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"Pseudo-Riemannian + Conformal
Dynamics", Srní, January 2014

Outline:

I. Totally geodesic lightlike foliations in Lorentzian geometry

- A. Lightlike boundaries in Minkowski space
- B. TGL foliations of the non-past lightcone
- C. The Heisenberg group as a past lightcone space

II. Applications of TGL foliations;
Introduction to conformal geometry

- A. D'Alambert theorem
- B. Many TGL foliations \Rightarrow contact-sectional
geometry (warped products)
- C. Classification of compact Lorentzian manifolds
actions of finite Lie groups
- D. Comparison with conformal actions

III. Local dynamical foliations for pseudo-Riemannian
conformal actions

- A. Smooth flows



- B. Localized normal flows (geometric, conformal
pseudo-Riemannian metrics)

I. Totally geodesic lightlike foliations in Lorentzian geometry

$$\text{Min}^n = \mathbb{R}^n \text{ with } Q^{1,n-1}(\vec{x}) = -x_1^2 + x_2^2 + \dots + x_n^2$$

$$\iff B_{1,n-1} = \begin{pmatrix} -1 & \\ & \text{Id}_{n-1} \end{pmatrix}$$

linear isometries $O(1,n-1) = \{M^* B_{1,n-1} M = B_{1,n-1}\}$

for $g_n \rightarrow \infty$ in $O(1,n-1)$, take KAK decomposition

$$K \cong O(1) \times O(n-1) \quad \text{and} \quad A \cong \mathbb{R}_{>0}^*$$

up to subsequence, $g_n' = j_n a_n K_n$
 with $j_n \rightarrow j$, $K_n \rightarrow k$ in K , $a_n \rightarrow \infty$

get

$$H^+ = \{v \in \mathbb{R}^{1,n-1} : \|g_n' v\| \text{ bounded}\} \quad \text{stable subspace}$$

$$U^-$$

$$L^+ = \{v \in \mathbb{R}^{1,n-1} : \|g_n' v\| \rightarrow 0\} \quad \text{strongly stable subspace} \quad \left. \begin{array}{l} \text{for} \\ dg_n' \end{array} \right\}$$

lightlike line \subseteq lightlike hyperplane

similarly

$$H^- = \{v \in \mathbb{R}^{1,n-1} : \|g_n'^{-1} v\| \text{ bounded}\} \quad \text{unstable subspace}$$

$$U^+$$

$$L^- = \{v \in \mathbb{R}^{1,n-1} : \|g_n'^{-1} v\| \rightarrow 0\} \quad \text{strongly unstable subspace}$$

(M^n, σ) Lorentzian manifold

$$\text{Isom } M = \{g \in \text{Diff } M : g^* \sigma = \sigma\}$$

$$x \in M, \quad \text{Stab}(x) = \{g \cdot x = x\} \subseteq \text{Isom } M$$

given $g_n \rightarrow \infty$ in $\text{Stab}(x)$

have $(g_n)_* x \rightarrow \infty$ in $O(T_x M) \cong O(1, n-1)$

passing to subsequence, get

$$L^+ \subset H^+ \subset T_x M \quad \begin{matrix} \text{strongly stable} \\ \text{subspaces} \end{matrix} \quad \begin{matrix} \text{stable} \\ \text{subspaces} \end{matrix}$$

$$\text{and } L^- \subset H^- \subset T_x M \quad \text{unstable}$$

There's more:

Proposition (Gromov): There is a totally geodesic lightlike hypersurface through x tangent to H^+ containing a lightlike geodesic line tangent to L^+ ; similarly for $L^- \subset H^-$.

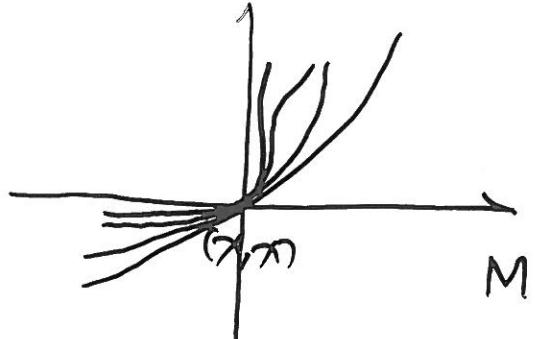
proof of Gromov proposition

Take $(M \times M, \sigma \oplus -\sigma)$. Connection is $\nabla \oplus \nabla$.

$\Delta \subseteq M \times M$ is totally isotropic and totally geodesic through (x, x)

$\Gamma(g_n) = \Gamma_n \subseteq M \times M$ totally isotropic and totally geodesic through (x, x)

$\Gamma_n \rightarrow \Gamma_\infty$ in a neighborhood of (x, x) M
 totally isotropic
 and totally geodesic

