Covariant differential operators as singular intertwining operators

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- I. Knapp-Stein intertwining operators and covariant differential operators on the sphere

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- II. Conformally invariant trilinear forms on the sphere
- III. Singular invariant trilinear forms and covariant (bi-)differential operators

Based on

- J-L. Clerc & B. Ørsted, *Conformally invariant trilinear forms on the sphere*, Ann. Instit. Fourier, **61** (2011), 1807–1838

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- J-L. Clerc, T. Kobayashi, B. Ørsted & M. Pevzner, *Generalized Bernstein-Reznikov integrals*, Math. Annalen **349** (2011), 395–431

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- R. Beckmann & J-L. Clerc, *Singular invariant trilinear forms and covariant (bi-)differential operators under the conformal group*, J. Funct. Anal. **262** (2012), 4341–4376

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- J-L. Clerc, *Singular conformally invariant trilinear forms on the sphere*, in preparation

Some related references

- J. Bernstein & A. Reznikov, *Estimates of automorphic functions*, Mosc. Math. J. **4** (2004), 19–37

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- T. Kobayashi and B. Speh, *Symmetry breaking for representations of rank one orthogonal groups*, posted on arXiv (2013)
- P. Somberg, *Rankin-Cohen brackets for orthogonal Lie algebras and bilinear conformally invariant differential operators*, posted on arXiv (2013)

Knapp-Stein intertwining operators and covariant differential operators on the sphere

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The geometric context

- $S = S^{n-1}$ the unit sphere in $E = \mathbb{R}^n$
  $\mathbf{1} = (1, 0, \ldots, 0)$
  $S \simeq K/M$, where $K = SO(n)$, $M \simeq SO(n-1)$

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  $S \simeq K/M$, where $K = SO(n)$, $M \simeq SO(n-1)$
- $\mathcal{S} =$ set of isotropic lines in the Lorentz space $E^{1,n}$.

$$S \ni x \longmapsto \mathbb{R}(1, x) \in \mathcal{S}$$

  is a 1-1 correspondence, and so G acts on S.
- The Lorentz group $G = SO_0(1, n)$ acts on $\mathcal{S}$, hence on S.
  $S \simeq G/P$, where P the stabilizer of $\mathbf{1}$ in G is a *parabolic subgroup* of G.

- The action of G is *conformal*, i.e. for any tangent vector ξ to S at x
$$|Dg(x)\xi| = \kappa(g,x)|\xi| \ ,$$
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where $\kappa(g,x)$ is the *conformal factor* of g at x.
- Covariance property of the Euclidean distance on S :
$$|g(x) - g(y)| = \kappa(g,x)^{\frac{1}{2}} \, |x-y| \, \kappa(g,y)^{\frac{1}{2}} \ ,$$
for $x, y \in S$ and $g \in G$.

The principal series

- The group G acts naturally on the spaces of smooth densities on S, thus defining the (scalar) *principal series* of representations of G.
- Let dx be the Lebesgue measure on the sphere, use it to identify smoth densities with $\mathcal{C}^\infty$ functions. For $\lambda \in \mathbb{C}$, we obtain the following representation of G on $\mathcal{C}^\infty(S)$

$$\pi_\lambda(g)f(x) = \kappa(g^{-1}, x)^{\frac{n-1}{2}+\lambda} f(g^{-1}(x)) \ .$$

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- The duality relation between π_λ and $\pi_{-\lambda}$

$$\int_S \pi_\lambda(g)\varphi(x)\psi(x)dx = \int_S \varphi(x)\pi_{-\lambda}(g^{-1})\psi(x)dx \ .$$

The Knapp-Stein intertwining operators

- Knapp-Stein operator (formal)

$$J_\lambda f(x) = \int_S |x-y|^{-(n-1)+2\lambda} f(y)dy$$

- intertwining relation

$$J_\lambda \circ \pi_\lambda(g) = \pi_{-\lambda}(g) \circ J_\lambda, \qquad \text{for any } g \in G\ .$$

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- convergent for $\Re\lambda > 0$
- look for analytic continuation of J_λ for $\lambda \in \mathbb{C}$

Bernstein-Sato identity

For f a given real analytic function everywhere ≥ 0

$$D(x, \frac{\partial}{\partial x}, s)\, f^{s+1}(x) = b(s)\, f^s(x) \ ,$$

valid on $\{x, f(x) \neq 0\}$.

