Towards New Formulation of Quantum field theory: Geometric Picture for Scattering Amplitudes

Part 3

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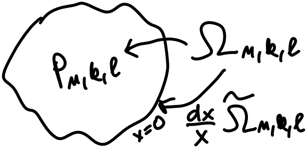
Work with Nima Arkani-Hamed, Jacob Bourjaily, Freddy Cachazo, Alexander Goncharov, Alexander Postnikov, arxiv: 1212.5605
Work with Nima Arkani-Hamed, arxiv: 1312.2007

The Amplituhedron

Review of the result

- For each amplitude $A_{n,k}^{\ell-loop}$ we define a positive space "Amplituhedron" $P_{n,k,\ell}$ - it is a generalization of the positive Grassmannian.
- For each positive space we associate a form $\Omega_{n,k,\ell}$ which has logarithmic singularities on the boundary of this region.

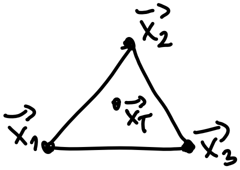


- From this form we can extract the amplitude $A_{n,k}^{\ell-loop}$.
- Calculating amplitudes: triangulation of the positive space $P_{n,k,\ell}$ in terms of building blocks for which the form is trivial.
- Set of on-shell diagrams (given by recursion relations) provides one particular triangulation.

The New Positive Region

Inside of the simplex

- Problem from classical mechanics: center-of-mass of three points

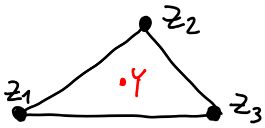


Imagine masses c_1, c_2, c_3 in the corners.

$$\vec{x_T} = \frac{c_1\vec{x_1} + c_2\vec{x_2} + c_3\vec{x_3}}{c_1 + c_2 + c_3}$$

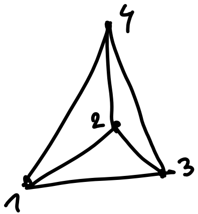
- Interior of the triangle: ranging over all positive c_1, c_2, c_3.
- Triangle in projective space $\mathbb{P}^2$.
  - Projective variables $Z_i = \begin{pmatrix} 1 \\ \vec{x_i} \end{pmatrix}$
  - Point Y inside the triangle.

$$Y = c_1 Z_1 + c_2 Z_2 + c_3 Z_3$$



Inside of the simplex

- Generalization to higher dimensions is straightforward.



Point Y inside tetrahedon in $\mathbb{P}^3$:

$$Y = c_1 Z_1 + c_2 Z_2 + c_3 Z_3 + c_4 Z_4$$

Ranging over all positive c_i spans the interior of the simplex.

- In general point Y inside a simplex in $\mathbb{P}^{m-1}$:

$$Y^I = C_{1a} Z_a^I \qquad \text{where } I = 1, 2, \ldots, m$$

- C is $(1 \times m)$ matrix of positive numbers,

$$C = (c_1\, c_2\, \ldots\, c_m)/GL(1) \qquad \text{which is } G_+(1, m)$$

Into the Grassmannian

- Generalization of this notion to Grassmannian.
- Let us imagine the same triangle and a line Y,

$$Y_1 = c_1^{(1)} Z_1 + c_2^{(1)} Z_2 + c_3^{(1)} Z_3$$
$$Y_2 = c_1^{(2)} Z_1 + c_2^{(2)} Z_2 + c_3^{(2)} Z_3$$

  writing in the compact form

$$Y_\alpha^I = C_{\alpha a} Z_a^I \qquad \text{where } \alpha = 1, 2$$

- The matrix C is a (2×3) matrix mod GL(2) - Grassmannian $G(2,3)$.
- Positivity of coefficients? No, minors are positive!

