# Fundamental theorems of invariant theory for classical and quantum groups - 1

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# SFT in endomorphism algebra setting

Based on the following joint papers with Gus Lehrer

[1] Gus Lehrer and Ruibin Zhang, The second fundamental theorem of invariant theory for the orthogonal group. Annals of Math. **176** (2012), no. 3, 2031-2054.

[2] Gus Lehrer and Ruibin Zhang, The Brauer category and invariant theory. arXiv (2012).

# Introduction

Basic problem in classical invariant theory for, e.g., a group $G$:
One wants to describe
*generators* (FFT) & *relations* (SFT)
of invariants of $G$ actions.

Typically, for some $G$-module $M$, find the generators & relations of

• $M^G$ as vector space,

• $S(M^*)^G$ as a commutative algebra,

• $\mathrm{End}_G(M) = \mathrm{End}(M)^G$ as an associative algebra,
i.e., find a presentation for $\mathrm{End}_G(M)$.

We mostly consider the third case, which involves some interesting algebras.

# Classical groups

Work over field $K$ of characteristic 0, e.g., $K = \mathbb{C}$.
$V$, finite dimensional vector space;
$V^{\otimes r} = \underbrace{V \otimes \cdots \otimes V}_{r}$.

The case of $G = GL(V)$.
$V^{\otimes r}$ is a $G$-module for any $r$, with the action defined, for any $g \in G$ and $\mathbf{v} = v_1 \otimes v_2 \otimes \cdots \otimes v_r \in V^{\otimes r}$, by

$$g.\mathbf{v} = gv_1 \otimes gv_2 \otimes \cdots \otimes gv_r.$$

The symmetric group $\mathrm{Sym}_r$ of degree $r$ acts on $V^{\otimes r}$ by

$$\sigma.\mathbf{v} = v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(r)}, \quad \sigma \in \mathrm{Sym}_r.$$

Denote by $\nu_r$ this representation of $\mathrm{Sym}_r$. Clearly

$$\begin{aligned} g.(\sigma.\mathbf{v}) &= gv_{\sigma(1)} \otimes gv_{\sigma(2)} \otimes \cdots \otimes gv_{\sigma(r)} \\ &= \sigma.(gv_1 \otimes gv_2 \otimes \cdots \otimes gv_r) \\ &= \sigma.(g.\mathbf{v}), \qquad \text{for all } g \in G, \sigma \in \mathrm{Sym}_r. \end{aligned}$$

Hence $\nu_r(K\mathrm{Sym}_r) \subseteq \mathrm{End}_G(V^{\otimes r})$.

• FFT for $G = GL(V)$ (Schur, 1901): There is a surjection of $K$-algebras

$$\nu_r : K\mathrm{Sym}_r \longrightarrow \mathrm{End}_G(V^{\otimes r}).$$

Let $m = \dim_K V$. As $GL(V) \times \mathrm{Sym}_r$-module,

$$V^{\otimes r} = \bigoplus_\lambda L(\lambda) \otimes D_\lambda,$$

where the sum is over partitions of $r$ of the form

$$\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m);$$

$D_\lambda$, simple $\mathrm{Sym}_r$-module associated with $\lambda$;
$L(\lambda)$, simple $GL(V)$-module with highest weight $\lambda$.

$K\mathrm{Sym}_r$ is semi-simple. Inspection of the simple $\mathrm{Sym}_r$-modules appearing in the decomposition leads to:
if $r \leq m$ then $\mathrm{Ker}(\nu_r) = 0$,
if $r > m$ then $\mathrm{Ker}(\nu_r) =$ sum of ideals corresponding to $D_\lambda$ with $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r), \quad \lambda_{m+1} > 0$.

Let

$$\Sigma(m+1) := \sum_{\sigma \in \mathrm{Sym}_{m+1}} (-1)^{\ell(\sigma)} \sigma,$$

where $\ell(\sigma)$ is the length of $\sigma$. Then

$$\Sigma(m+1)(V^{\otimes(m+1)}) = \{0\}.$$

$\mathrm{Sym}_{m+1} \hookrightarrow \mathrm{Sym}_r$ as the subgroup permuting the first $m+1$ letters for any $r > m+1$.