- $D(x, \frac{\partial}{\partial x}, s)$ is a differential operator with smooth coefficients, *polynomial* in the parameter $s \in \mathbb{C}$.
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- example : BS identity for $|\mathbf{1} - x|^2$ on S

$$\Big(\Delta + (\frac{s}{2}+1)(\frac{s}{2}+n-1)\Big)|\mathbf{1}-x|^{s+2} = (s+2)(s+n-1)\,|\mathbf{1}-x|^{s}$$

where Δ is the Laplacian on S.

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Hint : $|\mathbf{1} - x|^2 = 2(1 - x_1)$ and

$$\Delta\varphi(x_1) = (1 - x_1^2)\varphi''(x_1) - (n-1)\varphi'(x_1) \ .$$

Meromorphic continuation of $\int_S |x-y|^s f(y)dy$

- from the previous Bernstein-Sato identity

$$|x-y|^s = \frac{1}{(s+2)(s+n-1)}\left(\Delta_y + (\frac{s}{2}+1)(\frac{s}{2}+n-1)\right)|x-y|^{s+2}$$

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- this allows the meromorphic continuation, poles occurring for $s=-(n-1),-(n-1)-2,\ldots,-(n-1)-2k,\ldots$

Meromorphic continuation of J_λ

- Letting $s = -(n-1) + 2\lambda$, the operator-valued function $\lambda \to J_\lambda$ can be continued meromorphically to $\mathbb{C}$ with poles at $-k, k \in \mathbb{N}$.

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- for $\lambda \sim 0$, write $J_\lambda f(x) = \int_S |x-y|^{-(n-1)+2\lambda} f(y) dy$ as

$$\int_S |x-y|^{-(n-1)+2\lambda}\big(f(y)-f(x)\big)dy + f(x)\int_S |x-y|^{-(n-1)+2\lambda} dy \, .$$

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- $f(x) - f(y) = O(|x-y|)$, hence no problem with the first integral
- $$\int_S |x-y|^s dy = \int_S |\mathbf{1} - y|^s dy = (2\sqrt{\pi})^{n-1} 2^s \frac{\Gamma(\frac{s}{2}+\rho)}{\Gamma(\frac{s}{2}+2\rho)} \,,$$

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- hence

$$Res(J_\lambda f(x), \lambda = 0) = \frac{\pi^\rho}{\Gamma(\rho)} f(x) \ .$$

- for $\lambda \sim -1$, use BS identity, integrate by parts, take residues at -1 on both sides to get

$$Res(J_\lambda f(x), \lambda = -1) = \frac{\pi^\rho 4^{-1}}{\Gamma(\rho+1)} \Delta_1 f(x)$$

where $\Delta_1 = \Delta - \frac{1}{4}(n-1)(n-3)$ is the *conformal Laplacian* (Yamabe operator) on S.

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- iterating the argument

$$Res(J_\lambda, \lambda = -k) = \frac{\pi^\rho 4^{-k}}{\Gamma(\rho+k)} \Delta_k \ ,$$

where

$$\Delta_k = \prod_{j=1}^{k} \left(\Delta - (\rho+j-1)(\rho-j)\right) \ .$$

Covariant differential operators for densities on S

The intertwining property satisfied by J_λ between $\pi_\lambda, \pi_{-\lambda}$ extends to the residues and yields

$$\Delta_k \circ \pi_{-k}(g) = \pi_k(g) \circ \Delta_k \ ,$$

usually know as the *covariance property* for Δ_k.

It can be shown that there are no more covariant differential operators between spaces of densities. The proof uses results on homomorphisms of (generalized) Verma modules.

Sketch of a down-to-earth proof

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- Let D be a differential operator on $\mathbb{R}^{n-1}$ covariant w.r.t. (π_λ, π_μ).

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- The action of $M \simeq SO(n-1)$ is by rotation, hence D has to be of the form $D = \sum a_k \Delta^k$, where Δ is the Laplacian on $\mathbb{R}^{n-1}$.

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- A acts by dilation on $\mathbb{R}^{n-1}$, hence D has to be homogeneous. So $D = a_k \Delta^k$ for some $k \in \mathbb{N}$, and $\mu = \lambda + 2k$.

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- A acts by dilation on $\mathbb{R}^{n-1}$, hence D has to be homogeneous. So $D = a_k \Delta^k$ for some $k \in \mathbb{N}$, and $\mu = \lambda + 2k$.
- If D is non trivial, then the Casimir operator acts on both spaces by the same scalar. This forces $\lambda = -\mu = -k$. Q.E.D.

End of part I

Thank you for your attention !