Towards New Formulation of Quantum field theory: Geometric Picture for Scattering Amplitudes

Part 2

Jaroslav Trnka

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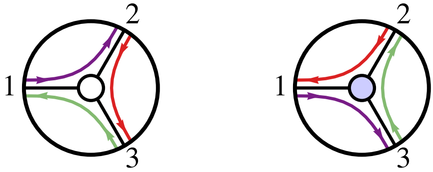
Work with Nima Arkani-Hamed, Jacob Bourjaily, Freddy Cachazo, Alexander Goncharov, Alexander Postnikov, arxiv: 1212.5605
Work with Nima Arkani-Hamed, arxiv: 1312.2007

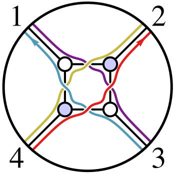
Review of the last lecture

Permutations

- Fundamental 3pt vertices:



represent permutations $(1,2,3) \to (2,3,1)$ and $(1,2,3) \to (3,2,1)$.

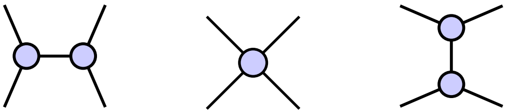


- Permutation $(1,2,3,4) \to (3,4,1,2)$.

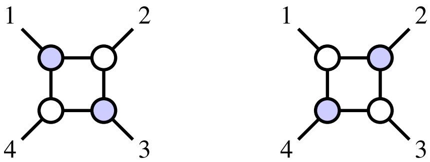
Permutations

There are two identity moves:

- merge-expand of black (or white) vertices



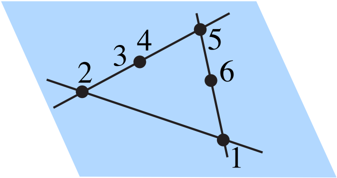
- square move



They preserve permutation.

Configuration of vectors

- Permutations $\leftrightarrow$ Configuration of n points $\mathbb{P}^{k-1}$ in with consecutive linear dependencies.
- Permutation $\sigma(i)$ means that $i \subset \mathrm{span}(i+1, \ldots \sigma(i))$
- Example: $n = 6, k = 3$, we have six points in $\mathbb{P}^2$.



$$1 \subset (2, 34, 5, 6) \to \sigma(1) = 6, \qquad 2 \subset (34, 5) \to \sigma(2) = 5,$$
$$3 \subset (4) \to \sigma(3) = 4, \qquad 4 \subset (5, 6, 1, 2) \to \sigma(4) = 2,$$
$$5 \subset (6, 1) \to \sigma(5) = 1, \qquad 6 \subset (1, 2, 3) \to \sigma(6) = 3.$$

- The permutation is $(1, 2, 3, 4, 5, 6) \to (6, 5, 4, 8, 7, 9)$.

The Positive Grassmannian

- Grassmannian $G(k,n)$: space of k-dimensional planes in n dimensions, represented by $k \times n$ matrix modulo $GL(k)$,

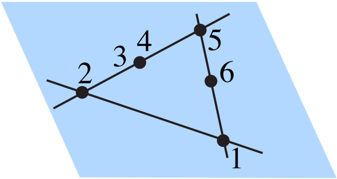
$$C = \begin{pmatrix} * & * & * & \dots & * & * \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ * & * & * & \dots & * & * \end{pmatrix} = \begin{pmatrix} v_1 \\ \vdots \\ v_k \end{pmatrix} = \begin{pmatrix} c_1 & c_2 & \dots & c_n \end{pmatrix}$$

- Positive part: all minors

$$(c_{i_1} \dots c_{i_k}) > 0 \quad \text{for} \quad i_1 < i_2 < \dots < i_k.$$

The Positive Grassmannian

- Back to 6pt example:



- Linear dependencies: fix points 1, 2, 3,

$$c_4 = a_{34}c_3 \qquad c_5 = a_{25}c_2 + a_{35}c_3$$

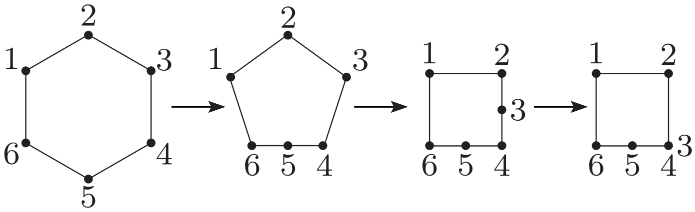
$$c_6 = a_{16}c_1 + zc_5 = a_{16}c_1 + za_{25}c_5 + za_{35}c_5$$

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & a_{16} \\ 0 & 1 & 0 & 0 & a_{25} & za_{25} \\ 0 & 0 & 1 & a_{34} & a_{35} & za_{35} \end{pmatrix}$$

- This is 5-dimensional cell in $G(2,6)$.

The Positive Grassmannian

- Positive Grassmannian $G_+(k, n)$: generalization of "convex" configurations of n points in $\mathbb{P}^{k-1}$.
- Top cell in $G_+(k, n)$: generic configuration of points.
- Example: top cell of $G_+(3, 6)$.

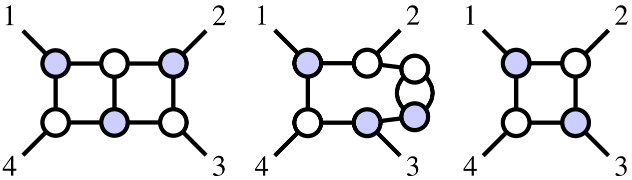


We send minor $(456) = 0$, then $(234) = 0$, then $(345) = 0$.

- Boundaries preserve convexity: all non-zero minors of $G_+(k, n)$ stay positive.
- This provides a stratification of $G_+(k, n)$.

Plabic graphs

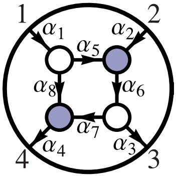
- Plabic graphs = diagrams with black and white vertices.
- Reduced graphs: no internal bubbles - their equivalence class is isomorphic to permutations and cells in $G_+(k,n)$.
- Generic diagram is not reduced: it contains internal bubbles.



- The diagram is reduced after all bubbles are removed.
- In order to find the Grassmannian matrix for each reduced diagram we have to choose variables.
  - Edge variables.
  - Face variables.

Edge variables

- Variables associated with edges, orientation for the graph.



- There is a $GL(1)$ redundancy in each vertex. The edge variables are "connections" on the graph.
- The rule for entries of the C matrix,

$$C_{iJ} = -\sum_{\text{paths } i \to J} \prod \alpha_i \quad \text{edges along path}$$

- For this example:

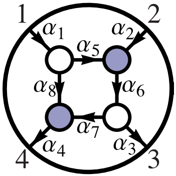
$$c_{11} = 1, \qquad c_{12} = 0, \qquad c_{21} = 0, \qquad c_{22} = 1$$

$$c_{13} = -\alpha_1\alpha_5\alpha_6\alpha_3, \qquad c_{14} = -\alpha_1(\alpha_5\alpha_6\alpha_7 + \alpha_8)\alpha_4$$

$$c_{23} = -\alpha_2\alpha_6\alpha_3, \qquad c_{24} = -\alpha_2\alpha_6\alpha_7\alpha_4$$

Edge variables

- Variables associated with edges, orientation for the graph.



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- The rule for entries of the C matrix,

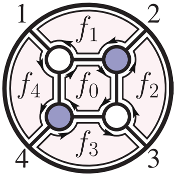
$$C_{iJ} = -\sum_{\text{paths } i \to J} \prod \alpha_i \quad \text{edges along path}$$

- For this example:

$$C = \begin{pmatrix} 1 & 0 & -\alpha_1\alpha_3\alpha_5\alpha_6 & -\alpha_1\alpha_4\alpha_5\alpha_6\alpha_7 - \alpha_1\alpha_4\alpha_8 \\ 0 & 1 & -\alpha_2\alpha_3\alpha_6 & -\alpha_2\alpha_4\alpha_6\alpha_7 \end{pmatrix}$$

Face variables

- Variables associated with faces.



