Conformally invariant trilinear forms on the sphere

Jean-Louis Clerc
(based on results by JLC and B. Ørsted)

Invariant trilinear forms

- Recall the definition of the principal series representation π_λ
$$\pi_\lambda(g)f(x) = \kappa(g^{-1}, x)^{\rho+\lambda} f(g^{-1}(x))$$
where $f \in \mathcal{C}^\infty(S), g \in G, x \in S$ and $\rho = \frac{n-1}{2}$.

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- Let $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}$. A continuous trilinear form $\mathcal{T}$ on $\mathcal{C}^\infty(S) \times \mathcal{C}^\infty(S) \times \mathcal{C}^\infty(S)$ is *invariant* with respect to $(\pi_{\lambda_1}, \pi_{\lambda_2}, \pi_{\lambda_3})$ if
$$\mathcal{T}\big(\pi_{\lambda_1}(g)f_1, \pi_{\lambda_2}(g)f_2, \pi_{\lambda_1}(g)f_3\big) = \mathcal{T}(f_1, f_2, f_3)$$
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for $f_1, f_2, f_3 \in \mathcal{C}^\infty(S)$ and $g \in G$.
- A trilinear form $\mathcal{T}$ can also be regarded as a distribution on $S \times S \times S$, and when convenient, we use the notation $\mathcal{T}(f)$ for $f \in \mathcal{C}^\infty(S \times S \times S)$ or $\mathcal{T}(f_1 \otimes f_2 \otimes f_3)$ instead of $\mathcal{T}(f_1, f_2, f_3)$.

Formal construction

Let $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{C}^3$, and let

$$\mathcal{K}_{\boldsymbol{\alpha}}(f_1, f_2, f_3) = \int_{S\times S\times S} |x-y|^{\alpha_3}|y-z|^{\alpha_1}|z-x|^{\alpha_2} f_1(x) f_2(y) f_3(z)\, dx\, dy\, dz$$

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Proposition

Let $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}$, and define $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ by

$$\begin{aligned}
\alpha_1 &= -\rho - \lambda_1 + \lambda_2 + \lambda_3 \\
\alpha_2 &= -\rho + \lambda_1 - \lambda_2 + \lambda_3 \\
\alpha_3 &= -\rho + \lambda_1 + \lambda_2 - \lambda_3 \ .
\end{aligned}$$

Then the trilinear form $\mathcal{K}_{\boldsymbol{\alpha}}$ is invariant w.r.t. $\pi_{\lambda_1}, \pi_{\lambda_2}, \pi_{\lambda_3}$, whenever it makes sense.

- The proof of the invariance amounts to the change of variable $x' = g^{-1}(x), y' = g^{-1}(y), z' = g^{-1}(z)$ in the integral. It uses the *covariance property* of the Euclidean distance on S, namely

$$|g(x) - g(y)| = \kappa(g, x)^{\frac{1}{2}} \, |x - y| \, \kappa(g, y)^{\frac{1}{2}} \ .$$

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- Let $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3)$. The correspondance $\boldsymbol{\lambda} \mapsto \boldsymbol{\alpha}$ can be inverted,

$$\lambda_1 = \rho + \frac{\alpha_2 + \alpha_3}{2}, \quad \lambda_2 = \rho + \frac{\alpha_3 + \alpha_1}{2}, \quad \lambda_3 = \rho + \frac{\alpha_3 + \alpha_1}{2} \ .$$

- We also use the notation $\mathcal{K}^{\boldsymbol{\lambda}} = \mathcal{K}_{\boldsymbol{\alpha}}$, and call $\boldsymbol{\alpha}$ the *geometric parameter* and $\boldsymbol{\lambda}$ the *spectral parameter* of the trilinear form.

Convergence, meromorphic continuation of $\mathcal{K}_{\boldsymbol{\alpha}}$

- The integral defining the trilinear form $\mathcal{K}_{\boldsymbol{\alpha}}$ is convergent iff
  - $\Re(\alpha_j) > -(n-1), \quad j = 1, 2, 3$
  - $\Re(\alpha_1 + \alpha_2 + \alpha_3) > -2(n-1)$

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- The map $\boldsymbol{\alpha} \longmapsto \mathcal{K}_{\boldsymbol{\alpha}}$ can be meromorphically extended to $\mathbb{C}^3$ with simple poles along four families of planes :
  - $\alpha_j = -(n-1) - 2k$, for $k \in \mathbb{N}$
  - $\alpha_1 + \alpha_2 + \alpha_3 = -2(n-1) - 2l$, for $l \in \mathbb{N}$

- The *renormalized form* $\widetilde{\mathcal{K}}_{\boldsymbol{\alpha}}$ defined by

$$\widetilde{\mathcal{K}}_{\boldsymbol{\alpha}} = \frac{1}{\Gamma(\frac{\alpha_1+\alpha_2+\alpha_3}{2} + 2\rho)\prod_{1\le j\le 3}\Gamma(\frac{\alpha_j}{2}+\rho)}\mathcal{K}_{\boldsymbol{\alpha}}$$

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- Question : for which values of $\boldsymbol{\alpha}$ is $\widetilde{\mathcal{K}}_{\boldsymbol{\alpha}} \not\equiv 0$?
  If $\boldsymbol{\alpha}$ is not a pole, then certainly $\widetilde{\mathcal{K}}_{\boldsymbol{\alpha}} \not\equiv 0$, by testing on functions (f_1, f_2, f_3) where $Supp(f_i) \cap Supp(f_j) = \emptyset$ for $1 \leq i \neq j \leq 3$.

