2 < \cdots < i_k.$$

- Cyclic structure: $c_1 \to c_2$, $c_2 \to c_3$, $\ldots$, $c_n \to (-1)^{k+1} c_1$.

The Positive Grassmannian

- We can think about C as collection n points in $\mathbb{P}^{k-1}$.
- Back to 6pt example:

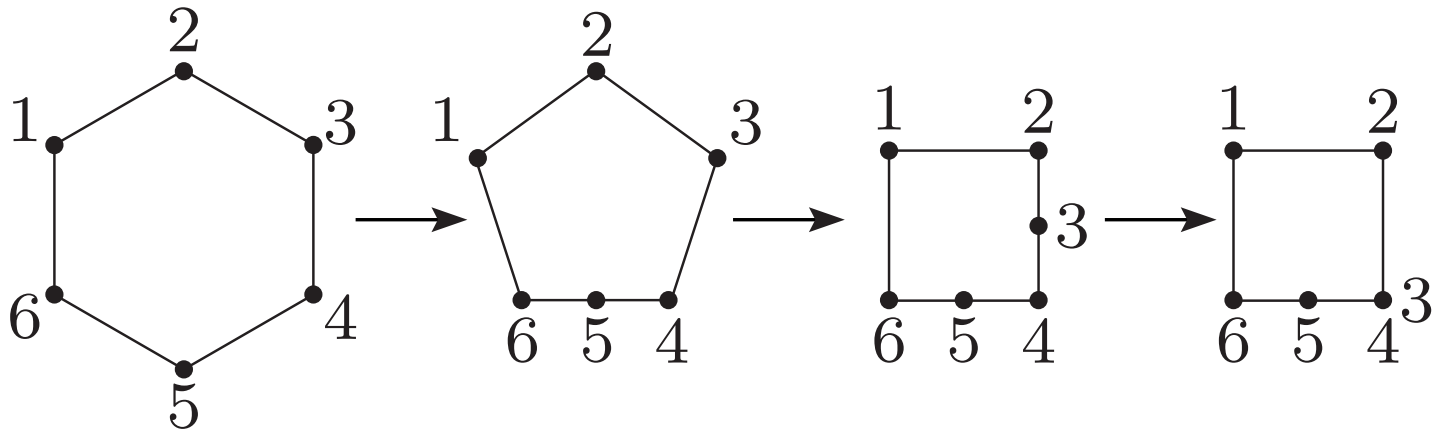


$$C = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & c_{16} \\ 0 & 1 & 0 & 0 & c_{25} & a \cdot c_{25} \\ 0 & 0 & 1 & c_{34} & c_{35} & a \cdot c_{35} \end{pmatrix}$$

Five-dimensional configuration in $G(3,6)$.

The Positive Grassmannian

- Positive part of $G(k,n)$: convex configurations of points.
- Top cell in the Grassmannian (no constraint imposed) $\rightarrow$ configuration of n generic points in $\mathbb{P}^{k-1}$.
- Stratification of the space is nicely provided by imposing linear dependencies between consecutive points



This corresponds to sending minors of $G_+(k,n)$ to zero.

- Boundaries preserve convexity: all minors of $G_+(k,n)$ stay positive (except the ones sent to zero).

Equivalence

Reduced graphs (mod identity moves)

