Singular trilinear forms and covariant (bi)differential operators

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Srni, January 18–25, 2014

Poles and Residues

Recall that for $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{C}^3$, $\mathcal{K}_{\boldsymbol{\alpha}}$ is the trilinear form (obtained by meromorphic continuation of)

$$\mathcal{K}_{\boldsymbol{\alpha}}(f_1, f_2, f_3) = \int_{S \times S \times S} |x-y|^{\alpha_3} |y-z|^{\alpha_1} |z-x|^{\alpha_2} f_1(x) f_2(y) f_3(z) \, dx \, dy \, dz$$

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The meromorphic continuation has simple poles along four families of planes in $\mathbb{C}^3$, given by

- $\alpha_j = -(n-1) - 2k$, for $k \in \mathbb{N}$, $j = 1, 2, 3$
- $\alpha_1 + \alpha_2 + \alpha_3 = -2(n-1) - 2l$, for $l \in \mathbb{N}$

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- A pole is said to be *generic* if it belongs to only *one* plane of poles.
- A (generic) pole is said to be *of type I* if it belongs to a plane from the three first families, and *of type II* if its belongs to a plane from the fourth family.

Residue at a pole of type I

Let $\boldsymbol{\alpha}^0 = (\alpha_1^0, \alpha_2^0, -(n-1) - 2k)$ be a generic pole of type I. The *residue* at α^0 is

$$Res(\mathcal{K}_{\boldsymbol{\alpha}}, \boldsymbol{\alpha}^0)(f_1, f_2, f_3) = \lim_{\alpha_3 \to -(n-1)-2k} (\frac{\alpha_3}{2} + \rho + k)\mathcal{K}_{\boldsymbol{\alpha}}(f_1, f_2, f_3)$$

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The singularity of $\mathcal{K}_{\boldsymbol{\alpha}}$ at $\boldsymbol{\alpha}^0$ comes from the factor $|x-y|^{\alpha_3}$ in the kernel of $\mathcal{K}_{\boldsymbol{\alpha}}$, which becomes singular on $\{x = y\}$, a submanifold of dimension $2(n-1)$. The residue is a distribution supported in this set.

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To express it at near a point (x^0, y^0, z^0) where $x^0 = y^0$, we need to chose a transverse submanifold . Our choice is

$$\mathcal{N}_{x^0,y^0,z^0} = \{(x, y^0, z^0), x \in S\} \ .$$

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So the residue will be "of the form"

$$\int_{S \times S} f_3(z) f_2(y) \big(D(y, z, \frac{\partial}{\partial x}) f_1\big)(y) \, dy \, dz \ .$$

Recall that Δ_k is the differential operator on S defined by

$$\Delta_k = \prod_{j=1}^{k} \big(\Delta - (\rho + j - 1)(\rho - j)\big)$$

where Δ is the Laplacian on S. It satisfies the covariance relation

$$\Delta_k \circ \pi_{-k}(g) = \pi_k(g) \circ \Delta_k \ .$$

For $\alpha_1, \alpha_2 \in \mathbb{C}^2$ set

$$\mathcal{T}^{3,k}_{\alpha_1,\alpha_2}(f_1, f_2, f_3) = \mathcal{T}^k(f_1, f_2, f_3) =$$

$$\int_{S\times S} f_3(z) f_2(y)\, \Delta_k[f_1(.)|z - .|^{\alpha_2}](y)\, |z - y|^{\alpha_1}\, dy\, dz\ .$$

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Theorem

The trilinear form $\mathcal{T}^k = \mathcal{T}^{3,k}_{\alpha_1,\alpha_2}$ originally defined as a convergent integral for $\Re\alpha_1$ and $\Re\alpha_2$ large enough, can be extended meromorphically to $\mathbb{C}^2$, with simple poles contained in the family of lines

$$\alpha_1 + \alpha_2 = -(n-1) + 2k - 2l, l \in \mathbb{N} \ .$$

The trilinear form $\mathcal{T}_k$ is invariant w.r.t. $(\pi_{\lambda_1}, \pi_{\lambda_2}, \pi_{\lambda_3})$, where $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3)$ is the spectral parameter associated to $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, -(n-1) - 2k)$.

Theorem

Let $\boldsymbol{\alpha} \in \mathbb{C}^3$ *such that* $\alpha_3 = -(n-1) - k$ *for some* $k \in \mathbb{N}$*, and* $\alpha_1 + \alpha_2 \notin -(n-1) + 2k - 2\mathbb{N}$*. Then the residue of* $\mathcal{K}_{\boldsymbol{\alpha}}$ *is equal to*

$$Res\big(\mathcal{K}_{\boldsymbol{\alpha}}, \alpha_3 = -(n-1) - 2k\big) \;=\; \frac{\pi^{\rho}\, 4^{-k}}{\Gamma(\rho + k)k!}\, \mathcal{T}^{3,k}_{\alpha_1,\alpha_2} \;.$$

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Remark 1. The condition $\alpha_1 + \alpha_2 = 2k - 2l$ is equivalent to $\alpha_1 + \alpha_2 + (-(n-1) - 2k) = -2(n-1) - 2l$. Otherwise said, poles of $\mathcal{T}^k$ correspond to non generic poles of $\mathcal{K}_{\boldsymbol{\alpha}}$ lying in the intersection of a plane of poles of type I and a plane of poles of type II.

