Fundamental theorems of invariant theory for classical and quantum groups - 3

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Invariant theory of Lie superalgebras

Based on joint work with Gus Lehrer.

Gus Lehrer and Ruibin Zhang, The first fundamental theorem of invariant theory for the orthosymplectic supergroup. Preprint, 2014.

$\mathbb{Z}_2$-graded vector spaces

A $\mathbb{Z}_2$-graded vector space V is

$$V = V_{\bar{0}} \oplus V_{\bar{1}}, \quad V_{\bar{0}}, \text{ even subspace}, \quad V_{\bar{1}}, \text{ odd subspace}.$$

If V and W are $\mathbb{Z}_2$-graded vector spaces, $V \otimes_{\mathbb{C}} W$ has a natural $\mathbb{Z}_2$-grading:

$$\begin{aligned}(V \otimes_{\mathbb{C}} W)_{\bar{0}} &= V_{\bar{0}} \otimes_{\mathbb{C}} W_{\bar{0}} \oplus V_{\bar{1}} \otimes_{\mathbb{C}} W_{\bar{1}},\\ (V \otimes_{\mathbb{C}} W)_{\bar{1}} &= V_{\bar{0}} \otimes_{\mathbb{C}} W_{\bar{1}} \oplus V_{\bar{1}} \otimes_{\mathbb{C}} W_{\bar{0}}.\end{aligned}$$

The space $\mathrm{Hom}_{\mathbb{C}}(V, W)$ of homomorphisms is also $\mathbb{Z}_2$-graded with

$$\begin{aligned}\mathrm{Hom}_{\mathbb{C}}(V, W)_{\bar{0}} &= \mathrm{Hom}_{\mathbb{C}}(V_{\bar{0}}, W_{\bar{0}}) \oplus \mathrm{Hom}_{\mathbb{C}}(V_{\bar{1}}, W_{\bar{1}}),\\ \mathrm{Hom}_{\mathbb{C}}(V, W)_{\bar{1}} &= \mathrm{Hom}_{\mathbb{C}}(V_{\bar{0}}, W_{\bar{1}}) \oplus \mathrm{Hom}_{\mathbb{C}}(V_{\bar{1}}, W_{\bar{0}}).\end{aligned}$$

Superdimension: $\mathrm{sdim}\, V = (\dim V_{\bar{0}} | \dim V_{\bar{1}})$.

Choose ordered homogeneous basis for V such that even elements precede odd ones. If $\mathrm{sdim}\, V = m|n)$, then $V \cong \mathbb{C}^{m|n}$.

Space $\mathcal{M}(k|l \times m|n; \mathbb{C})$ of $(k+l) \times (m+n)$ matrices is $\mathbb{Z}_2$-graded.

Write each matrix in block form $\begin{array}{c c} & \begin{array}{cc} m & n \end{array} \\ \begin{array}{c} k \\ l \end{array} & \begin{pmatrix} X & \Phi \\ \Psi & Y \end{pmatrix} \end{array}$, then

$$\mathcal{M}(k|l \times m|n; \mathbb{C})_{\bar{0}} = \left\{ \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \right\},$$

$$\mathcal{M}(k|l \times m|n; \mathbb{C})_{\bar{1}} = \left\{ \begin{pmatrix} 0 & \Phi \\ \Psi & 0 \end{pmatrix} \right\}.$$

Parity: $[v] = \bar{\alpha}$ if $v \in V_{\bar{\alpha}}$.