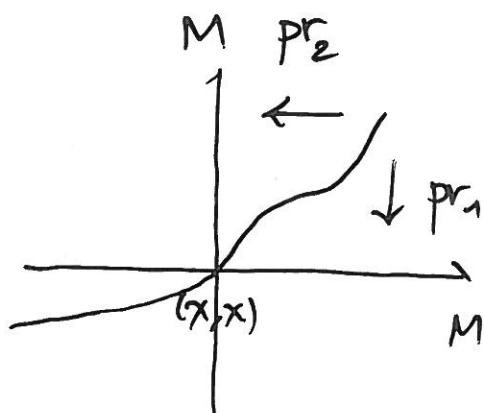


$$\text{pr}_1(\Gamma_\infty) = \mathcal{H}^+$$

$$\text{pr}_2(\Gamma_\infty) = \mathcal{H}^-$$

$$\bar{\text{pr}}_2^{-1}(\{x\}) = \mathcal{L}^+ \times \{x\}$$

$$\bar{\text{pr}}_1^{-1}(\{x\}) = \{x\} \times \mathcal{L}^-$$



(M^n, σ) compact Lorentzian, $G = \text{Isom } M$
auxiliary Riemannian metric $l \cdot l_x$

For $\{g_n\} \subseteq G$, $x \in M$

$$AS_x^+(\{g_n\}) = \left\{ v = \lim_{n \rightarrow \infty} v_n : dv_n \subseteq TM \text{ and } |g_n \cdot v_n| \text{ is bounded} \right\}$$

$$SAS_x^+(\{g_n\}) = \left\{ v = \lim_{n \rightarrow \infty} v_n : dv_n \subseteq TM \text{ and } |g_n \cdot v_n| \rightarrow 0 \right\}$$

(well-defined independently of $l \cdot l_x$)

$$AS_x^-(\{g_n\}) = AS_x^+(\{g_n^{-1}\})$$

$$SAS_x^-(\{g_n\}) = SAS_x^+(\{g_n^{-1}\})$$

"approximately asymptotically (un)stable set for $\{g_n\}$ at x "

"strongly approximately asymptotically (un)stable set for $\{g_n\}$ at x "

Example : $\hat{M} = \text{AdS}^3 = \text{PSL}(2, \mathbb{R})$ with the Cartan-Killing metric

$\Gamma < \text{PSL}(2, \mathbb{R})$ a lattice, cocompact

$$M = \hat{M}/\Gamma, \quad \text{PSL}(2, \mathbb{R}) \leq \text{Isom } M$$

- $g_n = \begin{pmatrix} \lambda^n & 0 \\ 0 & \bar{\lambda}^{-n} \end{pmatrix}, \quad \lambda > 1$

$$\text{SAS}_{[h]}^+(\{g_n\}) \subset \text{AS}_{[h]}^+(\{g_n\})$$

$$[U_+ \cdot h] \subset [B_+ \cdot h]$$

$$\text{SAS}_{[h]}^-(\{g_n\}) \subset \text{AS}_{[h]}^-(\{g_n\})$$

$$[U_- \cdot h] \subset [B_- \cdot h]$$

for $v \in \text{SAS}^+(\{g_n\}), \quad |g_{n+} v| \leq \bar{\lambda}^{-cn} |v|, \quad c > 0$

$v \in \text{AS}^+(\{g_n\}), \quad |g_{n+} v| \text{ is bounded}$

- $g_n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$

$$\text{SAS}_{[h]}^+(\{g_n\}) = \text{SAS}_{[h]}^-(\{g_n\}) \subset \text{AS}_{[h]}^+(\{g_n\}) = \text{AS}_{[h]}^-(\{g_n\})$$

$$[U_+ \cdot h] \subset [B_+ \cdot h]$$

for $v \in \text{SAS}^+(\{g_n\}), \quad |g_{n+} v| \text{ is bounded}$

$v \in \text{AS}^+(\{g_n\}), \quad |g_{n+} v| \rightarrow \infty$

Proof outline

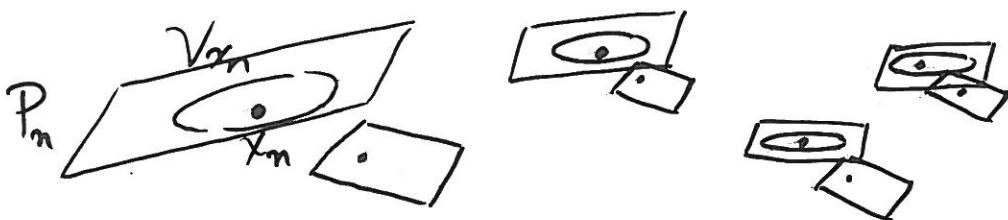
- 1) M compact and $g_n \rightarrow \infty$ in $\text{Isom } M \Rightarrow \forall x \in M, |g_{n+x}|$ unbounded
- 2) In a measurable, banded (piecewise continuous) trivialization of TM , get

$$g_{n+x} \leftrightarrow C_n(x) \rightarrow \infty \text{ in } O(n, n-1)$$

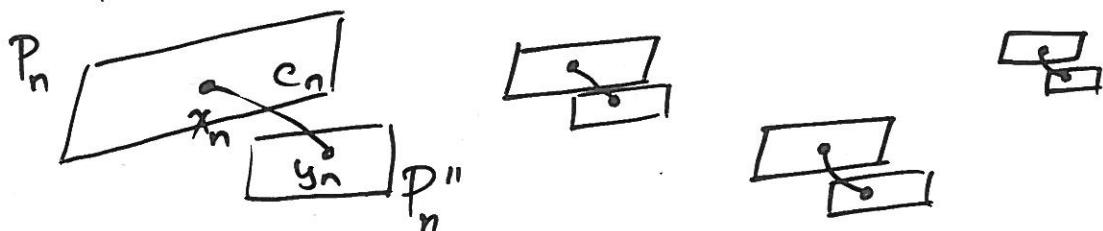
A subsequence $C_{n_k}(x)$ has stable & unstable hyperplanes

