

# Fundamental theorems of invariant theory for classical and quantum groups - 2

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# Unified treatment of FFTs old and new

Based on

- 
- G. I. Lehrer and R. B. Zhang, "Strongly multiplicity free modules for Lie algebras and quantum groups", *J. of Algebra* **306** (2006), 138–174.

# Casimir operators

- ▶  $\mathfrak{g}$ , a simple Lie algebra or  $\mathfrak{gl}_k$ .  
 $U(\mathfrak{g})$ , universal enveloping algebra of  $\mathfrak{g}$ .
- ▶  $U(\mathfrak{g})$  is a Hopf algebra with co-multiplication  $\Delta$  such that

$$\Delta(X) = X \otimes 1 + 1 \otimes X, \quad X \in \mathfrak{g} \subset U(\mathfrak{g}).$$

Let  $\Delta^{(r-1)} = (\Delta \otimes \text{id}^{\otimes(r-2)}) \cdots (\Delta \otimes \text{id})\Delta$ .

- ▶  $C$ , the quadratic Casimir element in the center of  $U(\mathfrak{g})$ .  
Exist bases  $\{X_\alpha | 1 \leq \alpha \leq \dim \mathfrak{g}\}$  and  $\{\tilde{X}_\alpha | 1 \leq \alpha \leq \dim \mathfrak{g}\}$  of  $\mathfrak{g}$  such that

$$C = \sum_{\alpha} X_{\alpha} \tilde{X}_{\alpha} = \sum_{\alpha} \tilde{X}_{\alpha} X_{\alpha}.$$

Let  $t := \frac{1}{2}(\Delta(C) - C \otimes 1 - 1 \otimes C)$ .

- ▶ Then

$$t = \sum_{\alpha} X_{\alpha} \otimes \tilde{X}_{\alpha} = \sum_{\alpha} \tilde{X}_{\alpha} \otimes X_{\alpha}.$$

- ▶ Define  $C_{ij} \in U(\mathfrak{g})^{\otimes r}$  ( $i, j \leq r$ ) by

$$C_{ij} = \sum_{\alpha} 1 \otimes \cdots \otimes 1 \otimes \underbrace{X_{\alpha}}_i \otimes 1 \otimes \cdots \otimes 1 \otimes \underbrace{\tilde{X}_{\alpha}}_j \otimes 1 \otimes \cdots \otimes 1;$$

let  $T_r \subset U(\mathfrak{g})^{\otimes r}$  be the subalgebra generated by all the  $C_{ij}$ .

- ▶ For all  $u \in T_r$ , we have  $[u, \Delta^{(r-1)}(x)] = 0$  for all  $x \in U(\mathfrak{g})$ .

Proof: It follows from  $[t, \Delta(x)] = 0$  for all  $x \in U(\mathfrak{g})$  that

$$[\Delta^{(r-1)}(x), C_{ij}] = 0, \quad \forall x \in U(\mathfrak{g}).$$

# Endomorphism algebras

- ▶  $V$ , finite dimensional  $U(\mathfrak{g})$ -module;  
 $\pi : U(\mathfrak{g}) \longrightarrow \text{End}_{\mathbb{C}}(V)$ , a rep of  $\mathfrak{g}$  on  $V$ .
- ▶ Actions of various algebras on  $V^{\otimes r}$ :

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  - ▶  $\psi_r := \pi^{\otimes r}|_{T_r} : T_r \longrightarrow \text{End}_{\mathbb{C}}(V^{\otimes r})$ .
- ▶ Lemma  
We have  $\psi_r(T_r) \subseteq \text{End}_{U(\mathfrak{g})}(V^{\otimes r})$ .