Classical Lie superalgebras

The general linear Lie superalgebra $\mathfrak{gl}(V)$ over $\mathbb{C}$ is $\mathrm{End}_{\mathbb{C}}(V)$ endowed with a bilinear Lie sperbracket

$$[X, Y] = XY - (-1)^{[X][Y]} YX, \quad X, Y \in \mathfrak{gl}(V).$$

For $\mathrm{sdim}\, V = (m|n)$, identify $\mathrm{End}_{\mathbb{C}}(V)$ with $\mathcal{M}(m|n) = \mathcal{M}(m|n \times m|n)$.

$$\mathfrak{gl}(V)_{\bar{0}} = \left\{ \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \right\} = \mathfrak{gl}(V_{\bar{0}}) \oplus \mathfrak{gl}(V_{\bar{1}}),$$

$$\mathfrak{gl}(V)_{\bar{1}} = \left\{ \begin{pmatrix} 0 & \Phi \\ \Psi & 0 \end{pmatrix} \right\}.$$

$GL(V)_0 := GL(V_{\bar{0}}) \times GL(V_{\bar{1}})$ acts on $\mathfrak{gl}(V)$ by conjugation; $GL(V)_0$ action on V is compatible with that of $\mathfrak{gl}(V)$. $(GL(V)_0, \mathfrak{gl}(V))$ is a "Harish-Chandra" pair.

Assume that V admits a non-degenerate even bilinear form

$$(\, . \, , \, .) : V \times V \longrightarrow \mathbb{C},$$

that is supersymmetric

$$(u, v) = (-1)^{[u][v]}(v, u), \quad u, v \in V.$$

Call this an orthosymplectic form.

The orthosymplectic Lie superalgebra $\mathfrak{osp}(V)$ is the Lie sub-superalgebra of $\mathfrak{gl}(V)$ given by

$$\mathfrak{osp}(V) = \{X \in \mathfrak{gl}(V) \mid (Xv, w) + (-1)^{[X][v]}(v, Xw) = 0, \ \forall v, w \in V\}.$$

Let $\mathrm{OSp}(V)_0 = \mathrm{O}(V_{\bar{0}}) \times \mathrm{Sp}(V_{\bar{1}})$. We have the Harish-Chandra pair $(\mathrm{OSp}(V)_0, \mathfrak{osp}(V))$.

FFT for $\mathfrak{gl}(V)$

$$V^{\otimes r} = \underbrace{V \otimes \cdots \otimes V}_{r}.$$

The $\mathrm{GL}(V)_0$ action on $V^{\otimes r}$:
for all $w = w_1 \otimes \ldots \otimes w_r$ and $g \in \mathrm{GL}(V)_0$,

$$g.w = gw_1 \otimes \ldots \otimes gw_r.$$

The $\mathfrak{gl}(V)$-action on $V^{\otimes r}$: for all $X \in \mathfrak{gl}(V)$,

$$\begin{aligned} X.w = {} & Xw_1 \otimes w_2 \otimes \cdots \otimes w_r + (-1)^{[w_1][X]} w_1 \otimes Xw_2 \otimes \cdots \otimes w_r \\ & + \cdots + (-1)^{[X]\sum_{i=1}^{r-1}[w_i]} w_1 \otimes w_2 \otimes \cdots \otimes Xw_r. \end{aligned}$$

Denote the associated rep of $(\mathrm{GL}(V)_0, \mathfrak{gl}(V))$ on $V^{\otimes r}$ by ρ_r.

The permutation

$$\tau : V \otimes V \longrightarrow V \otimes V, \quad v \otimes w \mapsto (-1)^{[v][w]} w \otimes v.$$

Define a representation ν_r of Sym_r on $V^{\otimes r}$ such that the simple reflections $s_i = (i, i+1)$ are mapped to

$$\nu_r(s_i) : w \mapsto w_1 \otimes \cdots \otimes \tau(w_i \otimes w_{i+1}) \otimes \cdots \otimes w_r.$$

Lemma

The actions of $\mathbb{C}\mathrm{Sym}_r$ *and* $(\mathrm{GL}(V)_0, \mathfrak{gl}(V))$ *on* $V^{\otimes r}$ *commute.*