$$L^\pm(x) \subset H^\pm(x) \Leftrightarrow SAS_x^\pm \subset AS_x^\pm$$

- 3) If $P_n \subset T_{x_n} M$ are approximately stable subspaces (with uniform bound on $|g_{n+x_n}|_{P_n}$) then \exists neighborhoods $0 \in V_{x_n} \subset P_n$ s.t. $\{g_n\}$ is app. stable on $P_n = \exp_{x_n}(\mathbb{V}_{x_n})$



- 4) For curves x_n to y_n in P_n , the parallel transports $P_n'' = J_{e_n}(P_n)$ are an approximately stable sequence of subspaces.



Proof outline cont'd

- 5) If $P \subseteq AS_x^+(\{g_n\})$ is a maximal subspace, then $\exp_x(P)$ is totally geodesic.
- 6) If $P = \lim P_n \subseteq AS_x^+(\{g_n\})$ is a hyperplane, then it equals $AS_x^+(\{g_n\})$



- 7) Take $M' \subset M$ countable and dense. Diagonal procedure gives $\{g_{n_k}\}$ and hyperplanes, lines
 $L_x^+ \subset H_x^+ \subset T_x M$ strongly app. stable/
 $\forall x \in M'$ app. stable
- * a codimension 1 geodesic lamination on compact M is Lipschitz
- 8) The plaques $fl_x^+ = \exp_x(H_x^+)$, $x \in M'$ fit together to geodesic foliation AS^+ (and AS^-).

Theorem (Zeghib 1999) : For any $g_n \rightarrow \infty$ in G ,
there is a subsequence $\{g_{n_k}\}$ for which

- $AS^+(\{g_{n_k}\})$ is an integrable $(n-1)$ -dimensional distribution; same for $AS^-(\{g_{n_k}\})$
- $SAS^+(\{g_{n_k}\}) \subseteq AS^+(\{g_{n_k}\})$ is a 1-dimensional distribution; similarly for $SAS^-(\{g_{n_k}\})$
- The resulting foliations
 $SAS^+(\{g_{n_k}\}) \subseteq Ad^+(\{g_{n_k}\})$
and $SAd^-(\{g_{n_k}\}) \subseteq Ad^+(\{g_{n_k}\})$
are lightlike and totally geodesic.

If $Lg_n = \{g^n\}$, then $SAd^+(\{g_n\}) \subseteq Ad^+(\{g_n\})$ and
 $SAd^-(\{g_n\}) \subseteq Ad^-(\{g_n\})$ are g -invariant.

II

Haefliger: M compact, simply connected and $C^\omega \Rightarrow M$ admits no codimension 1 C^ω foliation.

For M C^ω Lorentzian, can show Zeghib's foliations are also C^ω .

get Thm (D'Ambra '88): Let M be compact, simply connected, C^ω Lorentzian. Then $\text{Isom } M$ is compact.

Prop (Zeghib) M^n Lorentzian, $n \geq 3$. Suppose that $\forall x \in M, \exists$ TGL hypersurfaces H_1, \dots, H_n through x such that $\{T_x H_i^\perp : i=1, \dots, n\}$ span $T_x M$. Then M has constant sectional curvature.

Proof : Choose $0 \neq u_i \in H_i^\perp = T_x M_i^\perp$.

For $v \in T_x M$, let $A_v : w \mapsto R(v, w)v$

- For $v \in H_i$, $A_v(u_i) \in \mathbb{R}u_i$

Let $e_i \in \bigcap_{j \neq i} H_j$ s.t. $\langle e_i, e_i \rangle = 1$.

$$\text{so. } A_{e_i}(u_j) = \lambda_{ij} u_j \quad \forall j \neq i$$

- $\lambda_{ij} \langle u_j, u_k \rangle = \langle R(e_i, u_j)e_i, u_k \rangle = \langle R(e_i, u_k)e_i, u_j \rangle = \lambda_{ik} \langle u_j, u_k \rangle$
and $\langle u_j, u_k \rangle \neq 0 \Rightarrow \lambda_{ij} = \lambda_i \quad \forall j \neq i$

- For any plane P containing e_i ,
 $P = \text{span}\{e_i, v\}$ with $v \in \text{span}\{u_j : j \neq i\}$

$$\text{Sec}(P) = \frac{\langle A_{e_i}(v), v \rangle}{\langle e_i, e_i \rangle \langle v, v \rangle - \langle e_i, v \rangle^2} = \frac{\lambda_i \langle v, v \rangle}{\langle v, v \rangle}$$

$\Rightarrow \lambda_i = \lambda = \text{Sec}(P)$ for any
P containing e_i , any $i = 1, \dots, n$

- $\text{Sec}(P) = \lambda_x \quad \forall \text{planes } P \subseteq T_x M$

Schur : If Sec is constant $\lambda(x)$ on all non-degenerate planes $P \subseteq T_x M$, then $\text{Sec} \equiv \lambda$ on all M .