- ▶  $G$ , complex Lie group with Lie algebra  $\mathfrak{g}$ .
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- ▶ Theorem

$$\text{End}_G(V^{\otimes r}) = \psi_r(T_r), \quad \forall r \geq 2$$

for the groups and respective representations listed in the table:

$G$	$V$
$GL_k, SL_k$	$\mathbb{C}^k$ , natural module
$O_k$	$\mathbb{C}^k$ , natural module
$Sp_{2m}$	$\mathbb{C}^{2m}$ , natural module
$G_2$	7-dim'l irred. module
$GL_2, SL_2$	any finite dim'l simple module

# Relationship to usual FFT

Example:  $O_n(\mathbb{C})$  with  $V = \mathbb{C}^n$



$$V \otimes V = L(2\epsilon_1) \oplus L(\epsilon_1 + \epsilon_2) \oplus L(0),$$

where  $L(\lambda)$  denotes the simple module with highest weight  $\lambda$ .

Thus in  $\psi_2(T_2)$  ( $t = t_{12}$ ),

$$(\psi_2(t) - 1)(\psi_2(t) + 1)(\psi_2(t) - 1 + n) = 0.$$

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- The projection operators  $P[\lambda] : V \otimes V \longrightarrow L(\lambda)$  are given by

$$P[2\epsilon_1] = \frac{(\psi_2(t) + 1)(\psi_2(t) - 1 + n)}{2n},$$

$$P[\epsilon_1 + \epsilon_2] = -\frac{(\psi_2(t) - 1)(\psi_2(t) - 1 + n)}{2(n - 2)},$$

$$P[0] = \frac{(\psi_2(t) - 1)(\psi_2(t) + 1)}{n(n - 2)}.$$

► Let

$$\begin{aligned}s &= P[2\epsilon_1] + P[0] - P[\epsilon_1 + \epsilon_2], \quad e = nP[0], \\ s_i &= \text{id}_V^{\otimes(i-1)} \otimes s \otimes \text{id}_V^{\otimes(r-1-i)}, \\ e_i &= \text{id}_V^{\otimes(i-1)} \otimes e \otimes \text{id}_V^{\otimes(r-1-i)}, \quad 1 \leq i \leq r-1.\end{aligned}$$

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- ▶ The operators  $s_i, e_i$  satisfy the following relations:

$$s_i^2 = \text{id}_V^{\otimes r}, \quad s_i s_j = s_j s_i, \quad |i - j| > 1,$$

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1},$$

$$e_i^2 = \delta e_i, \text{ where } \delta = \dim V = n,$$

$$e_i e_{i \pm 1} e_i = e_i, \quad e_i e_j = e_j e_i, \quad |i - j| > 1,$$

$$s_i e_i = e_i s_i = e_i,$$

$$s_{i+1} e_i s_{i+1} = s_i e_{i+1} s_i,$$

$$s_i e_j = e_j s_i, \quad |i - j| > 1.$$

- ▶ These are the same as the defining relations of the Brauer algebra  $B_r(\delta)$  with  $\delta = n$ . Thus the operators

$$s_i, \quad e_i \quad (1 \leq i \leq r-1)$$

generate the representation

$$\nu_r : B_r(n) \longrightarrow \text{End}_{\mathbb{C}}(V^{\otimes r})$$

of  $B_r(n)$  on  $V^{\otimes r}$  discussed earlier.

Remarks.

- ▶ Proof of theorem is case by case.
- ▶ FFT for  $G_2$  was known in a different form (due to G. Schwartz).
- ▶ For all the cases in the table except  $O_{2m}$ ,

$$\text{End}_{U(\mathfrak{g})}(V^{\otimes r}) = \psi_r(T_r), \quad \forall r \geq 2.$$

- ▶ The reason for turning to groups is to take care of the even orthogonal group  $O_{2m}$ .

# Invariant theory of quantum groups

Questions in invariant theory make sense for all Hopf algebras, and quantum groups in particular.

Want to develop the invariant theory for quantum groups.

## References

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# Quantum groups

$U_q(\mathfrak{g})$ , quantum group of simple Lie algebra or  $\mathfrak{gl}_k$  defined over  $K = \mathbb{C}(q^{\frac{1}{2}})$ .  $U_q(\mathfrak{g})$  is a Hopf algebra with co-multiplication

$$\Delta : U_q(\mathfrak{g}) \longrightarrow U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g}).$$

Write  $\Delta(x) = \sum x_{(1)} \otimes x_{(2)}$  for any  $x \in U_q(\mathfrak{g})$ .