$\mathrm{End}_{(\mathrm{GL}(V)_0, \mathfrak{gl}(V))}(V^{\otimes r}) \subset \mathrm{End}_{\mathbb{C}}(V^{\otimes r})$ such that any element ϕ satisfies

$$\rho_r(g)\phi = \phi\rho_r(g), \qquad \rho_r(X)\phi = \phi\rho_r(X),$$
$$\forall g \in \mathrm{GL}(V)_0, \ X \in \mathfrak{gl}(V).$$

We have $\mathrm{End}_{(\mathrm{GL}(V)_0, \mathfrak{gl}(V))}(V^{\otimes r}) = \mathrm{End}_{\mathfrak{gl}(V)}(V^{\otimes r})$.

FFT for $\mathfrak{gl}(V)$

Theorem (Berele - Regev)

$$\mathrm{End}_{\mathfrak{gl}(V)}(V^{\otimes r}) = \nu_r(\mathbb{C}\mathrm{Sym}_r); \quad \mathrm{End}_{\mathrm{Sym}_r}(V^{\otimes r}) = \rho_r(U(\mathfrak{gl}(V))).$$

An application. Under the joint action of $\mathfrak{gl}(V)$ and Sym_r,

$$V^{\otimes r} = \bigoplus_{\lambda} L(\lambda^\sharp) \otimes D_\lambda,$$

$L(\lambda^\sharp)$, simple $\mathfrak{gl}(V)$-module with highest weight $\lambda^\sharp$,
D_λ, simple Sym_r-module associated with partition λ of r,

$$\lambda = (\lambda_1 \geq \lambda_2 \geq \dots);$$

for $\mathrm{sdim}\, V = (m|n)$, $\lambda \vdash r$ appears in the sum iff $\lambda_{m+1} \leq n$;
let $\mu = (\lambda_{m+1}, \lambda_{m+2}, \dots)$, and let μ' be its transpose; then
$\lambda^\sharp = (\lambda_1, \dots, \lambda_m, \mu')$ as $\mathfrak{gl}_m \times \mathfrak{gl}_n$-weight.

FFT for $\mathfrak{osp}(V)$

Assume that V has an orthsymplectic form and $\mathrm{sdim}\, V = (m|2n)$. Let $\{b_i\}$ be a homogeneous basis of V, and let $\{\bar{b}_i\}$ be the dual basis such that $(\bar{b}_i, b_j) = \delta_{ij}$. Then

$$c_0 = \sum_{i=1}^{m+2n} b_i \otimes \bar{b}_i$$

is an $(\mathrm{OSp}(V)_0, \mathfrak{osp}(V))$- inavraint in $V \otimes V$. Define linear maps

$$\begin{aligned} \check{C} &: \mathbb{C} \longrightarrow V \otimes V, \qquad 1 \mapsto c_0, \\ \hat{C} &: V \otimes V \longrightarrow \mathbb{C}, \qquad v \otimes w \mapsto (v, w). \end{aligned}$$

Both maps are clearly $(\mathrm{OSp}(V)_0, \mathfrak{osp}(V))$-module homomorphisms. Let

$$e = \check{C} \circ \hat{C} : V \otimes V \longrightarrow V \otimes V.$$

Note that $e^2 = (m - 2n)e$.

Lemma

There is a representation of the Brauer algebra $\nu_r : B_r(m-2n) \longrightarrow \mathrm{End}_{\mathbb{C}}(V^{\otimes r})$ *defined by*

$$\nu_r(s_i) = (\mathrm{id}_V)^{\otimes(i-1)} \otimes \tau \otimes (\mathrm{id}_V)^{\otimes(r-i-1)},$$
$$\nu_r(e_i) = (\mathrm{id}_V)^{\otimes(i-1)} \otimes e \otimes (\mathrm{id}_V)^{\otimes(r-i-1)}.$$

FFT for $\mathfrak{osp}(V)$

Theorem (Lehrer-Zhang)

Assume $\mathrm{sdim}\, V = (m|2n)$. *Then*

$$\mathrm{End}_{(\mathrm{OSp}(V)_0, \mathfrak{osp}(V))}(V^{\otimes r}) = \nu_r(B_r(m-2n))$$

as associative superalgebra for all r.