- "Gauge invariant" (fluxes) associated with faces of the graph. Only one condition $\prod f_i = -1$.
- The rule for entries of the C matrix,

$$C_{iJ} = -\sum_{\text{paths } i \to J} \prod (-f_j) \quad \text{faces right to the path}$$

- For this example:

$$C = \begin{pmatrix} 1 & 0 & f_0 f_3 f_4 & -f_0 f_4 + f_4 \\ 0 & 1 & -f_0 f_1 f_3 f_4 & -f_0 f_1 f_4 \end{pmatrix}$$

On-shell diagrams and Scattering amplitudes

Three point amplitudes

- We want to find an alternative to Feynman diagrams.
- Let us take physical three point amplitudes as our fundamental objects instead of Feynman vertices.
- On-shell conditions and momentum conservation:

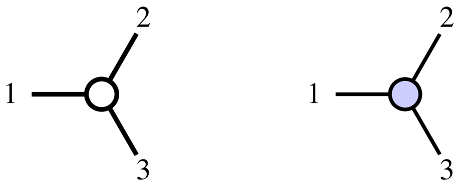
$$p_1 + p_2 + p_3 = 0, \qquad p_1^2 = p_2^2 = p_3^2 = 0$$

  No solution for real momenta!
- For complex momenta we get two different solutions
  - All λ are proportional, $\tilde{\lambda}$ are generic.
  - All $\tilde{\lambda}$ are proportional, λ are generic.

  Reminder: $\sigma^{\mu}_{a\dot{a}} p_\mu = \lambda_a \tilde{\lambda}_{\dot{a}}$.
- Two independent three point amplitudes ($k = 1$ and $k = 2$).

Three point amplitudes

- We graphically represent as



- They represent the expressions:

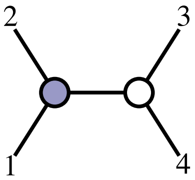
$$M_3^{(1)} = \frac{1}{[12][23][31]}\delta^4(\tilde{\eta}_1[23] + \tilde{\eta}_2[31] + \tilde{\eta}_3[12])\delta^4(p_1 + p_2 + p_3)$$

$$M_3^{(2)} = \frac{1}{\langle 12\rangle\langle 23\rangle\langle 31\rangle}\delta^8(\lambda_1\tilde{\eta}_1 + \lambda_2\tilde{\eta}_2 + \lambda_3\tilde{\eta}_3)\delta^4(p_1 + p_2 + p_3)$$

where $\langle 12\rangle = \epsilon_{ab}\lambda_1^a\lambda_2^b$, $[12] = \epsilon_{\dot{a}\dot{b}}\tilde{\lambda}_1^{\dot{a}}\tilde{\lambda}_2^{\dot{b}}$

On-shell gluing

- Glue two three point vertices into four point diagram



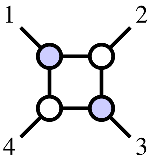
- We solve for the internal λ and $\tilde{\eta}$ and get

$$\frac{1}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}\delta^8(\lambda_1\tilde{\eta}_1+\lambda_2\tilde{\eta}_2+\lambda_3\tilde{\eta}_3+\lambda_4\tilde{\eta}_4)\delta^4(p_1+p_2+p_3+p_4)$$
$$\times\delta((p_1+p_2)^2)$$

- This is a factorization channel of 4pt tree-level amplitude, $(p_1+p_2)^2=0$.

On-shell gluing

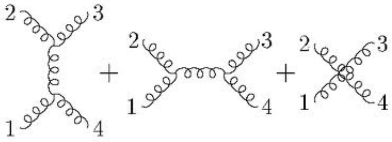
- Glue four three point vertices into four point diagram



which is a 4pt tree level amplitude!