If  $V$  and  $W$  are  $U_q(\mathfrak{g})$ -modules, then  $U_q(\mathfrak{g})$  acts on  $V \otimes_K W$ :

$$x.(v \otimes w) = \sum x_{(1)} v \otimes x_{(2)} w$$

Example:  $U_q(\mathfrak{sl}_2)$  is generated by  $e, f, k, k^{-1}$  with relations

$$kk^{-1} = k^{-1}k = 1,$$

$$kek^{-1} = q^2e, \quad kfk^{-1} = q^{-2}f,$$

$$ef - fe = \frac{k - k^{-1}}{q - q^{-1}}.$$

Co-multiplication

$$\Delta(k) = k \otimes k$$

$$\Delta(e) = e \otimes k + 1 \otimes e,$$

$$\Delta(f) = f \otimes 1 + k^{-1} \otimes f.$$

2-dimenional irreducible representation

$$e \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad k \mapsto \begin{pmatrix} q & o \\ 0 & q^{-1} \end{pmatrix}.$$

# The universal $R$ -matrix

Exists an invertible element  $R \in U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$  such that

$$\begin{aligned} R\Delta(x) &= \Delta^{op}(x)R, \quad \forall x \in U_q(\mathfrak{g}), \\ (\Delta \otimes id)R &= R_{13}R_{23}, \quad (id \otimes \Delta)R = R_{13}R_{12}. \end{aligned} \tag{1}$$

Write  $R = \sum \alpha_t \otimes \beta_t$ , then

$$R_{12} = R \otimes 1, \quad R_{23} = 1 \otimes R, \quad R_{13} = \sum \alpha_t \otimes 1 \otimes \beta_t.$$

Equation (1) implies that  $R$  satisfies the celebrated Yang-Baxter equation

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}.$$

# Braid group representations

- ▶ The braid group  $\mathcal{B}_r$  on  $r$  strings is generated by  $b_1^{\pm 1}, \dots, b_{r-1}^{\pm 1}$  subject to the standard relations

$$b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1}, \quad b_i b_j = b_j b_i, \quad \text{if } |i - j| > 1.$$

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- ▶  $V_q$ , finite dim'l  $U_q(\mathfrak{g})$ -module of type  $(1, 1, \dots, 1)$ ,

$$\check{R} := \tau R : V_q \otimes V_q \longrightarrow V_q \otimes V_q,$$

where  $\tau : v \otimes w \mapsto w \otimes v$ .

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- ▶ It follows from  $\tau R \Delta(x) = \tau \Delta^{op}(x) R = \Delta(x) \tau R$  that

$$\check{R} \in \text{End}_{U_q(\mathfrak{g})}(V_q \otimes V_q).$$

The Yang-Baxter equation

$$(\check{R} \otimes \text{id}_{V_q})(\text{id}_{V_q} \otimes \check{R})(\check{R} \otimes \text{id}_{V_q}) = (\text{id}_{V_q} \otimes \check{R})(\check{R} \otimes \text{id}_{V_q})(\text{id}_{V_q} \otimes \check{R}).$$

## Theorem

There exists rep  $\psi_r : \mathcal{B}_r \longrightarrow GL(V_q)$  of  $\mathcal{B}_r$  defined by

$$\psi_r(b_i) = R_i, \quad \forall i,$$

$$R_i := (id_{V_q})^{\otimes(i-1)} \otimes \check{R} \otimes (id_{V_q})^{\otimes(r-i-1)}.$$

This immediately follows from the Yang-Baxter equation for  $\check{R}$ .