$$\Updownarrow$$

Permutations

$$\Updownarrow$$

Configurations of vectors with linear dependencies

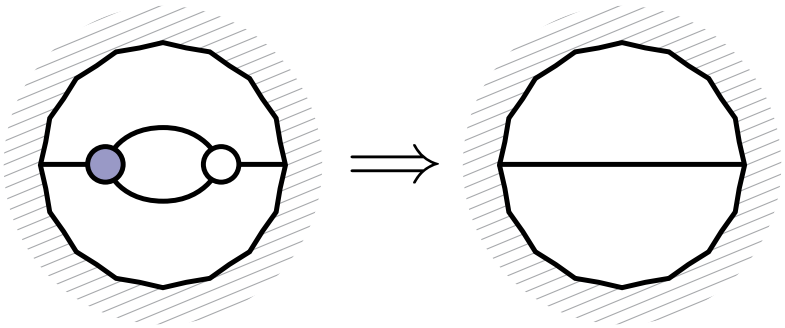
$$\Updownarrow$$

Cells of Positive Grassmannian

Plabic graphs and Positive Grassmannian

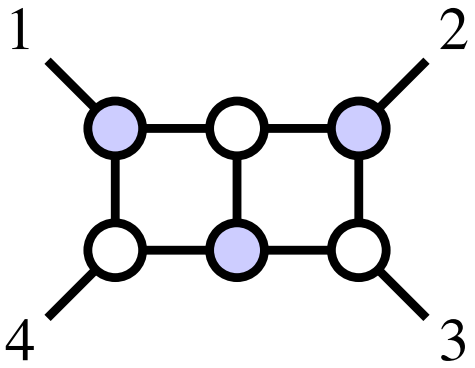
Plabic graphs

- These diagrams are known in the literature as "plabic graphs" and were extensively studied by Alexander Postnikov (math/0609764).
- He established the connection to the positive Grassmannian and showed how to construct explicitly a matrix for each reduced diagram.
- There is a precise definition what the "reduced" means but the in practice it means that the diagram does not have any bubbles.
- Bubble reduction:



Plabic graphs

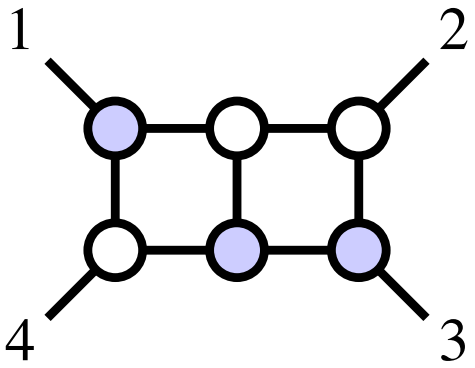
- Example:



- Postnikov proved proved isomorphism between permutations and reduced plabic graphs (modulo identity moves).
- In order to find the Grassmannian matrix for each reduced diagram we have to choose variables.
  - Edge variables.
  - Face variables.
- Orientation: choose an arrow for each edge.

Plabic graphs

- Example:



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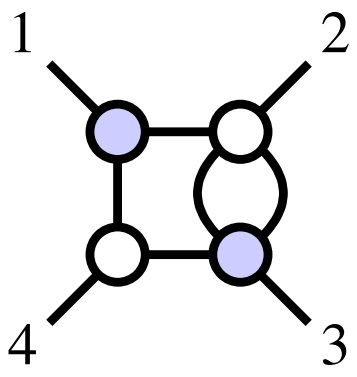

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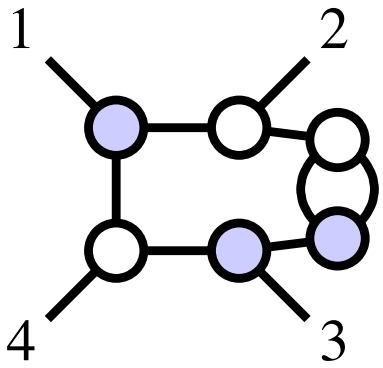

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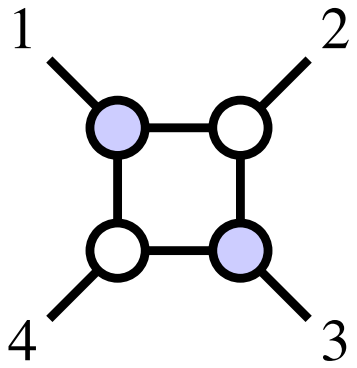

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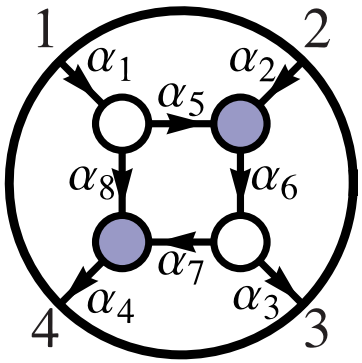
Edge variables

- Variables associated with edges.