- Remark. FFTs for O_n and Sp_{2n} are special cases of this result.
- Remark. The rest of the lecture explains the proof of the result. For this, we need to work with Lie supergroups.

Classical supergroups

$\Lambda(N)$, the exterior algebra of an N-dimensional vector space;
an asssociative superalgebra with
$\Lambda(N)_{\bar{0}}$ consisting of the even degree elements,
$\Lambda(N)_{\bar{1}}$ consisting of the odd degree elements.
Call this superalgebra the Grassmann algebra of degree N.

The Grassmann algebra of infinite degree is the direct limit

$$\Lambda := \varinjlim \Lambda(N).$$

Let $V_\Lambda = V \otimes \Lambda$ for any $\mathbb{Z}_2$-graded vector space V; regard V_Λ as a Λ-bimodule: for $v \in V$, $\lambda, \mu \in \Lambda$,

$$(v \otimes \mu)\lambda \quad = v \otimes \mu\lambda = (-1)^{[\lambda]([v]+[\mu])}\lambda(v \otimes \mu).$$

$\mathrm{Hom}_\Lambda(V_\Lambda, W_\Lambda)$ consists of Λ-bimodule maps $\phi : V_\Lambda \longrightarrow W_\Lambda$

$$\phi(v\lambda) = \phi(v)\lambda, \quad v \in V_\lambda,\ \lambda \in \Lambda.$$

$V_\Lambda^* := \mathrm{Hom}_\Lambda(V_\Lambda, \Lambda) = V^* \otimes \Lambda.$

The general linear supergroup $\mathrm{GL}(V_\Lambda)$ is

$$\mathrm{GL}(V_\Lambda) = \{g \in \mathrm{End}_\Lambda(V_\Lambda)_{\bar{0}} \mid g \ \text{ invertible}\}$$

with multiplication being the composition of endomorphisms.

If $\mathrm{sdim}\, V = (m|n)$, then $\mathrm{End}_\Lambda(V_\Lambda)_{\bar{0}} \cong \mathcal{M}(m|n;\Lambda)_{\bar{0}}$

$$\mathcal{M}(m|n;\Lambda)_{\bar{0}} = \left\{ \begin{array}{c} \\ m \\ n \end{array} \begin{array}{c} \begin{array}{cc} m & n \end{array} \\ \begin{pmatrix} A & \Phi \\ \Psi & B \end{pmatrix} \end{array} \middle| \begin{array}{ll} A, B & \text{entries are in } \Lambda_{\bar{0}} \\ \Phi, \Psi & \text{entries are in } \Lambda_{\bar{1}} \end{array} \right\}.$$

Matrix $\begin{pmatrix} A & \Phi \\ \Psi & B \end{pmatrix} \in \mathcal{M}(m|n;\Lambda)_{\bar{0}}$ is invertible iff A^{-1} and B^{-1} exist.

If V admits an orthosymplectic form $(\ .\ ,\ .)$, extend it to a Λ-bilinear form $(-,-) : V_\Lambda \times V_\Lambda \longrightarrow \Lambda$, by

$$(\lambda v, w) = \lambda(v, w), \quad (v, \lambda w) = (-1)^{[\lambda][v]}\lambda(v, w).$$

The orthosymplectic supergroup of V_Λ is the subgroup of $\mathrm{GL}(V_\Lambda)$ given by

$$\mathrm{OSp}(V_\Lambda) := \{g \in \mathrm{GL}(V_\Lambda) \mid (gv, gw) = (v, w),\ \forall v, w \in V_\Lambda\}$$

Invariant theory over Λ

Sym_r acts on $T^r(V_\Lambda) := V_\Lambda^{\otimes_\Lambda r}$ and $T^r(V_\Lambda^*) := V_\Lambda^{*\otimes_\Lambda r}$ by permuting factors with appropriate signs.