•SFT: if $r \leq m$, then $\nu_r$ is bijective; if $r > m$, then $\mathrm{Ker}(\nu_r)$ as a 2-sided ideal of $K\mathrm{Sym}_r$ is generated by $\Sigma(m+1)$.

• FFT for $G = O(V)$ or $Sp(V)$ (Brauer, 1937): there is a surjection of $K$-algebras

$$\nu_r : B_r(\epsilon m) \longrightarrow \mathrm{End}_G(V^{\otimes r}),$$

where $B_r(\epsilon m)$ is the Brauer algebra of degree $r$ with parameter $\epsilon m$,
$m = \dim_K V, \quad \epsilon = \epsilon(G), \quad \epsilon(O) = 1, \quad \epsilon(Sp) = -1$

• SFT: describe generators of $\mathrm{Ker}(\nu_r)$ as an ideal of $B_r(\epsilon m)$. This is only known recently [Lehrer-Zhang].

Remarks:
(1). Weyl's SFT for $S(V \oplus \cdots \oplus V)^G$ is not enough for describing $\mathrm{Ker}(\nu_r)$ as an ideal of $B_r(\epsilon m)$.
(2). $B_r(\epsilon m)$ is not semi-simple unless $r$ is very small [Rui-Si].

# The Brauer diagrams

We describe the Brauer algebra categorically.

### Definition

For any pair $k, \ell \in \mathbb{N}$, a $(k, \ell)$ Brauer diagram is a partitioning of the set $\{1, 2, \ldots, k+\ell\}$ as a disjoint union of pairs.

This is thought of as a diagram where $k+\ell$ points are placed on two parallel horizontal lines, $k$ on the lower line and $\ell$ on the upper, with arcs drawn to join points which are paired. The pairs will be called *arcs*.

A $(6,4)$ Brauer diagram:

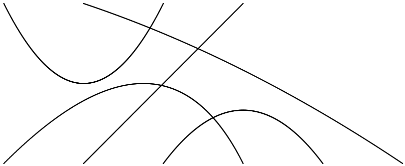


Figure :

There is just one Brauer $(0,0)$-diagram, the empty diagram.

Composition and tensor product

Let $K$ be a commutative ring with identity, and fix $\delta \in K$.
$B_k^\ell(\delta)$, free $K$-module with basis consisting of $(k, \ell)$ diagrams.
$B_k^\ell(\delta) \neq 0$ if and only if $k + \ell$ is even,
$B_0^0(\delta) = K$.

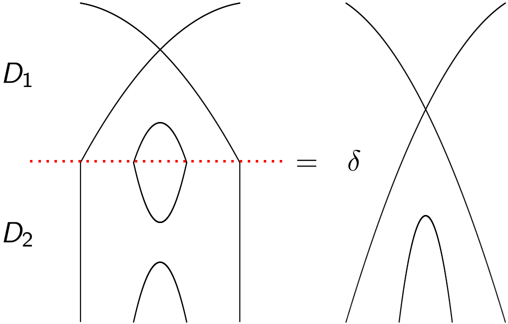
Two $K$-bilinear operations on diagrams.

$$\begin{aligned} \text{composition} \quad & \circ : \quad B_\ell^p(\delta) \times B_k^\ell(\delta) \longrightarrow B_k^p(\delta), \\ \text{tensor product} \quad & \otimes : \quad B_p^q(\delta) \times B_k^\ell(\delta) \longrightarrow B_{k+p}^{q+\ell}(\delta) \end{aligned}$$

Composition of diagrams:

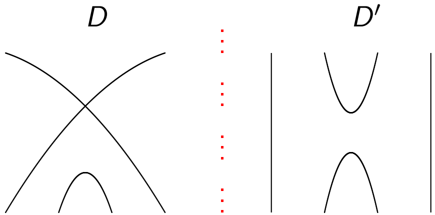
The composite $D_1 D_2$ of the Brauer diagrams $D_1 \in B_\ell^p(\delta)$ and $D_2 \in B_k^\ell(\delta)$ is defined as follows. First, the concatenation $D_1 \# D_2$ is obtained by placing $D_1$ above $D_2$, and identifying the $\ell$ lower nodes of $D_1$ with the corresponding upper nodes of $D_2$. Then $D_1 \# D_2$ is the union of a Brauer $(k, p)$ diagram $D$ with a certain number, $f(D_1, D_2)$ say, of free loops. The composite $D_1 D_2$ is the element $\delta^{f(D_1, D_2)} D \in B_k^p(\delta)$.