# Quantum group $U_q(\mathfrak{o}_k)$

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  - ▶ if  $k = 2m$ , let  $E_{m-1}, F_{m-1}, k_{m-1}^{\pm 1}$  and  $E_m, F_m, k_m^{\pm 1}$  be the generators of  $U_q(\mathfrak{so}_{2m})$  associated with the simple roots  $\alpha_{m-1} = \epsilon_{m-1} - \epsilon_m$  and  $\alpha_m = \epsilon_{m-1} + \epsilon_m$ . Then  $\sigma E_{m-1} \sigma^{-1} = E_m$ ,  $\sigma F_{m-1} \sigma^{-1} = F_m$ ,  $\sigma k_{m-1}^{\pm 1} \sigma^{-1} = k_m^{\pm 1}$ , and  $\sigma$  commutes with all the other generators of  $U_q(\mathfrak{so}_{2m})$ .

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- ▶ Let  $V_q = K^k$  be the natural module for  $U_q(\mathfrak{o}_k)$ . Extend it to a  $U_q(\mathfrak{o}_n)$ -module by requiring that
  - (i) For odd  $k$ ,  $\sigma$  acts on the highest weight vector of  $V_q$  by  $-1$ ;
  - (ii) For even  $k$ ,  $\sigma$  acts on the highest weight vector of  $V_q$  by  $1$ .

# FFTs in endomorphism algebra setting

- ▶ Consider the following cases:

$U_q(\mathfrak{g})$	$V_q$
$U_q(\mathfrak{gl}_k), U_q(\mathfrak{sl}_k)$	$K^k$ , natural module
$U_q(\mathfrak{o}_k)$	$K^k$ , natural module
$U_q(\mathfrak{sp}_{2m})$	$K^{2m}$ , natural module
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## ▶ Theorem

*For the quantum groups and respective modules listed in the table,*

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$$\text{End}_{U_q(\mathfrak{g})}(V_q^{\otimes r}) \cong \psi_r(K\mathcal{B}_r).$$

- ▶ Results for the quantum groups associated with classical series of Lie algebras were known (Jimbo, Wenzl, ...).

# The Hecke algebra and $U_q(\mathfrak{gl}_k)$

- ▶ The Hecke algebra  $H_r(q)$  of type  $A_{r-1}$  with parameter  $q$ :

$$H_r(q) = K\mathcal{B}_r/\mathcal{J}, \quad \mathcal{J} = ((b_i - q)(b_i + q^{-1}), \forall i)$$

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- ▶ For  $U_q(\mathfrak{gl}_k)$  with  $V_q = K^k$ ,

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$$V_q \otimes V_q = L_q(2\epsilon_1) \oplus L_q(\epsilon_1 + \epsilon_2).$$

- ▶ Let  $P[\lambda] : V_q \otimes V_q \longrightarrow L_q(\lambda)$  be the projections. Then

$$\check{R} = qP[2\epsilon_1] - q^{-1}P[\epsilon_1 + \epsilon_2],$$

hence

$$(R_i - q)(R_i + q^{-1}) = 0, \quad \forall i.$$

A basis of  $H_r(q)$ :  $\{T_w \mid w \in Sym_r\}$ . Define

$$E = \sum_{w \in Sym_{k+1}} (-q)^{-\ell(w)} T_w.$$

$H_{k+1}(q) \hookrightarrow H_r(q)$  for any  $r \geq k+1$ .

### Theorem

In the case of  $U_q(\mathfrak{gl}_k)$  with  $V_q = K^k$ ,

- (1). the rep  $\psi_r$  of  $K\mathcal{B}_r$  factors through the Hecke algebra  $H_r(q)$ ;
- (2).

$$\text{End}_{U_q(\mathfrak{gl}_k)}(V_q^{\otimes r}) \cong H_r(q) \text{ if } r \leq k;$$

$$\text{End}_{U_q(\mathfrak{gl}_k)}(V_q^{\otimes r}) \cong H_r(q)/(E) \text{ if } r > k.$$

# The BMW algebra and quantum groups of $\mathfrak{o}_k$ and $\mathfrak{sp}_{2\ell}$

The Birman-Wenzl-Murakami algebra  $BMW_r(z, y)$  with parameters  $z$  and  $y$  is generated by  $g_1^{\pm 1}, \dots, g_{r-1}^{\pm 1}$  and  $e_1, \dots, e_{r-1}$ , subject to the following relations:

- ▶ The braid relations for the  $g_i$ :

$$g_i g_j = g_j g_i \text{ if } |i - j| \geq 2$$

$$g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1} \text{ for } 1 \leq i \leq r-1;$$

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- ▶ The Kauffman skein relations:

$$g_i - g_i^{-1} = z(1 - e_i) \text{ for all } i;$$

# The BMW algebra and quantum groups of $\mathfrak{o}_k$ and $\mathfrak{sp}_{2\ell}$

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$$g_i - g_i^{-1} = z(1 - e_i) \text{ for all } i;$$

- ▶ The de-looping relations:

$$g_i e_i = e_i g_i = y e_i;$$

$$e_i g_{i-1}^{\pm 1} e_i = y^{\mp 1} e_i;$$

$$e_i g_{i+1}^{\pm 1} e_i = y^{\mp 1} e_i.$$

- ▶ For  $U_q(\mathfrak{o}_k)$  with  $V_q = K^k$ ,

$$V_q \otimes V_q = L_q(2\epsilon_1) \oplus L_q(\epsilon_1 + \epsilon_2) \oplus L_q(0),$$

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- ▶ Let  $P[\lambda] : V_q \otimes V_q \longrightarrow L_q(\lambda)$  be the projections. Then

$$\check{R} = qP[2\epsilon_1] - q^{-1}P[\epsilon_1 + \epsilon_2] + q^{-k+1}P[0].$$

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- ▶ Let

$$e = D_q P[0] \quad \text{with} \quad D_q = 1 + \frac{q^{k-1} - q^{-k+1}}{q - q^{-1}}.$$

Then

$$\check{R} - \check{R}^{-1} = (q - q^{-1})(1 - e),$$

$$\check{R}e = e\check{R} = q^{-k+1}e.$$

Thus

$$g_i \mapsto R_i, \quad e_i \mapsto (\text{id}_{V_q})^{\otimes(i-1)} \otimes e \otimes (\text{id}_{V_q})^{\otimes(r-i-1)}$$

is a representation of  $BMW_r(q - q^{-1}, q^{-k+1})$ .

### Theorem

Let  $V_q = K^k$  be the natural module for  $U_q = U_q(\mathfrak{o}_k)$ .

- ▶ (1). The rep  $\psi_r$  of  $K\mathcal{B}_r$  factors through the BMW algebra  
 $BMW_r = BMW_r(q - q^{-1}, q^{-k+1})$ .

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- ▶ (1). The rep  $\psi_r$  of  $K\mathcal{B}_r$  factors through the BMW algebra  $BMW_r = BMW_r(q - q^{-1}, q^{-k+1})$ .
- ▶ (2).

If  $r \leq k$ ,  $\text{End}_{U_q}(V_q^{\otimes r}) = BMW_r$ .

If  $r > k$ , there exists a single element  $\Phi_q \in BMW_r$  such that

$$\text{End}_{U_q}(V_q^{\otimes r}) = BMW_r / (\Phi_q).$$

We have a similar result for  $U_q(\mathfrak{sp}_{2\ell})$ .

### Theorem

Let  $V_q = K^{2\ell}$  be the natural module for  $U_q = U_q(\mathfrak{sp}_{2\ell})$ .

- ▶ (1). The rep  $\psi_r$  of  $K\mathcal{B}_r$  factors through the BMW algebra  $BMW_r = BMW_r(q - q^{-1}, q^{2\ell+1})$ .
- ▶ (2). If  $r \leq \ell$ ,  $\text{End}_{U_q}(V_q^{\otimes r}) = BMW_r$ ;  
If  $r > \ell$ , there exists a single element  $\Phi_q \in BMW_r$  such that

$$\text{End}_{U_q}(V_q^{\otimes r}) = BMW_r / (\Phi_q).$$

**Remark:** We only know a formula for  $\Phi_q$  in the case of the 3-dim'l  $U_q(\mathfrak{o}_3)$ -module.