$GL(V_\Lambda)$-action on V_Λ^*: for $\bar{v} \in V_\Lambda^*$, $g \in \mathrm{GL}(V_\Lambda)$,

$$g.\bar{v}(w) = \bar{v}(g^{-1}w), \quad \forall w \in V_\Lambda.$$

This extends to $T^r(V_\Lambda^*)$ diagonally. Denote the corresponding representation by ρ_r^*.

Denote by $\gamma : T^r(V_\Lambda^*) \otimes_\Lambda T^r(V_\Lambda) \longrightarrow \Lambda$ the natural pairing:

$$\gamma : \bar{\mathbf{v}} \otimes \mathbf{w} \mapsto (-1)^{J(v,w)} \bar{v}_1(w_1)\bar{v}_2(w_2)...\bar{v}_r(w_r),$$

for $\bar{\mathbf{v}} = \bar{v}_1 \otimes ... \otimes \bar{v}_r$ and $\mathbf{w} = w_1 \otimes ... \otimes w_r$, where $J(v,w) = \sum_{\mu=1}^r d_\mu$ with $d_r = 0$ and $d_\mu = [w_\mu]([\bar{v}_{\mu+1}] + [\bar{v}_{\mu+2}] + \cdots + [\bar{v}_r])$.

FFT for $GL(V_\Lambda)$

FFT for $GL(V_\Lambda)$

Theorem

Let $\Phi : T^r(V_\Lambda^*) \otimes_\Lambda T^s(V_\Lambda) \longrightarrow \Lambda$ be a Λ-linear $\mathrm{GL}(V_\Lambda)$-invariant function. Then $\Phi \neq 0$ only when $r = s$, and in this case, Φ belongs to the Λ-span of functions $\gamma_\pi = \gamma \circ (\mathrm{id} \otimes \pi)$, where $\pi \in \mathrm{Sym}_r$.

This can be easily deduced from FFT for $\mathfrak{gl}(V)$.

Key Lemma for $\mathrm{OSp}(V_\Lambda)$

For any W_Λ, regard $(W_\Lambda^*)_{\bar{0}}$ as $\Lambda_{\bar{0}}$-module. Let $S((W_\Lambda^*)_{\bar{0}})$ be the symmetric algebra of $(W_\Lambda^*)_{\bar{0}}$ over $\Lambda_{\bar{0}}$. Let

$$\mathcal{P}[(W_\Lambda)_{\bar{0}}] := S((W_\Lambda^*)_{\bar{0}}) \otimes_{\Lambda_{\bar{0}}} \Lambda.$$

For $f \in \mathcal{P}[(W_\Lambda)_{\bar{0}}]$ homogeneous of degree r,

$$f : w \mapsto f(\underbrace{w \otimes \cdots \otimes w}_{r}), \quad w \in (W_\Lambda)_{\bar{0}}.$$

Call $\mathcal{P}[(W_\Lambda)_{\bar{0}}]$ the superalgebra of polynomial functions on $(W_\Lambda)_{\bar{0}}$.

Assume that V_Λ is an orthosymplectic superspace.

$\mathrm{OSp}(V_\Lambda)$ action on $\mathcal{P}[\mathrm{End}_\Lambda(V_\Lambda)_{\bar{0}}]$:
for any $f \in \mathcal{P}[\mathrm{End}_\Lambda(V_\Lambda)_{\bar{0}}]$ and $g \in \mathrm{OSp}(V_\Lambda)$,

$$(g \cdot f)(A) = f(g^{-1}A), \quad \forall A \in \mathrm{End}_\Lambda(V_\Lambda)_{\bar{0}},$$

where $g^{-1}A$ is the composition of g^{-1} and A.