$D_1 D_2$



Tensor product of diagrams:

The tensor product $D \otimes D'$ of any two Brauer diagrams $D \in B_p^q(\delta)$ and $D' \in B_k^\ell(\delta)$ is the $(p+k, q+\ell)$ diagram obtained placing $D'$ on the right of $D$ without overlapping.



## Definition

The category $\mathcal{B}(\delta)$ of Brauer diagrams is the following tensor category (with tensor product denoted by $\otimes$):

1. the set of objects is $\mathbb{N} = \{0, 1, 2, \ldots\}$;
   $\mathrm{Hom}_{\mathcal{B}(\delta)}(k, l) = B_k^l(\delta)$ for any $k, l \in \mathbb{N}$;
   composition of morphisms is given by the composition of Brauer diagrams;
2. the identity object is $\emptyset$;
   tensor product of objects: $k \otimes l = k + l$;
   tensor product of morphisms is given by the tensor product of Brauer diagrams.

## Remark

$\mathcal{B}(\delta)$ is a quotient category of the category of tangles defined by Turaev and others.

## Involutions

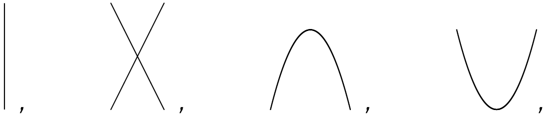
Duality functor $^* : \mathcal{B}(\delta) \to \mathcal{B}(\delta)^{\rm op}$, taking each object to itself, and each diagram to its reflection in a horizontal line.

Involution $^\sharp : \mathcal{B}(\delta) \to \mathcal{B}(\delta)$, taking objects to themselves, but a diagram $D$ to its reflection in a vertical line.

Contravariant functor $D \mapsto *D := D^{*\circ\sharp}$.

## Theorem

1. *The four elementary Brauer diagrams*

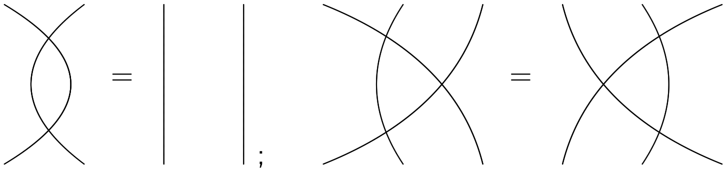


*generate all Brauer diagrams by composition and tensor product.*
*Denote these generators by $I$, $X$, $A$ and $U$ respectively.*

2. *A complete set of relations among these four generators is given by the following, and their transforms under $^*$ and $^\sharp$.*

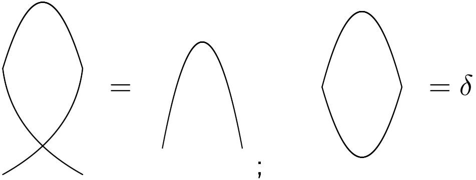
$$I \circ I = I, \quad (I \otimes I) \circ X = X, \tag{1}$$
$$A \circ (I \otimes I) = A, \quad (I \otimes I) \circ U = U, \tag{2}$$
$$X \circ X = I \otimes I, \tag{3}$$
$$(X \otimes I) \circ (I \otimes X) \circ (X \otimes I) = (I \otimes X) \circ (X \otimes I) \circ (I \otimes X), \tag{4}$$
$$A \circ X = A, \tag{5}$$
$$A \circ U = \delta, \tag{6}$$
$$(A \otimes I) \circ (I \otimes X) = (I \otimes A) \circ (X \otimes I) \tag{7}$$
$$(A \otimes I) \circ (I \otimes U) = I. \tag{8}$$

The relations (3)-(8) are depicted diagrammatically below.