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Remark 2. There is no pole of $\mathcal{T}^k$ corresponding to α in the intersection of two planes of poles of type I.

Residue at a pole of type II

- At a generic pole of type II, the residue of $\mathcal{K}_{\boldsymbol{\alpha}}$, viewed as a distribution on $S \times S \times S$ is supported on the diagonal $\mathcal{O}_4 = \{(x, x, x), x \in S\}$.

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- A *bi-differential operator* D on S is a continuous map $D : \mathcal{C}^\infty(S \times S) \longrightarrow \mathcal{C}^\infty(S)$ which can be written locally as

$$D(f)(z) = \sum_{I,J} a_{I,J}(z) \left(\partial_x^I \partial_y^J f\right)(z, z)$$

where the $a_{I,J}$ (uniquely determined) are smooth.

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  where the $a_{I,J}$ (uniquely determined) are smooth.
- Let $\lambda_1, \lambda_2, \mu \in \mathbb{C}$. A bi-differential operator D on S is said to be *covariant* w.r.t. $(\pi_{\lambda_1} \otimes \pi_{\lambda_2}, \pi_\mu)$ if

$$D \circ (\pi_{\lambda_1} \otimes \pi_{\lambda_2})(g) = \pi_\mu(g) \circ D$$

Proposition

Let $\mathcal{T}$ be a trilinear form on $\mathcal{C}^\infty(S)$, invariant w.r.t. $(\pi_{\lambda_1}, \pi_{\lambda_2}, \pi_{\lambda_3})$. Assume that, as a distribution on $S \times S \times S$, $\mathcal{T}$ is supported in the diagonal $\mathcal{O}_4$. Then there exists a bi-differential operator D, covariant w.r.t. $(\pi_{\lambda_1} \otimes \pi_{\lambda_2}, \pi_{-\lambda_3})$ such that

$$\mathcal{T}(f \otimes g) = \int_S Df(x)g(x)dx$$

for $f \in \mathcal{C}^\infty(S \times S), g \in \mathcal{C}^\infty(S)$.

Remark. The converse statement is clear.

Residue at a "first pole" of type II

We say that $\boldsymbol{\alpha}$ is a first pole of type II if

$$\alpha_1 + \alpha_2 + \alpha_3 = -2(n-1)$$

Proposition

Let $\boldsymbol{\alpha}$ be a generic first pole of type II. Then

$$Res(\mathcal{K}_{\boldsymbol{\alpha}} f, \boldsymbol{\alpha}) = c(\boldsymbol{\alpha}) \int_S f(x,x,x)dx$$

where

$$c(\boldsymbol{\alpha}) = c_n \frac{\Gamma(\frac{\alpha_1}{2}+\rho)\Gamma(\frac{\alpha_2}{2}+\rho)\Gamma(\frac{\alpha_3}{2}+\rho)}{\Gamma(-\frac{\alpha_1}{2})\Gamma(-\frac{\alpha_2}{2})\Gamma(-\frac{\alpha_3}{2})}$$

A Bernstein-Sato identity

- Use a stereographic projection to move to the noncompact realization of the principal series. The kernel of the basic invariant trilinear form becomes (for $x, y, z \in \mathbb{R}^{n-1}$)

$$I_{\boldsymbol{\alpha}}(x, y, z) = |x - y|^{\alpha_3}|y - z|^{\alpha_1}|z - x|^{\alpha_2}$$

- For $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$, let $\boldsymbol{\alpha} + 2_1 = (\alpha_1 + 2, \alpha_2, \alpha_3)$.

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- For $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$, let $\boldsymbol{\alpha} + 2_1 = (\alpha_1 + 2, \alpha_2, \alpha_3)$.
- Bernstein-Sato identity

$$B(y, z, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \boldsymbol{\alpha})\, I_{\boldsymbol{\alpha}+2_1} = b(\boldsymbol{\alpha})\, I_{\boldsymbol{\alpha}}$$

where

$$B(y,z,\frac{\partial}{\partial z},\frac{\partial}{\partial y},\boldsymbol{\alpha}) \quad = \quad |y-z|^2\Delta_y\Delta_z$$

$$+2(\alpha_3+\alpha_1+2\rho)\sum_{j=1}^{n-1}(z_j-y_j)\frac{\partial}{\partial y_j}\Delta_z + 2(\alpha_2+\alpha_1+2\rho)\sum_{j=1}^{n-1}(y_j-z_j)\frac{\partial}{\partial z_j}\Delta_y$$

$$+(\alpha_3+\alpha_1+2\rho)(\alpha_3+\alpha_1+2)\Delta_z + (\alpha_2+\alpha_1+2\rho)(\alpha_2+\alpha_1+2)\Delta_y$$

$$-2(\alpha_3+\alpha_1+2\rho)(\alpha_2+\alpha_1+2\rho)\sum_{j=1}^{n-1}\frac{\partial^2}{\partial y_j\partial z_j}\ ,$$

$$b(\boldsymbol{\alpha}) = (\alpha_1+2\rho)(\alpha_1+2)(\alpha_1+\alpha_2+\alpha_3+4\rho)(\alpha_1+\alpha_2+\alpha_3+2\rho+2)\,.$$

Sketch of the proof

- For $\lambda \in \mathbb{C}$, denote by $\mathcal{F}_\lambda$ the image of $\mathcal{C}^\infty(S)$ (viewed as $(\frac{1}{2} + \frac{\lambda}{n-1})$-densities) under the stereographic projection.