Cor (Zeghib '04): If M^n , $n \geq 3$ is an irreducible Lorentzian symmetric space, then M has constant sectional curvature.

More generally,

Thm: For M Lorentzian, $x \in M$, let E_x be the span of all $v \in T_x M$ such that $v^\perp = T_x H$ for a TGL hypersurface H . For any $U \subseteq M$ open on which $\{E_x : x \in U\}$ is a smooth distribution of $\dim \geq 3$,

$$U \cong L^* f N$$

for L Riemannian

$f : L \rightarrow \mathbb{R}_{>0}^*$ smooth

N Lorentzian constant curvature

$(Mink^k, dS^k, AdS^k)$

M connected Lorentzian, $\dim M \geq 3$

$G = \text{Isom } M$ semisimple with no local $SL_2(\mathbb{R})$ factors
 $|Z(G)| < \infty$

Thm (Arouche + Deffaf + Zeghib '06): Suppose that M has a Lorentzian G -orbit with noncompact stabilizer, say θ . Then there is a G -invariant neighborhood

$$(*) \quad \theta \subseteq U \cong L \times_f N, \quad N \text{ Lorentzian, const. curv.}$$

(**) and $G \cong G_1 \times_{loc} G_2$ (dS^k or AdS^k)

$$\text{where } G_1 \cong \text{Isom}^\circ(N) \cong O(1, k) \quad k \geq 3 \\ \text{or } \cong O(2, k-1) \quad k \geq 4$$

Thm (Deffaf + Me. + Zeghib '08): If M contains a G -fixed point or a degenerate G -orbit with noncompact stabilizer, then any neighborhood of this orbit contains a Lorentzian G -orbit with noncompact stabilizer.

Summary Thm (DMZ '08): Assume G acts nonproperly on M . Then

$$M = U \amalg \partial U \amalg V$$

where $U \neq \emptyset$ is as in (*)

and G satisfies (**)

• If $G_1 \cong O(2, k-1)$, then $V = \emptyset$

• If $G_1 \cong O(1, k)$, then $V \cong Q \times_{f_h} H^k$
and ∂U consists of fixed points and degenerate orbits.

Role of TGL hypersurface in DMZ theorem:

- N. Kowalsky: nonproper G -action \rightsquigarrow noncompact stabilizer G_x

and \mathfrak{g}_x contains $\bigoplus_{\alpha \in S} \mathfrak{g}_\alpha = \boxed{n_x}$ $\dim \geq 2$
 $(S \subseteq \Delta)$

Assume $\Theta = G \cdot x$ degenerate, with such $n_x \subseteq \mathfrak{g}_x$

- There is a TGL hypersurface H through x with $T_x H^\perp$ isotropic subspace of $T_x \Theta := \mathbb{R} r_0$ and H is foliated by null geodesic curves \mathcal{L}
- G_x preserves $\mathbb{R} r_0 = T_x H^\perp \Rightarrow G_x$ preserves H
- n_x trivial on H/\mathcal{L} . $\dim n_x \geq 2 \Rightarrow$ generic $y \in H$ have $n_x \cap \mathfrak{g}_y \neq 0 \Rightarrow$ noncompact stabilizer
- $\Theta = G \cdot x$ cannot be contained in H (no sl_2 -factors!)
- $G \cdot y$ degenerate or Lorentzian.
 degenerate on nbhd of $y \Rightarrow G \cdot y \in H \#$

Example : $\text{Conf}(\text{Ein}^{1,n-1}) \cong \text{PO}(2,n)$

$$\text{for } Q^{2,n}(\vec{x}) = 2x_0x_{n+1} + 2x_1x_n + \sum_{i=2}^{n-1} x_i^2$$

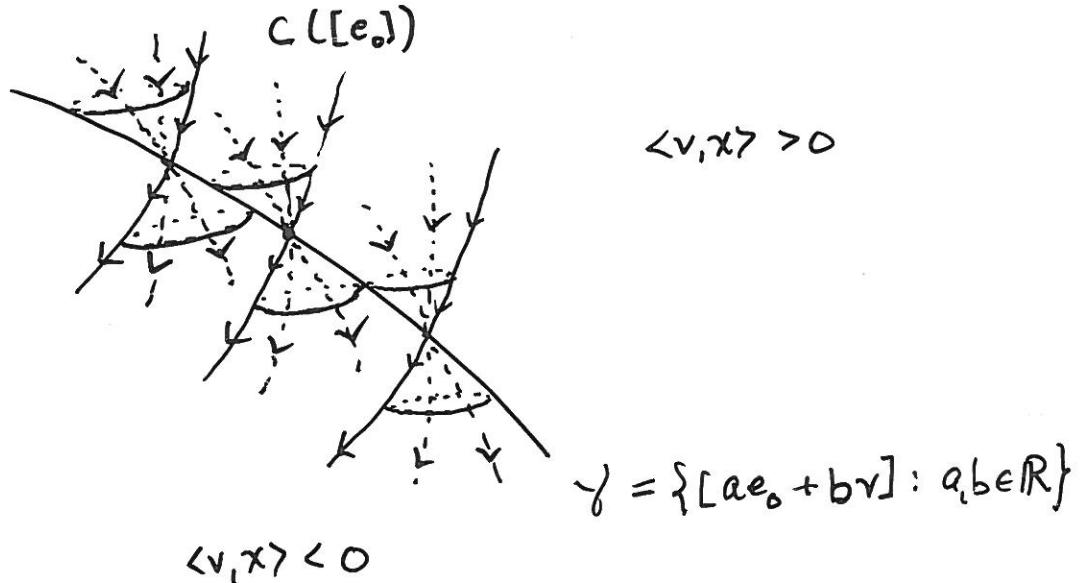
$$g_t = \begin{pmatrix} 1 & tv^* & 0 \\ & \ddots & -tv \\ & & 1 \end{pmatrix} \in \text{O}(2,n) \quad v \in \mathbb{R}^{1,n-1}$$