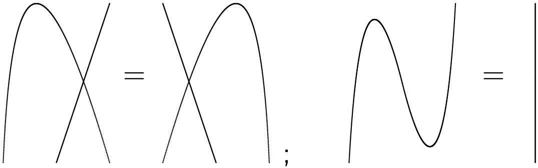

Double crossing Braid relation

Figure : Relations (3) and (4)




De-looping Loop Removal

Figure : Relations (5) and (6)



Sliding Straightening

Figure : Relations (7) and (8)

# The Brauer algebra

The Brauer algebra $B_r(\delta)$ is $B_r^r(\delta)$ with multiplication given by composition of diagrams. For $i = 1, \ldots, r-1$, let

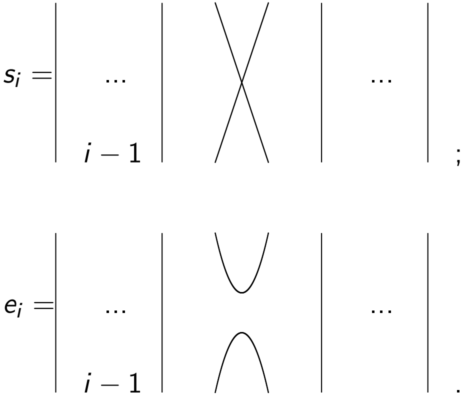



Figure :

$B_r(\delta)$ has the following presentation as $K$-algebra with anti-involution $*$:
The generators are $\{s_i, e_i \mid i = 1, 2, \ldots, r-1\}$;
the relations are the following (and their images under $*$)

$$
\begin{aligned}
& s_i s_j = s_j s_i, \ s_i e_j = e_j s_i, \ e_i e_j = e_j e_i, \quad \text{if } |i-j| \geq 2, \\
& s_i^2 = 1, \ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \\
& s_i e_i = e_i s_i = e_i, \\
& e_i^2 = \delta e_i, \\
& e_i e_{i\pm 1} e_i = e_i, \\
& s_i e_{i+1} e_i = s_{i+1} e_i,
\end{aligned}
$$

where the last five relations being valid for all applicable $i$.

Remarks:

The elements $s_1, \ldots, s_{r-1}$ generate a $K\mathrm{Sym}_r$ subalgebra of $B_r(\delta)$.

If $\delta$ is integral, $B_r(\delta)$ is not semi-simple unless $r$ is very small [Rui-Si].

# Category of tensor representations

Non-degenerate bilinear form $(-,-): V \times V \longrightarrow K$, which is either symmetric or skew symmetric.

$G = \{g \in \mathrm{GL}(V) \mid (gv, gw) = (v, w), \forall v, w \in V\}$ is the orthogonal group $\mathrm{O}(V)$ if the form is symmetric, the symplectic group $\mathrm{Sp}(V)$ if the form is skew symmetric.

**Definition**

Denote by $\mathcal{T}_G(V)$ the full subcategory of $G$-modules with objects $V^{\otimes r}$ $(r \in \mathbb{N})$, where $V^{\otimes 0} = K$.
The usual tensor product of $G$-modules and of $G$-equivariant maps is a bi-functor $\mathcal{T}_G(V) \times \mathcal{T}_G(V) \longrightarrow \mathcal{T}_G(V)$.
Call $\mathcal{T}_G(V)$ the *category of tensor representations of $G$*.

This is a tensor category with $K$ being the identity object.
Note that $\mathrm{Hom}_G(V^{\otimes r}, V^{\otimes t}) = 0$ unless $r + t$ is even;
the zero module is not an object of $\mathcal{T}_G(V)$.

# Some morphisms

$G$-module isomorphisms

$$\mathrm{End}_K(V) \xrightarrow{\sim} V \otimes V^* \xrightarrow{\sim} V \otimes V.$$

Let $c_0 \in V \otimes V$ be the image of $\mathrm{id}_V$.