$$C = \begin{pmatrix} 1 & 0 & -a \\ 0 & 1 & b \end{pmatrix}$$

Into the Grassmannian

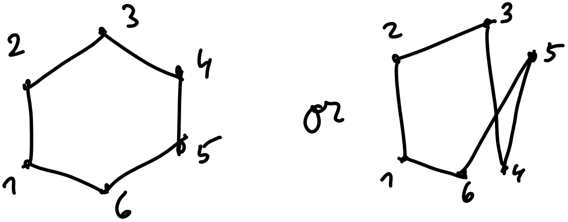
- In the general case we have

$$Y_\alpha^I = C_{\alpha\,a} Z_a^I$$

  where $\alpha = 1, 2, \ldots, k$, ie. it is a k-plane in $(k+m)$ dimensions, $a, I = 1, 2, \ldots k+m$. Simplex has $\alpha = 1$, for triangle also $m = 2$.
- The matrix C is a 'top cell' (no constraint imposed) of the positive Grassmannian $G_+(k, k+m)$, it is $k \cdot m$ dimensional.
- We know exactly what these matrices are!

Beyond triangles

- Let us go back to point inside a triangle.
- Points Z_i did not play role, we could always choose the coordinate system such that Z is identity matrix, then $Y \sim C$.
- If the number of points $n > k+m$ the position of vertices is crucial. For a given cyclic ordering we consider the interior of the polygon in $\mathbb{P}^2$.



We need a convex polygon!

Beyond triangles

- Convexity = positivity of Z's. They form a $(3 \times n)$ matrix with all ordered minors being positive,

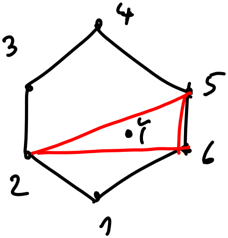
$$\langle Z_i Z_j Z_k \rangle > 0 \qquad \text{for all } i < j < k$$

- The point Y inside this polygon is

$$Y^I = c_1 Z_1 + \cdots + c_n Z_n = C_{1a} Z_a^I$$

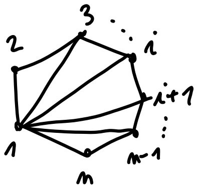
where $C \in G_+(1, n)$ and $Z \in M_+(3, n)$.

- Note that point Y is also inside some triangle



Beyond triangles

- Triangulation: set of non-intersecting triangles that cover the region.



$$P_n = \sum_{i=2}^{n} [1\,i\,i{+}1]$$

- The generic point Y is inside one of the triangles. The matrix:

$$C = \begin{pmatrix} 1 & 0 & \dots & 0 & c_i & c_{i+1} & 0 & \dots & 0 \end{pmatrix}$$

- Two descriptions:
  - Top cell $(n{-}1)$-dimensional of $G_+(1,n)$ - redundant.
  - Collection of 2-dimensional cells of $G_+(1,n)$ - triangulation.

Tree Amplituhedron

- In general case we have
  - A k-plane Y in $(k+m)$-dimensional space.
  - Positive region given by n points Z_i with positive constraints,

$$\langle Z_{a_1} Z_{a_2} Z_{a_3} \dots Z_{a_{k+m}} \rangle > 0 \qquad \text{for } a_1 < a_2 < a_3 < \cdots < a_{k+m}$$

  - The definition of the space:

$$Y^I_\alpha = C_{\alpha\, a} Z^I_a \qquad \alpha = 1, \dots, k, \quad I = 1, \dots, k+m, \quad a = 1, \dots, n$$

- Note that the geometric statement being "inside" does not generalize while the positivity of all $(k \times k)$ minors does.
- It is a map that defines a positive region $P_{n,k,m}$,

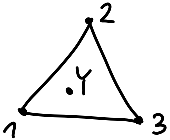
$$G_+(k,n) \times M_+(k+m,n) \to G(k,k+m)$$

- The tree-level amplitude for n, k corresponds to the case n, k and $m = 4$.