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- By the conformal covariance property of the Euclidean distance

$$M : \mathcal{F}_\lambda \otimes \mathcal{F}_\mu \longrightarrow \mathcal{F}_{\lambda-1} \otimes \mathcal{F}_{\mu-1}$$

is an intertwining operator for $(\pi_\lambda \otimes \pi_\mu, \pi_{\lambda-1} \otimes \pi_{\mu-1})$

- Define $N_{\lambda,\mu}$ by the following diagram :

$$\begin{tikzcd}
\mathcal{F}_{-\lambda} \otimes \mathcal{F}_{-\mu} \arrow[r, "M"] & \mathcal{F}_{-\lambda-1} \otimes \mathcal{F}_{-\mu-1} \arrow[d, "J_{-\lambda-1} \otimes J_{-\mu-1}"] \\
\mathcal{F}_{\lambda} \otimes \mathcal{F}_{\mu} \arrow[u, "J_{\lambda} \otimes J_{\mu}"] \arrow[r, "N_{\lambda,\mu}"] & \mathcal{F}_{\lambda+1} \otimes \mathcal{F}_{\mu+1}
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- Fact : $N_{\lambda,\mu}$ is a differential operator on $\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}$!
- The proof is by Euclidean Fourier transform. The Knapp-Stein operators are convolution operators, hence correspond to multiplications on the Fourier side and M corresponds to the constant coefficient differential operator

$$-\Delta_x - \Delta_y + 2\sum_{j=1}^{n-1} \frac{\partial^2}{\partial x_j \partial y_j}$$

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- Let $\boldsymbol{\alpha}$ be the geometric parameter associated to $\boldsymbol{\lambda}$. Then $(\lambda_1, \lambda_2 + 1, \lambda_3 + 1)$ is the spectral parameter associated to $\boldsymbol{\alpha} + 2_1$.
- Viewing trilinear forms as distributions, this yields

$$N^t_{\lambda_2,\lambda_3}\, k_{\boldsymbol{\alpha}+2_1} = b(\boldsymbol{\alpha})\, k_{\boldsymbol{\alpha}}$$

  which is essentially the desired Bernstein-Sato identity.

Application to covariant bi-differential operators

- Let $\boldsymbol{\alpha}$ be of type II, say $\alpha_1 + \alpha_2 + \alpha_3 = -2(n-1) - 2k$. Then $(\alpha_1 + 2k, \alpha_2, \alpha_3)$ belongs to the first plane of poles of type II. The residue at $\boldsymbol{\alpha}$ is obtained by k repeated integration by parts using the Bernstein-Sato identity.

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- This yields a bi-differential operator covariant w.r.t. $(\pi_{\lambda_1} \otimes \pi_{\lambda_2}, \pi_{\lambda_1+\lambda_2+\rho+2k})$.

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- Explicitly, let

$$
\begin{aligned}
F_{\lambda,\mu} = \; & |y-z|^2 \Delta_y \Delta_z \\
& + 4(\mu+1) \sum_{j=1}^{n-1} (z_j - y_j) \frac{\partial}{\partial z_j} \Delta_y + 4(\lambda+1) \sum_{j=1}^{n-1} (y_j - z_j) \frac{\partial}{\partial y_j} \Delta_z \\
& + 4(\mu+1)(\mu+\rho)\Delta_y + 4(\lambda+1)(\lambda+\rho)\Delta_z \\
& - 8(\lambda+1)(\mu+1) \sum_{j=1}^{n-1} \frac{\partial}{\partial y_j} \frac{\partial}{\partial z_j} \; .
\end{aligned}
$$

- Denote by restr (=restriction to the diagonal) the operator from $\mathcal{C}^\infty(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1})$ to $\mathcal{C}^\infty(\mathbb{R}^{n-1})$ given by $(\operatorname{restr} f)(x) = f(x, x)$.

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- $$F^{(k)}_{\lambda,\mu} = \operatorname{restr} \circ F_{\lambda+k-1,\,\mu+k-1} \circ \cdots \circ F_{\lambda,\,\mu}$$
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- Example : for $k = 1$, with $R = \sum_{j=1}^{n-1} \frac{\partial^2}{\partial y_j \partial z_j}$

  $$F^{(1)}_{\lambda,\mu} = 4(\mu+1)(\mu+\rho)\Delta_y - 8(\lambda+1)(\mu+1)R + 4(\lambda+1)(\lambda+\rho)\Delta_z$$

  is covariant w.r.t. $(\pi_\lambda \otimes \pi_\mu, \pi_{\lambda+\mu+\rho+2})$.

Thank you for your attention !