Define maps

$$\begin{aligned} P : V \otimes V \longrightarrow V \otimes V, &\quad v \otimes w \mapsto w \otimes v, \\ \check{C} : K \longrightarrow V \otimes V, &\quad 1 \mapsto c_0, \\ \hat{C} : V \otimes V \longrightarrow K, &\quad v \otimes w \mapsto (v, w). \end{aligned} \tag{9}$$

### Lemma

*The maps $P$, $\check{C}$ and $\hat{C}$ are all $G$-equivariant, and*

$$P^2 = \mathrm{id}^{\otimes 2}, \tag{10}$$

$$(P \otimes \mathrm{id})(\mathrm{id} \otimes P)(P \otimes \mathrm{id}) = (\mathrm{id} \otimes P)(P \otimes \mathrm{id})(\mathrm{id} \otimes P), \tag{11}$$

$$P\check{C} = \epsilon\check{C}, \quad \hat{C}P = \epsilon\hat{C}, \tag{12}$$

$$\hat{C}\check{C} = \epsilon m, \tag{13}$$

$$(\hat{C} \otimes \mathrm{id})(\mathrm{id} \otimes \check{C}) = \mathrm{id} = (\mathrm{id} \otimes \hat{C})(\check{C} \otimes \mathrm{id}), \tag{14}$$

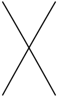
$$(\hat{C} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes P) = (\mathrm{id} \otimes \hat{C}) \circ (P \otimes \mathrm{id}), \tag{15}$$

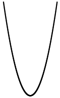
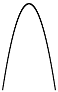
$$(P \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \check{C}) = (\mathrm{id} \otimes P) \circ (\check{C} \otimes \mathrm{id}). \tag{16}$$

# A tensor functor

**Theorem**

*There exists a tensor functor* $F : \mathcal{B}(\epsilon m) \longrightarrow \mathcal{T}_G(V)$ *such that*
$F$ *sends* *object* $r$ *to* $V^{\otimes r}$,
*morphism* $D : k \to \ell$ *to* $F(D) : V^{\otimes k} \longrightarrow V^{\otimes l}$,
*where* $F$ *is defined on the elementary diagrams by*

$F\Bigg($  $\Bigg) = \mathrm{id}_V$, $F\Bigg($  $\Bigg) = \epsilon P$,

$F\Bigg($  $\Bigg) = \check{C}$, $F\Bigg($  $\Bigg) = \hat{C}$. (17)

As a tensor functor, $F$ respects tensor product.
For any objects $r, r'$ and morphisms $D, D'$ in $\mathcal{B}(\epsilon m)$,

$$F(r \otimes r') = V^{\otimes r} \otimes V^{\otimes r'} = F(r) \otimes F(r'),$$
$$F(D \otimes D') = F(D) \otimes F(D').$$

# Fundamental theorems

We assume that $K$ is a field of characteristic zero.

Let $\Sigma_\epsilon(r) \in B_r(\delta)$ with $\epsilon = \pm 1$ be defined by

$$\Sigma_\epsilon(r) = \sum_{\sigma \in \mathrm{Sym}_r} (-\epsilon)^{\ell(\sigma)} \sigma.$$

Represent $\Sigma_\epsilon(r)$ pictorially by

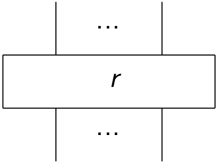



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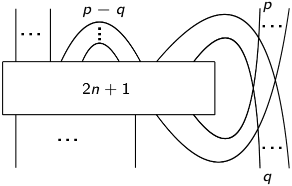
Figure : $\Sigma_\epsilon(r)$

Denote by $\langle \Sigma_\epsilon(m+1) \rangle$ the subspace of $\oplus_{k,\ell} B_k^\ell(\epsilon m)$ spanned by the morphisms generated by $\Sigma_\epsilon(m+1)$ by composition and tensor product.

Set $\langle \Sigma_\epsilon(m+1) \rangle_k^\ell = \langle \Sigma_\epsilon(m+1) \rangle \cap B_k^\ell(\epsilon m)$.

E.g., when $\epsilon = -1$,

$D(p, q) =$ 


belongs to $\langle \Sigma_\epsilon(m+1) \rangle_k^k$ with $k = 2n - p + q$.