- There is a $GL(1)$ redundancy in each vertex.
- The rule for entries of the C matrix,

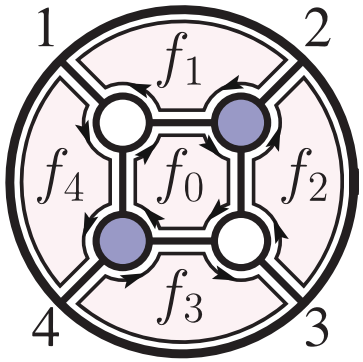
$$C_{iJ} = -\sum_{\text{paths } i \to J} \prod e_i \quad \text{edges along path}$$

- For this example (positive matrix for fixed signs of α_i):

$$C = \begin{pmatrix} 1 & 0 & -\alpha_1\alpha_3\alpha_5\alpha_6 & -\alpha_1\alpha_4\alpha_5\alpha_6\alpha_7 - \alpha_1\alpha_4\alpha_8 \\ 0 & 1 & -\alpha_2\alpha_3\alpha_6 & -\alpha_2\alpha_4\alpha_6\alpha_7 \end{pmatrix}$$

Face variables

- Variables associated with faces.



- "Gauge invariant" (fluxes) associated with faces of the graph. Only one condition $\prod f_i = -1$.
- The rule for entries of the C matrix,

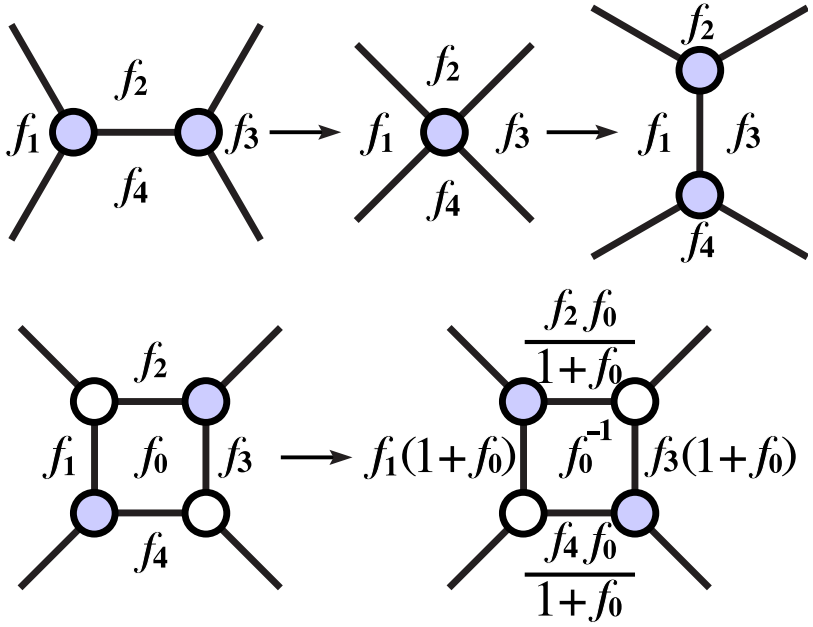
$$C_{iJ} = -\sum_{\text{paths } i \to J} \prod (-f_j) \quad \text{faces left to the path}$$

- For this example:

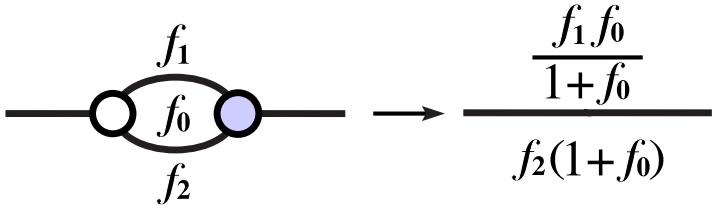
$$C = \begin{pmatrix} 1 & 0 & f_0 f_3 f_4 & -f_0 f_4 + f_4 \\ 0 & 1 & -f_0 f_1 f_3 f_4 & -f_0 f_1 f_4 \end{pmatrix}$$

Face variables

- Moves and face variables



- Reduction: eliminate irrelevant variable



- Face (or edge) variables are cluster variables and these are cluster transformations.