The generic uniqueness theorem

Theorem

Let $n \geq 3$. Let $\boldsymbol{\lambda} \in \mathbb{C}^3$, not a pole of $\mathcal{K}^{\boldsymbol{\lambda}}$. Then a continuous trilinear form on $\mathcal{C}^\infty(S)$ which is invariant w.r.t. $(\pi_{\lambda_1}, \pi_{\lambda_2}, \pi_{\lambda_3})$ is proportional to $\mathcal{K}^{\boldsymbol{\lambda}}$.

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Sketch of the proof.

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  Consequence : viewing a continuous trilinear form as a distribution (say T) on $S \times S \times S$, on $\mathcal{O}_0$ T has to coincide with a multiple of $\mathcal{K}^{\boldsymbol{\lambda}}$.
- Have to prove : there is no invariant distribution supported on the (closed) subset ${\mathcal{O}_0}^c$.

- There are four other G-orbits in $S \times S \times S$:

$$\mathcal{O}_1 = \{x \neq y = z\}, \quad \mathcal{O}_2 = \{y \neq z = x\}, \quad \mathcal{O}_3 = \{z \neq y = x\}$$

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- Now $\mathcal{O}_1$ is a closed submanifold of the open subset $\mathcal{O}_0 \cup \mathcal{O}_1$. If T is a distribution on $S \times S \times S$, supported in $\cup_{1 \leq j \leq 4} \mathcal{O}_j$, then the restriction of T to $\mathcal{O}_0 \cup \mathcal{O}_1$ is supported in $\mathcal{O}_1$.

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- Using *Bruhat's necessary condition* for the existence of an invariant distribution supported on a closed submanifold, one obtain that $T_{|\mathcal{O}_0 \cup \mathcal{O}_1}$ has to be 0.
- Same procedure for $\mathcal{O}_2$ and $\mathcal{O}_3$, and finally for $\mathcal{O}_4$ which is a closed orbit in $S \times S \times S$.

A consequence of the generic uniqueness

For $\lambda_1, \lambda_2, \lambda_3$ and $f_1, f_2, f_3 \in \mathcal{C}^\infty(S)$

$$\widetilde{\mathcal{K}}^{(-\lambda_1,\lambda_2,\lambda_3)}(\widetilde{J}_{\lambda_1} f_1, f_2, f_3) = \frac{\pi^\rho}{\Gamma(-\lambda_1+\rho)} \widetilde{\mathcal{K}}^{(\lambda_1,\lambda_2,\lambda_3)}(f_1, f_2, f_3) \ .$$

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Remark.

$$\frac{1}{\Gamma(-\lambda_1+\rho)} = 0 \iff \lambda_1 = \rho + k$$

for some $k \in \mathbb{N}$. In which case $\pi_{-\lambda_1}$ is *reducible* and admits a finite dimensional invariant subspace.

Evaluation of an integral

$$\mathcal{K}_{\boldsymbol{\alpha}}(1,1,1) = \int_{S\times S\times S} |x-y|^{\alpha_3}|y-z|^{\alpha_1}|z-x|^{\alpha_2} dx\, dy\, dz$$

$$= \left(\frac{\pi}{2}\right)^{\frac{3}{2}(n-1)} 2^{\alpha_1+\alpha_2+\alpha_3} \frac{\Gamma\left(\frac{\alpha_1+\alpha_2+\alpha_3}{2}+2\rho\right) \circlearrowleft \Gamma\left(\frac{\alpha_1}{2}+\rho\right)}{\circlearrowleft \Gamma\left(\frac{\alpha_1+\alpha_2}{2}+2\rho\right)}$$

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- When $n=2$ the computation is due to I. Bernstein & A. Reznikov (using mainly geometry!), the general case was obtained by A. Deitmar (using changes of variables and induction over n) and by JLC & B. Ørsted (using spherical harmonic analysis), further generalized by JLC, T. Kobayashi, B. Ørsted and M. Pevzner.

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- $\mathcal{K}_{\boldsymbol{\alpha}}(1,1,1)$ has exactly the poles predicted by the general theory for $\mathcal{K}_{\boldsymbol{\alpha}}$.

- $\mathcal{K}_{\boldsymbol{\alpha}}(1,1,1) = 0$ if and only if, up to permutation of the indices, $\frac{\alpha_1+\alpha_2}{2} + 2\rho \in -\mathbb{N}$, which is equivalent to $\lambda_3 \in -\rho - \mathbb{N}$, i.e. π_{λ_3} is reducible and admits a finite dimensional subspace.