## Theorem

*Assume that $K$ has characteristic $0$. Write $d = m$ if $G = \mathrm{O}(V)$, and $d = \frac{m}{2}$ if $G = \mathrm{Sp}(V)$.*

1. *The functor $F : \mathcal{B}(\epsilon m) \longrightarrow \mathcal{T}_G(V)$ is full. That is, $F$ is surjective on $\mathrm{Hom}$ spaces.*
2. *The map $F_k^\ell = F|_{B_k^\ell(\epsilon m)}$ is bijective if $k + \ell \leq 2d$, and $\mathrm{Ker} F_k^\ell = \langle \Sigma_\epsilon(m+1) \rangle_k^\ell$ if $k + \ell > 2d$.*

## Proof.

Part 1 follows from Brauer's result.
Part 2 is deduced from Weyl's SFT. $\square$

# Lecture 2

# Summary of Lecture 1

- $G = O(V)$ or $Sp(V)$. There exists a surjection of algebras $\nu_r : B_r(\epsilon m) \longrightarrow \mathrm{End}_G(V^{\otimes r})$. We want to describe $Ker(\nu_r)$.
- Introduced the category $\mathcal{B}(\delta)$ of Brauer diagrams such that $B_r(\delta) = Hom(r, r)$; constructed a tensor functor $F : \mathcal{B}(\epsilon m) \longrightarrow \mathcal{T}_G(V)$, where $\mathcal{T}_G(V)$ is the category of tensor modules for $G$.

- Theorem

  1. *The functor $F : \mathcal{B}(\epsilon m) \longrightarrow \mathcal{T}_G(V)$ is full. That is, $F$ is surjective on* $\mathrm{Hom}$ *spaces.*
  2. *The map $F_k^\ell = F|_{B_k^\ell(\epsilon m)}$ is bijective if $k + \ell \leq 2d$, and* $\mathrm{Ker}F_k^\ell = \langle \Sigma_\epsilon(m+1)\rangle_k^\ell$ *if $k + \ell > 2d$.*

Problem. The construction of $\langle \Sigma_\epsilon(m+1)\rangle_k^k$ used operations not defined in the Brauer algebra $B_k(\epsilon m)$.

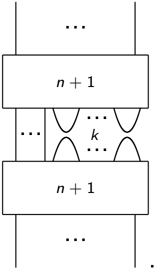
# Endomorphism algebras in $Sp(V)$ case

Take $G = \mathrm{Sp}(V)$ with $m = \dim V = 2n$.

The element $\Phi \in B_{n+1}^{n+1}(-2n)$:

$$\Phi = \sum_{k=0}^{\left[\frac{n+1}{2}\right]} a_k \Xi_k \quad \text{with} \quad a_k = \frac{1}{(2^k k!)^2 (n+1-2k)!}, \tag{18}$$

$\Xi_k \quad =$



Note that

$$\Xi_k = \Sigma_{-1}(n+1)E(k)\Sigma_{-1}(n+1),$$

$$E(k) = \prod_{j=1}^{k} e_{n+2-2j}, \quad E(0) = 1.$$

### Lemma

*The element $\Phi$ is the sum of all $(n+1, n+1)$ Brauer diagrams. It has the following properties:*

1. $e_i\Phi = \Phi e_i = 0$ *for all* $e_i \in B_{n+1}^{n+1}(-2n)$*;*
2. $\Phi^2 = (n+1)!\Phi$*;*
3. $\Phi \in \mathrm{Ker} F_{n+1}^{n+1}$.

For $s < r$, natural embedding of associative algebras:

$$B_s^s(-2n) \hookrightarrow B_r^r(-2n), \quad b \mapsto b \otimes I_{r-s}.$$

Regard $B_s^s(-2n)$ as the subalgebra of $B_r^r(-2n)$.

### Definition

Let $\langle \Phi \rangle_r$ be the 2-sided ideal of $B_r^r(-2n)$ generated by $\Phi$ if $r > n$.