$$\langle v, v \rangle = 0$$

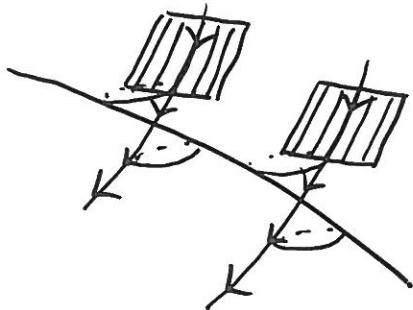
$$t \in \mathbb{R}$$

In affine chart $x_0 = 1$, g_t acts by

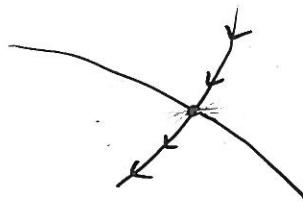
$$\left[1 : x : -\frac{\langle x, x \rangle}{2} \right] \mapsto \left[1 : \frac{x + \frac{t}{2}\langle x, x \rangle v}{1 + t\langle v, x \rangle} : \frac{-\langle x, x \rangle}{2(1 + t\langle v, x \rangle)} \right]$$



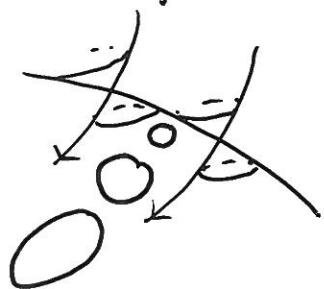
- SAS foliation singular along γ



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- Stability not uniform on neighborhood of γ



Equicontinuity: $\{f_k : U \rightarrow M\}$ conformal immersions is equicontinuous at $x \in U$ if $\exists y \in M$ s.t.
 $\forall x_k \rightarrow x, f_k(x_k) \rightarrow y$

Remark: Conformal transformations are not "linearizable" in general.

M^n semi-Riemannian, $n \geq 3$

Thm (C. Frances '12): Suppose $\{f_k\} \subseteq \text{Conf } M$ is equicontinuous at x . After passing to a subsequence, there is a neighborhood U of x and a set of asymptotically distinct sequences

$$\mu_1(k) \leq \dots \leq \mu_s(k) \leq 1 \quad \left(\begin{array}{l} \lim_{k \rightarrow \infty} \mu_j(k) = 0 \quad j=1, \dots, s-1 \\ \lim_{k \rightarrow \infty} \mu_s(k) \in [0,1] \end{array} \right)$$

for which:

① The sets $L_j(y)$, $y \in U$, $j=1, \dots, s-1$, defined by

- $v = \lim v_k$, $\exists v_k \in T_y U$, $\|f_{k+y}(v_k)\| = \Theta(\mu_j(k))$
AND

$$\forall v_k \rightarrow v \text{ in } T_y U, \|f_{k+y}(v_k)\| = O(\mu_j(k))$$

form integrable distributions $L_1 \subset \dots \subset L_{s-1} \subset TU$

Moreover, $v \in T_y U \setminus L_{s-1}(y) \Leftrightarrow \forall v_k \rightarrow v \text{ in } T_y U, \|f_{k+y}(v_k)\| = \Theta(\mu_s(k))$

② The resulting foliations $L_1 \subset \dots \subset L_{s-1} \subset U$ are characterized by: $\forall y \in U$

- $z \in L_j(y) \setminus L_{j-1}(y)$, $j < s$, iff:

$$\exists z_k \rightarrow z \text{ s.t. } d(f_k(z_k), f_k(y)) = \Theta(\mu_j(k)) \text{ AND } d(f_k(z_k), f_k(y)) = O(\mu_j(k))$$

- $z \in U \setminus L_{s-1}(y)$ iff $\forall z_k \rightarrow z \text{ in } U$

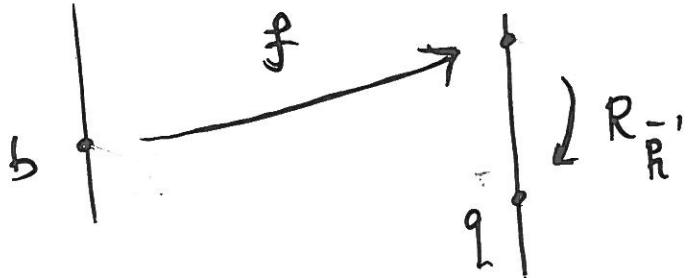
$$d(f_k(z_k), f_k(y)) = \Theta(\mu_s(k))$$