Canonical forms

Canonical form

- How to get the actual formula from the positive region?
- We define a canonical form Ω_P which has **logarithmic** singularities on the boundaries of P.
- Example of triangle in $\mathbb{P}^2$:



$$\Omega_P = \frac{\langle Y\, dY\, dY\rangle\langle 123\rangle^2}{\langle Y12\rangle\langle Y23\rangle\langle Y31\rangle}$$

- We parametrize $Y = Z_1 + c_2 Z_2 + c_3 Z_3$ and get

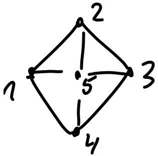
$$\Omega_P = \frac{dc_2}{c_2}\frac{dc_3}{c_3} = \mathrm{dlog}\, \mathrm{c}_2\ \mathrm{dlog}\, \mathrm{c}_3$$

- This is just a form for the top cell of $G_+(1,3)$,

$$C = \begin{pmatrix} 1 & c_2 & c_3 \end{pmatrix}$$

Canonical form

- Simplex in $\mathbb{P}^4$ - this is relevant for physics.



$$\Omega_P = \frac{\langle Y\, dY\, dY\, dY\, dY \rangle \langle 12345 \rangle^2}{\langle Y1234 \rangle \langle Y2345 \rangle \langle Y3451 \rangle \langle Y4512 \rangle \langle Y5123 \rangle}$$

For $Y = Z_1 + c_2 Z_2 + c_3 Z_3 + c_4 Z_4 + c_5 Z_5$:

$$\Omega_P = \mathrm{dlog}\, \mathrm{c}_2 \ \mathrm{dlog}\, \mathrm{c}_3 \ \mathrm{dlog}\, \mathrm{c}_4 \ \mathrm{dlog}\, \mathrm{c}_5$$

- This is a 4-dimensional (top) cell in $G_+(1,5)$.

$$C = \begin{pmatrix} 1 & c_2 & c_3 & c_4 & c_5 \end{pmatrix}$$

- The form Ω_P gives the $n = 5$, $k = 1$ tree-level scattering amplitude.

Canonical form

- For general D-dimensional cell of Positive Grassmannian there is an on-shell diagram associated with each cell.
- We parametrize the C matrix with D positive parameters. and the logarithmic form is

$$\Omega_P = \text{dlog}\, c_1 \ \text{dlog}\, c_2 \ \ldots \ \text{dlog}\, c_D$$

- Then we can solve for parameters c_j using

$$Y_\alpha^I = C_{\alpha a} Z_a^I$$

and get the form in Y.

Canonical form

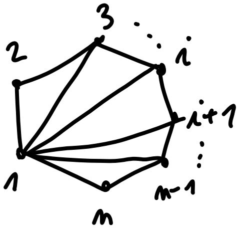
- This is not enough for us. Our space is km dimensional and it is an image of $k(n-k)$-dimensional top cell of $G_+(k,n)$ under the map

$$Y_\alpha^I = C_{\alpha\, a} Z_a^I \qquad \alpha = 1,\ldots,k, \quad I = 1,\ldots,k{+}m, \quad a = 1,\ldots,n$$

- There is a form Ω with logarithmic singularities on the boundaries of this space.
- It is not the dlog of all variables unlike for the cell of $G_+(k,n)$. Matrix C is parametrized using $k(n-k)$ parameters but the space is only km dimensional. We wish to find the form just from definition.
- Alternative method: triangulate the space - find a collection of km dimensional cells of $G_+(k,n)$ that cover the whole region for Y like in polygon case.
- Then the form is just a sum of dlog forms for each cell (after solving for parameters c_j in terms of Y and Z_i).

Canonical form

- Example: Polygon



$$\Omega_P = \sum_{i=2}^{n} \frac{\langle Y\, dY\, dY\rangle \langle 1\, i\, i{+}1\rangle^2}{\langle Y\, 1\, i\rangle \langle Y\, 1\, i{+}1\rangle \langle Y\, i\, i{+}1\rangle}$$

Spurious poles $\langle Y\, 1\, i\rangle$ cancel in the sum.

- We can also look directly for the form

$$\Omega = \frac{\langle Y\, dY\, dY\rangle\, \mathcal{N}(Y, Z_i)}{\langle Y 12\rangle \langle Y 23\rangle \langle Y 34\rangle \ldots \langle Y n1\rangle}$$

and fix the numerator from the condition that singularities are logarithmic only on the boundaries of the polygon.