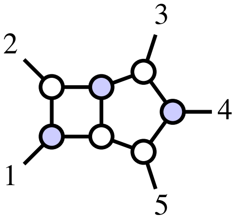
$$\frac{1}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}\delta^8(\lambda_1\tilde{\eta}_1+\lambda_2\tilde{\eta}_2+\lambda_3\tilde{\eta}_3+\lambda_4\tilde{\eta}_4)\delta^4(p_1+p_2+p_3+p_4)$$

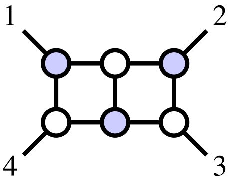
- This is equal to three Feynman diagrams.



On-shell gluing

- We glue arbitrary number of three point vertices and get **on-shell diagrams**: our new building blocks





- It is product of three point amplitudes where we solve (integrate) for internal data

$$\int \frac{d^2\lambda\, d^2\tilde{\lambda} d^4\tilde{\eta}}{\mathrm{GL}(1)}$$

- In general, it is a differential form.

On-shell diagrams

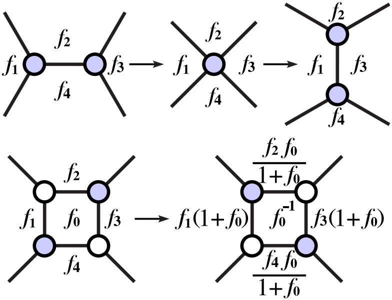
- These diagrams are identical to plabic graphs, they look identical and they satisfy the same identity moves!
- How to use the cell of Positive Grassmannian $G_+(k,n)$ associated with the diagram to get the function?
- We define a form with logarithmic singularities,

$$\int \frac{df_1}{f_1}\frac{df_2}{f_2}\cdots\frac{df_d}{f_d}\prod_{\alpha=1}^{k}\delta^{4|4}\left[C_{\alpha a}(f_i)\mathcal{W}_a\right]$$

  where C is the Grassmannian matrix parametrized by f_i.
- $\mathcal{W}$ carries the information about external data.
- There are different kinematical variables to choose: $\mathcal{W}=(\lambda,\tilde{\lambda},\tilde{\eta})$ or $\mathcal{W}=(Z,\tilde{\eta})$.
- Delta functions localize variables in the form.

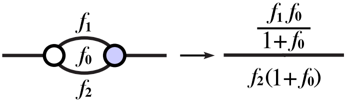
On-shell diagrams

- This form is invariant identity moves on diagrams:



- For reduction we get

$$\Omega \to \int \frac{df_0}{f_0}\frac{df_1'}{f_1'}\frac{df_2'}{f_2'}\cdots\frac{df_d}{f_d}\delta^{4|4}\left(C(f_1',f_2',f_3\dots f_d)_{\alpha a}\mathcal{W}_a\right)$$



Relations between on-shell diagrams

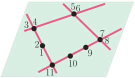
- All relations between on-shell diagrams are generated by

$$\partial\Omega_{D+1} = 0$$

  where Ω_{D+1} is $D+1$ dimensional cell in the Positive Grassmannian.
- This is extremely simple in terms of configurations of points in $\mathbb{P}^{k-1}$ but it generates non-trivial identities between functions.

Relations between on-shell diagrams

- Example: $n = 11$, $k = 5$ - identity involving higher roots



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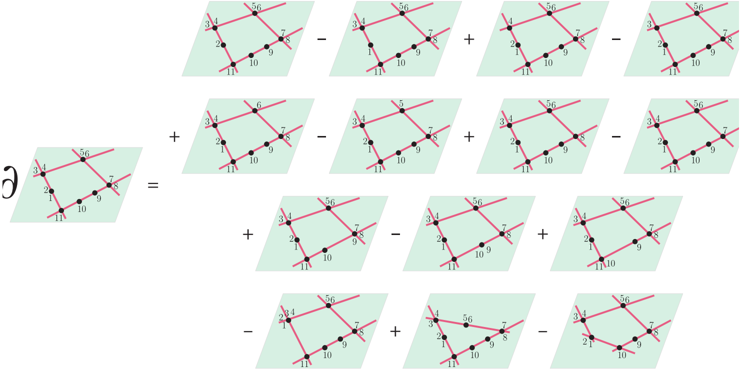