III Conformal local dynamical foliations

conformal automorphisms: $f \in \text{Conf } M$ lifts to
 bundle automorphism of B
 with $\boxed{f^* \omega = \omega}$

$$\text{so } f(\exp_b v) = \exp_{f(B)} v$$

Moreover, if $f \cdot b \cdot h' = g$ for $h \in P$,
 then $\omega_g(R_{h'}^{-1} \cdot f_* \cdot v) = \text{Ad } h \circ \omega_b(v)$

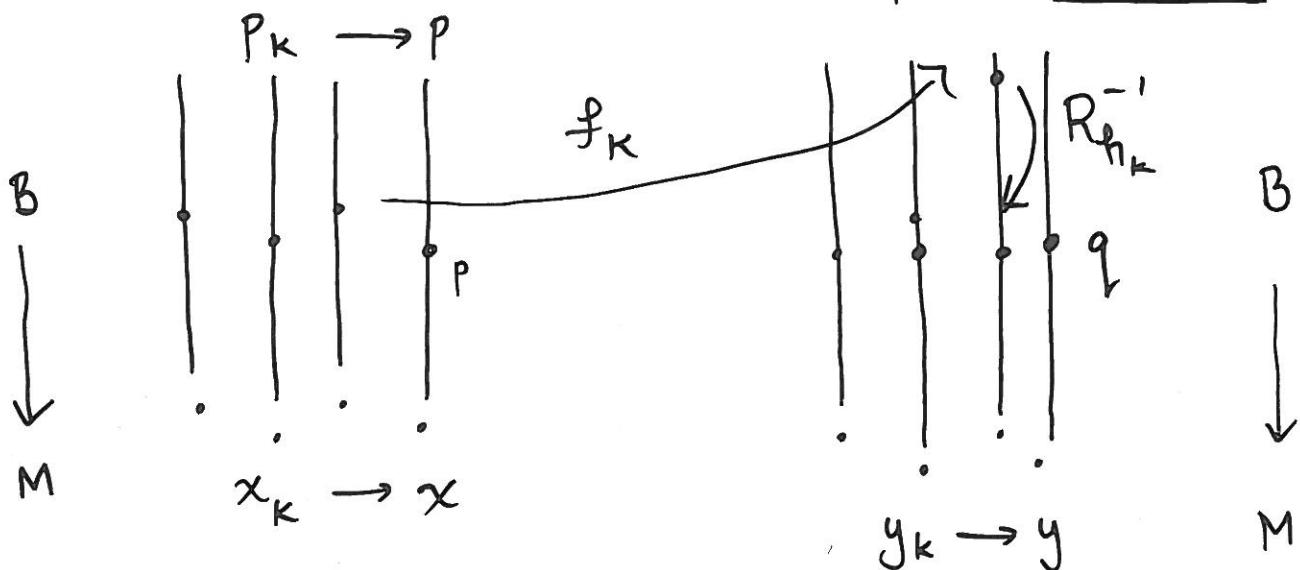


so

$$\boxed{f(\exp_b X) \cdot h^{-1} = \exp_q (\text{Ad } h X)}$$

picture of equicontinuous $\{f_k\}$

$$f_* p_k h_k^{-1} = q_k \rightarrow q, \quad \text{expect } h_k \rightarrow \infty$$



III. Conformal local dynamical foliations

M^n conformal type (p,q) semi-Riem. manifold

associated Cartan geometry (M, B, ω)

modeled on $E(n^{p,q}) = SO(p+q+1)/P$

$$P \cong \text{Conf}(R^{p,q}) \cong R^* \times O(p,q) \times R^{p,q}$$

(assume $n = p+q \geq 3$)

$\mathfrak{g} \cong \mathfrak{o}(p+q+1)$, B principal P -bundle

$$\pi: B \rightarrow M$$

Axioms for $\omega \in \Omega^1(B, \mathfrak{g})$

$$1) \forall b \in B, \quad \omega_b: T_b B \xrightarrow{\sim} \mathfrak{g}$$

$$2) \forall p \in P, \quad R_p^* \omega = \text{Ad } p^{-1} \circ \omega$$

$$3) \forall X \in \mathfrak{p}, \quad \omega_b\left(\frac{d}{dt}\Big|_0 b \cdot e^{tX}\right) \equiv X$$

exponential map : $\exp: B \times \mathfrak{g} \rightarrow B$
 (actually, on nbhd of $B \times \mathfrak{g}$)

$$X \in \mathfrak{g}, \quad \exp_b(X) = \hat{\gamma}_X(1)$$

$$\text{where } \boxed{\omega(\hat{\gamma}'_X(t)) = X}$$

$$\hat{\gamma}_X(0) = b$$

III. Conformal local dynamical foliations

$$f_k \cdot p_k \cdot h_k^{-1} = q_k \rightarrow q, \quad h_k \rightarrow \infty \text{ in } P$$

equivicontinuity $\Rightarrow h_k \rightarrow \infty$ "only in diagonal subgroup
 $A \subseteq P"$

Thm (C. Frances '12): M^n semi-Riemannian, $n \geq 3$.