Canonical form

- The case of physical relevance is $m = 4$.
- Tree-level Amplituhedron $P_{n,k,4}$ defined by

$$Y_\alpha^I = C_{\alpha\,a} Z_a^I \qquad \alpha = 1, \ldots, k, \quad I = 1, \ldots, k{+}4, \quad a = 1, \ldots, n$$

- There is a form with logarithmic singularities on the boundary of this space.
- Recursion relations provide a triangulation of this space in terms of building blocks, $4k$-dimensional cells of Positive Grassmannian $G_+(k,n)$.
- There exist many other triangulations.

From canonical forms to amplitudes

- How to extract the amplitude from Ω_P?
- Look at the example of simplex in $\mathbb{P}^4$.

$$\Omega_P = \frac{\langle Y\, dY\, dY\, dY\, dY\rangle\langle 12345\rangle^4}{\langle Y1234\rangle\langle Y2345\rangle\langle Y3451\rangle\langle Y4512\rangle\langle Y5123\rangle}$$

- Note that the data are five-dimensional, it is purely bosonic and it is a form rather than function.
- Let us rewrite Z_i as four-dimensional part and its complement

$$Z_i = \begin{pmatrix} z_i \\ \delta z_i \end{pmatrix} \qquad \text{where} \quad \delta z_i = (\eta_i \cdot \phi)$$

We define a reference point Y^* which is in the complement of 4d data z_i,

$$Y^* = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

From canonical forms to amplitudes

- We integrate the form, using $\langle Y^* 1234\rangle = \langle 1234\rangle$, etc. we get

$$\int d^4\phi \int \delta(Y - Y^*)\,\Omega_P = \frac{(\langle 1234\rangle\eta_5 + \langle 2345\rangle\eta_1 + \cdots + \langle 5123\rangle\eta_4)^4}{\langle 1234\rangle\langle 2345\rangle\langle 3451\rangle\langle 4512\rangle\langle 5123\rangle}$$

- For higher k we have $(k+4)$ dimensional external Z_i,

$$Z_i = \begin{pmatrix} z_i \\ (\eta_i \cdot \phi_1) \\ \vdots \\ (\eta_i \cdot \phi_k) \end{pmatrix} \qquad Y^* = \begin{pmatrix} \vec{0} & \vec{0} & \dots & \vec{0} \\ 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}$$

- Reference k-plane Y^* orthogonal to external z_i. We consider integral

$$A_{n,k} = \int d^4\phi_1 \dots d^4\phi_k \int \delta(Y - Y^*)\,\Omega_{P_{n,k}}$$

Loop amplitudes

MHV amplitudes

- The simplest case is $k = 0$ where there is no Y-plane – MHV amplitudes. The space is just $\mathbb{P}^3$ and the tree-level form is just $\Omega_P = 1$.
- The loop momentum is represented by a line $Z_A Z_B$. We consider all configurations of this line.
- For one line we can parametrize

$$A^I_\alpha = C_{\alpha\, a} Z^I_a, \qquad \text{where } \alpha = 1, 2$$

  where $A_\alpha = (A, B)$.
- We demand the matrix of coefficients to be positive, ie. $C \in G_+(2, n)$ and $Z \in M_+(4, n)$.
- The form with logarithmic singularities on the boundaries of this space is MHV 1-loop amplitude.
- We know the explicit form of the result by triangulation.

MHV amplitudes

- "Triangles" are just 4-dimensional cells of $G_+(2,n)$: "kermits"

$$C_{1,i,i+1;1,j,j+1} = \begin{pmatrix} 1 & 0 & 0 & c_i & c_{i+1} & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & c_j & c_{j+1} & 0 & 0 \end{pmatrix}$$

- Triangulation

$$P_n = \sum_{i<j} [1, i, i+1; 1, j, j+1]$$

where each kermit has a simple form

$$\Omega_P = \mathrm{dlog}\, c_i \;\, \mathrm{dlog}\, c_{i+1} \;\, \mathrm{dlog}\, c_j \;\, \mathrm{dlog}\, c_{j+1}$$