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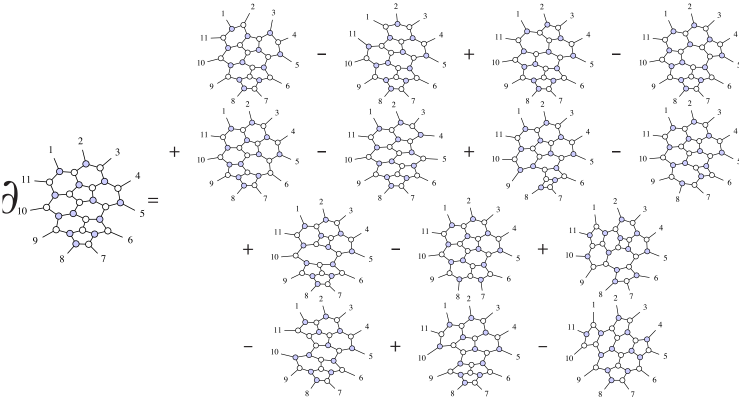
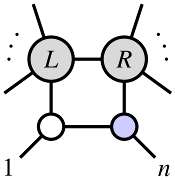
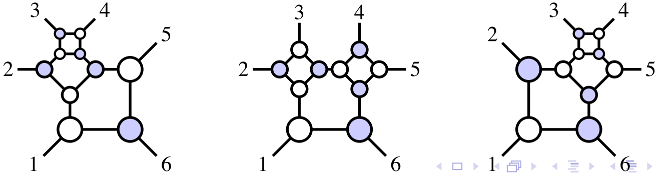


From on-shell diagrams to amplitude

- Each diagram is a potential building block for the amplitude. Label n is given by external legs and $k = W + 2B - E$.
- Recursion relations give us the expansion of the amplitude as a sum of on-shell diagrams.

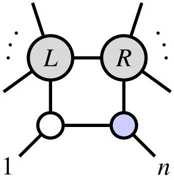


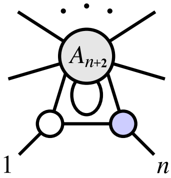
- Example: 6pt NMHV amplitude, $n = 6$, $k = 1$, there are 3 on-shell diagrams vs 220 Feynman diagrams



From on-shell diagrams to amplitude

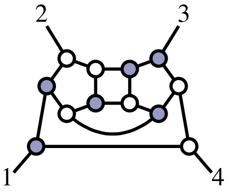
- The particular sum is dictated by physical properties of the amplitude - locality and unitarity.
- For tree-level amplitudes we always get reduced diagrams – invariant information is just a list of permutations.
- For loop amplitudes the diagrams are not reduced. At L-loops each diagram contains $4L$ irrelevant variables, each for one bubbles.
- Recursion relations:





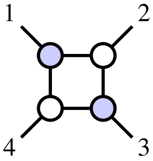
From on-shell diagrams to amplitude

- Example: 4pt one-loop amplitude



- It contains four bubbles = four irrelevant variables,

$$= \int \frac{df_1}{f_1} \frac{df_2}{f_2} \frac{df_3}{f_3} \frac{df_4}{f_4} \times$$



Conclusion

- On-shell diagrams provide a new basis of objects for scattering amplitudes (at least in our toy model).
- Each diagram corresponds to the cell in the Positive Grassmannian and its value is a canonical logarithmic form.
- It is possible to show that each diagram makes the hidden Yangian symmetry of our theory manifest – it is a positive diffeomorphism on positive part of Grassmannian.
- Scattering amplitude $M_{m,k,\ell}$ is a particular sum of on-shell diagrams.
- It is not a complete reformulation of QFT: amplitude is still a sum of pieces rather than a unique object, to get a sum we need a physical information (recursion relations) to construct the amplitude.