Where does $\widetilde{\mathcal{K}}_{\boldsymbol{\alpha}} \equiv 0$?

Theorem

$\widetilde{\mathcal{K}}_{\boldsymbol{\alpha}} \equiv 0$ if and only if either of the following two (non exclusive) possibilities is satisfied :

- *$\boldsymbol{\alpha}$ belong to two planes of poles of type I*
- *$\boldsymbol{\alpha}$ is a pole of type II*

$$\alpha_1 + \alpha_2 + \alpha_3 = -2(n-1) - 2k, \quad \textit{for some } k \in \mathbb{N}$$

and (up to permutation of $1, 2, 3$)

$$\alpha_1 + \alpha_2 = -2(n-1) - 2l, \quad \textit{for some } l \in \mathbb{N}$$

with $k \leq l$.

Theorem (bis)

Let $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3)$. *Then* $\widetilde{\mathcal{K}}^{\boldsymbol{\lambda}} \equiv 0$ *if and only if (at least) one of the following properties (up to permutation of the indices) is satisfied*

- $\lambda_3 = -\rho - p, \quad \lambda_1 - \lambda_2 = m,$
  $p \in \mathbb{N}, m \in \mathbb{Z}, \quad |m| \leq p, \quad p \equiv m \ (2).$

- $\lambda_3 = -\rho - p, \quad \lambda_1 + \lambda_2 = m,$
  $p \in \mathbb{N}, m \in \mathbb{Z}, \quad |m| \leq p, \quad p \equiv m \ (2).$

Elements of the proof

- An invariant trilinear functional $\mathcal{T}$ is K-invariant, hence it is enough to test it against K-invariant functions on $S \times S \times S$.
- K-invariant polynomial functions ($=$ restriction to $S \times S \times S$ of polynomials on $E \times E \times E$) are dense in K-invariant functions.

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- K-invariant polynomial functions ($=$ restriction to $S \times S \times S$ of polynomials on $E \times E \times E$) are dense in K-invariant functions.
- As a consequence of the *first fundamental theorem*, the algebra of K-invariant polynomial functions is generated by the restrictions to $S \times S \times S$ of

$$|x|^2, |y|^2, |z|^2, \quad < x, y >, \quad < y, z >, \quad < z, x > ,$$

or equivalently

$$1, \quad |x-y|^2, \quad |y-z|^2, \quad |z-x|^2 .$$

- For $a_1, a_2, a_3 \in \mathbb{N}$, let

$$p_{a_1,a_2,a_3}(x,y,z) = |x-y|^{2a_3}|y-z|^{2a_1}|z-x|^{2a_2} \ .$$

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- $$\mathcal{T} \equiv 0 \iff \mathcal{T}(p_{a_1,a_2,a_3}) = 0 \text{ for all } a_1, a_2, a_3 \in \mathbb{N} \ .$$

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- $$\mathcal{K}_{\boldsymbol{\alpha}}(p_{a_1,a_2,a_3}) = \mathcal{K}_{\alpha_1+2a_1,\alpha_2+2a_2,\alpha_3+2a_3}(1,1,1) \ ,$$

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-
$$\widetilde{\mathcal{K}}_{\boldsymbol{\alpha}}(p_{a_1,\,a_2,\,a_3}) = \quad \left(\frac{\pi}{2}\right)^{\frac{3}{2}(n-1)} 2^{\alpha_1+\alpha_2+\alpha_3}$$
$$\frac{\left(\frac{\alpha_1+\alpha_2+\alpha_3}{2}+2\rho\right)_{a_1+a_2+a_3} \circlearrowleft \left(\frac{\alpha_1}{2}+\rho\right)_{a_1}}{\circlearrowleft \Gamma\left(\frac{\alpha_1+\alpha_2}{2}+2\rho+a_1+a_2\right)} \ .$$
where $(x)_k = x(x+1)\dots(x+k-1)$ (Pochhammer's symbol).

To conclude, and for further study...

- For $\boldsymbol{\lambda}$ outside of a denumerable union of complex lines, there is a non trivial trilinear form $\widetilde{\mathcal{K}}^{\boldsymbol{\lambda}}$ which is invariant w.r.t. $(\pi_{\lambda_1}, \pi_{\lambda_2}, \pi_{\lambda_3})$.
- If the three representations $\pi_{\lambda_1}, \pi_{\lambda_2}, \pi_{\lambda_3}$ are irreducible, then $\widetilde{\mathcal{K}}^{\boldsymbol{\lambda}} \not\equiv 0$.
- For $\boldsymbol{\lambda}$ such that $\widetilde{\mathcal{K}}^{\boldsymbol{\lambda}} \equiv 0$, then any partial derivative $\frac{\partial}{\partial \lambda_j}\widetilde{\mathcal{K}}^{\boldsymbol{\lambda}}$ is still invariant w.r.t. $(\pi_{\lambda_1}, \pi_{\lambda_2}, \pi_{\lambda_3})$. This should yield (generically) two linearly independent invariant forms...