- The form for the full space is then

$$\Omega_P = \sum_{i<j} \frac{\langle ABd^2A\rangle\langle ABd^2B\rangle\langle AB(i{-}1\,i\,i{+}1)\bigcap(j{-}1\,j\,j{+}1)\rangle^2}{\langle AB\,1\,i\rangle\langle AB\,1\,i{+}1\rangle\langle AB\,i\,i{+}1\rangle\langle AB\,1\,j\rangle\langle AB\,1\,j{+}1\rangle\langle AB\,j\,j{+}1\rangle}$$

MHV amplitudes

- In the next case we consider two lines in $\mathbb{P}^3$, denoted as $Z_A Z_B$, $Z_C Z_D$,

$$A^{(1)\ I}_{\alpha} = C^{(1)}_{\alpha\, a} Z^I_a$$
$$A^{(2)\ I}_{\alpha} = C^{(2)}_{\alpha\, a} Z^I_a$$

We combine matrices into

$$C = \left(\begin{array}{c} C^{(1)} \\ C^{(2)} \end{array} \right)$$

- We demand $C^{(1)}, C^{(2)}$ to be both $G_+(2,n)$. This is a "square" of one-loop problem: $(A_n^{1-loop})^2$.
- Additional constraint: All (4×4) minors of C are positive!
- The form with log singularities is MHV two-loop amplitude.
- The space of C matrices is a generalization of positive Grassmannian and has not been studied in the literature - we do not know the stratification (even before applying Z-projection).

MHV amplitudes

- At L-loop we have L lines A^I_α.

$$A^{(1)\ I}_{\alpha} = C^{(1)}_{\alpha\, a} Z^I_a \qquad\qquad C = \begin{pmatrix} C^{(1)} \\ \vdots \\ C^{(L)} \end{pmatrix}$$
$$\vdots$$
$$A^{(L)\ I}_{\alpha} = C^{(L)}_{\alpha\, a} Z^I_a$$

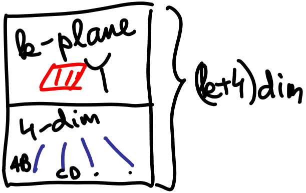
- Positivity constraints:
  - External data Z are positive.
  - All minors of $C^{(j)}$ are positive.
  - All (4×4) minors made of $C^{(i)}$, $C^{(j)}$ are positive, all (6×6) minors of $C^{(i)}$, $C^{(j)}$, $C^{(k)}$, etc. are also positive.
- The form is the integrand of n-pt L-loop MHV amplitude.

The Amplituhedron

In the general case of n-pt L-loop N^{k}MHV amplitude we have

- Positive $k+4$-dimensional external data Z.
- k-plane Y in $k+4$ dimensions
- L lines in 4-dimensional complement to Y plane

$$
\begin{aligned}
Y_\sigma^I &= C_{\sigma\, a} Z_a^I \\
A_\alpha^{(1)\ I} &= C_{\alpha\, a}^{(1)} Z_a^I \\
&\vdots \\
A_\alpha^{(L)\ I} &= C_{\alpha\, a}^{(L)} Z_a^I
\end{aligned}
\qquad
C = \begin{pmatrix} C \\ C^{(1)} \\ \vdots \\ C^{(L)} \end{pmatrix}
$$



Positivity constraints:

- C is positive.
- C + any combination of $C^{(i)}$'s is positive.

Application: Four point amplitudes

Four point amplitude

- We have L lines in $\mathbb{P}^3$ represented by L positive Grasmannians $G_+(2,4)$. All 4×4 minors made of pairs of $G_+(2,4)$ are positive. Zs are totally irrelevant here, $n=m=4$.
- In one parametrization,

$$C_1 = \begin{pmatrix} 1 & x_1 & 0 & -v_1 \\ 0 & y_1 & 1 & u_1 \end{pmatrix} \quad \dots \quad C_L = \begin{pmatrix} 1 & x_L & 0 & -v_L \\ 0 & y_L & 1 & u_L \end{pmatrix}$$