$f_k \in \text{Conf } M$ equivcontinuous at $x \in M$. After passing to a subsequence, there exist $b_k \rightarrow b$ in $\bar{\pi}'(x)$ and $h_k \in P$ such that

$$f_k b_k h_k^{-1} = q_k \rightarrow q \in B$$

Moreover, h_k have the form

$$h_k = \begin{pmatrix} \lambda_1(k) \text{Id}_{m_1} & & \\ & \ddots & \\ & & \lambda_s(k) \text{Id}_{m_s} \end{pmatrix} \in \text{Co}(p, q)$$

- $1 \leq \lambda_s(k) \leq \dots \leq \lambda_1(k)$
- $\lim_{k \rightarrow \infty} \frac{\lambda_i(k)}{\lambda_{i+1}(k)} = 0 \quad i=1, \dots, s-1$ (asymptotically distinct)
- $\lim_{k \rightarrow \infty} \frac{1}{\lambda_i(k)} = 0 \quad i=4, \dots, s-1$
and $\lim_{k \rightarrow \infty} \frac{1}{\lambda_s(k)} \in [0, 1]$

III. Conformal local dynamical foliations

$$h_k = \begin{pmatrix} \lambda_1(k) \text{Id}_{m_1} & & \\ & \ddots & \\ & & \lambda_s(k) \text{Id}_{m_s} \end{pmatrix} \in \text{Col}(p, q) \subset G$$

$1 \leq \lambda_s \leq \dots \leq \lambda_1$

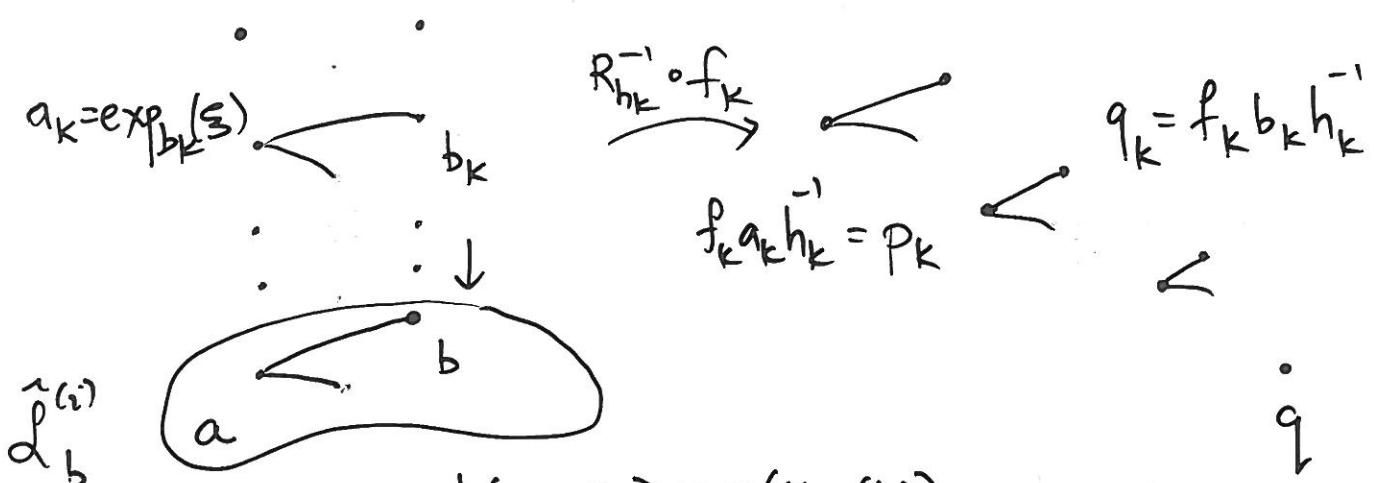
Let $\mu_i(k) = \frac{1}{\lambda_i(k)}$, $E_i \subset \mathfrak{g}$ be corresponding eigenspaces for $\text{Ad } h_k$

Let $V^{(i)} = \bigoplus_{j \leq i} E_j$ (note $V^{(s)} \hookrightarrow \mathfrak{g} \rightarrow \mathfrak{g}/\mathfrak{p}$ onto)

For $a \in B$, let $\hat{\mathcal{L}}_a^{(i)} = \omega_a^{-1}(V^{(i)}) \subseteq T_a B$

Claim: $\hat{\mathcal{L}}_b^{(i)} = \exp_b(V^{(i)} \cap B)$ ($B \subset \mathfrak{g}$ suff. small around 0) is an integral manifold for $\hat{\mathcal{L}}^{(i)}$.

Proof idea: fix norm on \mathfrak{g} , get d on nbhds of $\xi \in V^{(i)} \cap B$, $b, q \in B$.



$$d(q_k, p_k) = o(\mu_{i+1}(k))$$

$$\Rightarrow d(p_k, p'_k) = o(\mu_{i+1}(k))$$

$$\Rightarrow \text{Ad } h_k(\exp_{a_k}^{-1}(p'_k)) = o(\mu_{i+1}(k))$$

$$\therefore \omega_a(T_a \hat{\mathcal{L}}_b^{(i)}) = V^{(i)}$$

PROJECT
TO M

III. Conformal local dynamical foliations

Thm (Frances-Me. '13) Let M^n be C^ω Lorentzian manifold, $n \geq 3$. Let X be a C^ω conformal vector field on M with $X(p) = 0$. Then

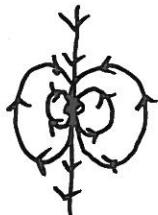
- the flow $\{\varphi_x^t\}$ is linearizable near p ; or
- M is conformally flat.