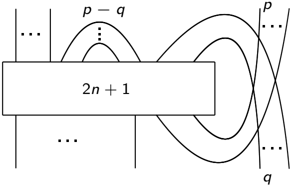
### Theorem

- *If $r \leq n$, $F_r^r : B_r^r(-2n) \longrightarrow \mathrm{End}_{\mathrm{Sp}(V)}(V^{\otimes r})$ is bijective.*
- *If $r > n$, $\mathrm{Ker} F_r^r$ is generated by $\Phi$ as a 2-sided ideal of $B_r^r(-2n)$.*

In particular, $\Sigma_{-1}(2n+1) \in \langle \Phi \rangle_{2n+1}$.

## Sketch of proof of main theorem

$D(p,q) \in B_k^k(-2n)$ with $k = 2n+1-p+q$:

$D(p,q) =$ 


### Proposition

*Assume that $r > n$.*
*(I). As a two-sided ideal of the Brauer algebra $B_r^r(-2n)$, $\mathrm{Ker} F_r^r$ is generated by $D(p,q)$ and $*D(p,q)$ with $p+q \leq r$ and $p \leq n$.*
*(II). $D(p,q)$ and $*D(p,q)$ all belong to $\langle \Phi \rangle_r$.*

# Endomorphism algebras in $O(V)$ case

$\epsilon = +1$

For $p = 0, 1, \ldots, m+1$, let $E_{m+1-p} \in B_{m+1}^{m+1}(m)$ be defined by

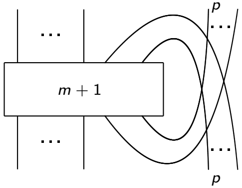
$E_{m+1-p} =$ 


Figure :

The elements $E_k$ are $\mathbb{Z}$-linear combinations of Brauer diagrams.

Formulae for $E_k$ are given in next slide.

If $1 \leq k < l$, write $A(k,l) := \Sigma_{+1}(\mathrm{Sym}_{\{k,k+1,\ldots,l\}})$ for the total antisymmetriser in $\mathrm{Sym}_{\{k,k+1,\ldots,l\}}$. Let

$$F_p = A(1, m+1-p)A(m+2-p, m+1).$$

For $j = 0, 1, 2, \ldots, i$, define

$$e_i(j) = e_{i,i+1}e_{i-1,i+2}\ldots e_{i-j+1,i+j}.$$

## Lemma

*For $i = 0, 1, \ldots, m+1$, let $min_i = min(i, m+1-i)$. Then*

$$E_i = \sum_{j=0}^{min_i} (-1)^j c_i(j)\Xi_i(j) \quad \textit{with} \quad \Xi_i(j) = F_i e_i(j) F_i, \tag{19}$$

*where $c_i(j) = \left((i-j)!(m+1-i-j)!(j!)^2\right)^{-1}$.*

## Lemma

1. $*E_p = E_{m+1-p}$ *for all* $p$.
2. $F_pE_p = E_pF_p = p!(m+1-p)!E_p$.
3. $e_iE_p = E_pe_i = 0$ *for all* $i \leq m$.
4. $E_p^2 = p!(m+1-p)!E_p$.

Note that $E_0 = E_{m+1} = \Sigma_{+1}(m+1)$.

## Lemma

*Let* $\ell = \left[\frac{m+1}{2}\right]$. *As two-sided ideals in* $B_r^r(m)$,

$$\langle E_0\rangle \subset \langle E_1\rangle \subset \cdots \subset \langle E_{\ell-1}\rangle \subset \langle E_\ell\rangle \supset \langle E_{\ell+1}\rangle \supset \cdots \supset \langle E_m\rangle \supset \langle E_{m+1}\rangle.$$

## The main theorem

### Theorem

*If $r \leq m$, $F_r^r : B_r^r(m) \longrightarrow \mathrm{End}_{\mathrm{O}(V)}(V^{\otimes r})$ is bijective.*

*If $r > m$, $\mathrm{Ker} F_r^r$ as a two-sided ideal of $B_r^r(m)$ is generated by $E_\ell$.*

This follows from the Lemma and

### Proposition

*Assume $r > m$. $\mathrm{Ker} F_r^r$, as a two-sided ideal of $B_r^r(m)$, is generated by $E_p$ for all $0 \leq p \leq m+1$.*

Continue to invariant theory of quantum groups...