  All coefficients $x_i, y_i, u_i, v_i > 0$ and there is a set of inequalities

$$(x_i - x_j)(u_i - u_j) + (y_i - y_j)(v_i - v_j) < 0$$

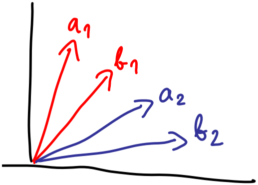
  for all pairs i, j.
- There is a way how to phrase this problem purely geometrically. Let us define

$$a_i = \begin{pmatrix} x_i \\ y_i \end{pmatrix} \qquad b_i = \begin{pmatrix} v_i \\ u_i \end{pmatrix}$$

Four-point amplitude

We have 2d vectors a_i, b_i for $i = 1, \ldots, L$ and we demand

- They all live in the first quadrant.
- For any pair $(a_i - a_j) \cdot (b_i - b_j) < 0$.
- Triangulation: Find all possible configurations of vectors!



We do not know how to solve it in general but we have some partial results.

One-loop amplitude

- This is a "free" case, $G_+(2,4)$. We have a single line in $\mathbb{P}^3$ parametrized by

$$C_1 = \begin{pmatrix} 1 & x_1 & 0 & -v_1 \\ 0 & y_1 & 1 & u_1 \end{pmatrix}$$

with $x_1, y_1, u_1, v_1 > 0$.

- Therefore the form with logarithmic singularities on the boundaries (0 and ∞ for all variables), is

$$\Omega = \frac{dx_1}{x_1}\frac{dy_1}{y_1}\frac{du_1}{u_1}\frac{dv_1}{v_1}$$

Two-loop amplitude

- This is a first "interacting" case.

$$C_1 = \begin{pmatrix} 1 & x_1 & 0 & -v_1 \\ 0 & y_1 & 1 & u_1 \end{pmatrix} \qquad C_2 = \begin{pmatrix} 1 & x_2 & 0 & -v_2 \\ 0 & y_2 & 1 & u_2 \end{pmatrix}$$

where all variables are positive and satisfy

$$(x_1 - x_2)(u_1 - u_2) + (y_1 - y_2)(v_1 - v_2) < 0$$

- This is a quadratic condition and in order to write bounds for all variables we have to triangulate the region – there are several top form.
- The final result is then:

$$\frac{(u_2 x_1 + u_1 x_2 + v_1 y_2 + v_2 y_1)}{x_1 x_2 y_1 y_2 u_1 u_2 v_1 v_2 [(x_1 - x_2)(u_1 - u_2) + (y_1 - y_2)(v_1 - v_2)]}$$

Cuts of the amplitude

- We do not know how to triangulate the amplitude in general. The system of inequalities is too hard for us at the moment.
- However we can solve some special cases - cuts of the amplitude: we can generate many data not accessible using any other approach.
- One example: all $v_i = 0$

$$C_1 = \begin{pmatrix} 1 & x_1 & 0 & 0 \\ 0 & y_1 & 1 & u_1 \end{pmatrix} \quad \ldots \quad C_L = \begin{pmatrix} 1 & x_L & 0 & 0 \\ 0 & y_L & 1 & u_L \end{pmatrix}$$

- It is the single cut of the amplitude $\langle (AB)_i 12 \rangle = 0$

$$(x_i - x_j)(u_i - u_j) < 0$$

- Solution: we consider all orderings of x's: $x_1 < x_2 < \ldots x_L$ and then y's are ordered in the opposite order $y_1 > y_2 > \ldots y_L$.

Conclusion

Conclusion

Mathematics

- Amplituhedron is a significant generalization of positive Grassmannian which is of high interest in QFT.
- Interesting directions: triangulation of this space in terms of more elementary building blocks, understanding stratification of this space.

Physics

- Amplituhedron is a new geometric picture for scattering amplitudes in planar $\mathcal{N} = 4$ SYM.
- It is defined purely in geometric terms with no reference to usual concepts of QFT: no Lagrangians, locality or unitarity which are all emergent properties from geometry.
- Interesting directions: calculate all-loop order results, generalization of the picture to other theories.