In either case, X is conjugate near p to a local conformal flow on $E^{n-1,n-1}$.

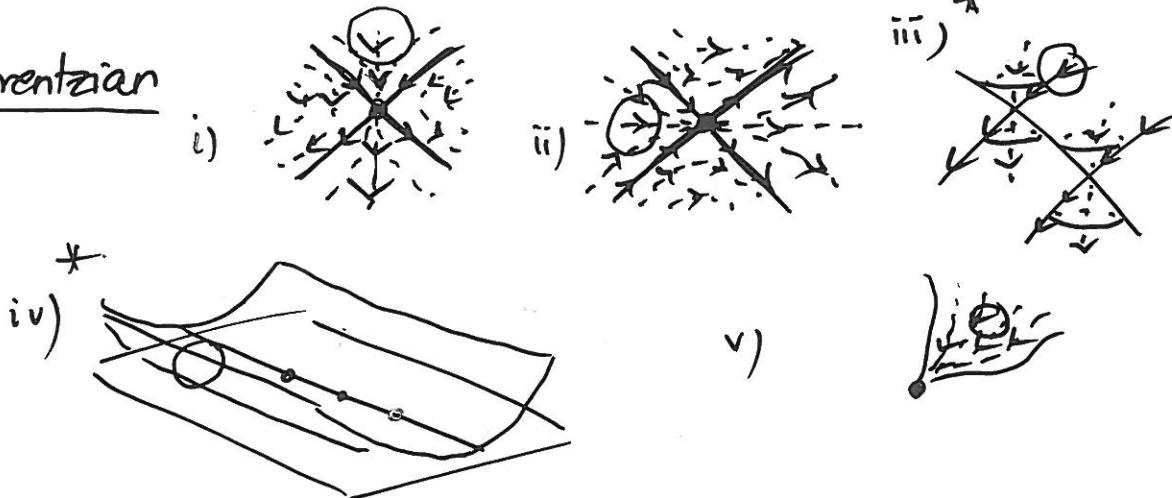
(Riemannian analogue : proved by C. Frances '12,
true for C^∞ ~~D. Alekseevski '72~~)

Nonlinearizable flows on the model spaces

Riemannian



Lorentzian



* nontrivial local foliations

III Conformal local dynamical foliations

$$\{f_k\} = \{\varphi_{x_k}^t\}$$

Find an equicontinuous point q_h
near $p \in M$.

Associate isotopy sequence at p : $\{g_k\} \subseteq P$
Get $\{h_k\} \subset C^1(\mathbb{H}^{n-1})$ from local behavior
of $\{g_k\}$ on E^{n-1} .

- could be $h_k = \begin{pmatrix} \lambda_k & & \\ & \ddots & \\ & & \lambda_k \end{pmatrix} \quad \lambda_k \rightarrow \infty$

get uniform contraction

vanishing of Weyl: $\frac{1}{\lambda_k} |W_q(u, v, w)| \underset{k \rightarrow \infty}{\approx} |W_{\varphi_{x_k}^t}(u_k, v_k, w_k)|$

same behavior near $q_h \Rightarrow W=0$ on nbhd of q_h .

OR

- $h_k = \begin{pmatrix} 1 & & \\ \lambda_k & \ddots & \\ & & \lambda_k^{-2} \end{pmatrix} \quad \lambda_k \rightarrow \infty$

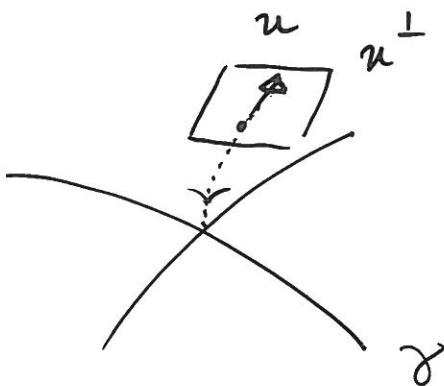
$\Rightarrow \exists u_k \rightarrow u \in T_q M$ isotropic s.t.

$W_q(x, v, w) = 0$ if $x, v, w \in u^\perp$

and $\text{Im } W_q \subseteq u^\perp$

III. Conformal local dynamical foliations

$$\text{Im } W_q \subseteq u^\perp$$



$\varphi_x^{t_k} \cdot q \rightarrow \text{point of } \gamma$
near p

at p



$$W_p \subseteq \bigcap_{u \in N} u^\perp = 0$$

$$\Rightarrow \varphi_x^{t_k} W_q(v, w, x) \rightarrow 0$$

$$\Rightarrow \text{Im } W_q \in \mathbb{R} u$$

Let $v \in T_q M$ s.t. $\langle v, u \rangle = 1$, $v \notin u^\perp$

$$\langle W_q(v, w)v, v \rangle = 0 \Rightarrow W_q \equiv 0.$$

(same behavior nearby, so $W \equiv 0$ on nbhd of q)