The Key Lemma

Lemma (Lehrer-Zhang)

Let $G = \mathrm{OSp}(V_\Lambda)$ *and* $\mathcal{P} = \mathcal{P}[\mathrm{End}_\Lambda(V_\Lambda)_{\bar{0}}]$. *Denote by* $\mathcal{P}^G$ *the subalgebra of* $\mathcal{P}$ *consisting of* G*-invariant functions. Given any* $f \in \mathcal{P}^G$, *we have a polynomial function* Φ *such that* $f(A) = \Phi(A^\dagger A)$, *where* $A^\dagger$ *is the adjoint of* A *wrt the orthosymplectic form.*

Remarks:

- Most of the work in proving FFT for $\mathfrak{osp}(V)$ lies in establishing the Key Lemma.
- One proof generalises to $OSp(V_\Lambda)$ an analogous result for the orthogonal group in Atiyah-Bott-Patodi (1973).
- This proof will be sketched at the end.

FFT for $\mathrm{OSp}(V_\Lambda)$

Define the Λ-linear map

$$\delta : T^{2d}(V_\Lambda) \longrightarrow \Lambda, \quad \mathbf{v} \mapsto \overrightarrow{\prod_{1\le i\le d}} (v_{2i-1}, v_{2i}),$$

where

$\mathbf{v} = v_1 \otimes v_2 \otimes \cdots \otimes v_{2d}$,

$\overrightarrow{\prod}_{1\le i\le d} (v_{2i-1}, v_{2i}) = (v_1, v_2)(v_3, v_4)(v_{2d-1}, v_{2d})$.

Corresponding to each $\pi \in \mathrm{Sym}_{2d}$, we have the function

$$\delta \circ \pi : T^{2d}(V_\Lambda) \longrightarrow \Lambda.$$

FFT for $\mathrm{OSp}(V_\Lambda)$

Theorem (Lehrer-Zhang, Sergeev)

Let $G = \mathrm{OSp}(V_\Lambda)$. Set $W = T^r(V_\Lambda^*)$ and denote by W^G the Λ-submodule of G-invariants of W.

1. *If r is odd, then $W^G = 0$.*
2. *If $r = 2d$ is even, then W^G is spanned over Λ by $\{\delta \circ \pi \mid \pi \in \mathrm{Sym}_{2d}\}$.*

Remark. The theorem is proved by combining the Key Lemma with FFT for $GL(V_\Lambda)$ then doing gymnastics with $\mathrm{OSp}(V_\Lambda)$-actions and unusal $GL(V_\Lambda)$-actions on $\mathcal{P}[\mathrm{End}_\Lambda(V_\Lambda)_{\bar{0}}]$ and $\mathcal{P}[\mathrm{End}_\Lambda(V_\Lambda)_{\bar{0}}] \otimes T(V_\Lambda^*)$.

On the proof of FFT for $\mathfrak{osp}(V)$

Stripping off Λ from FFT for $\mathrm{OSp}(V_\Lambda)$, we obtain

$$\left(V^{*\otimes 2d}\right)^{(G_0,\mathfrak{osp}(V))} = \mathbb{C}\text{-span}\{\delta_0 \circ \pi \mid \pi \in \mathrm{Sym}_{2d}\},$$

where δ_0 is the function δ restricted to $V^{\otimes r}$.

As vector spaces

$$\mathrm{End}_{(G_0,\mathfrak{osp}(V))}(V^{\otimes d}) \cong \left(V^{*\otimes 2d}\right)^{(G_0,\mathfrak{osp}(V))}.$$

This enables us to prove FFT for $\mathfrak{osp}(V)$.

Some Remarks

- Neither the $\mathfrak{osp}(V)$-action nor the $Br(m-2n)$-action on $V^{\otimes r}$ is semi-simple in general. Thus the decomposition of $V^{\otimes r}$ with respect to the $(\mathfrak{osp}(V), Br(m-2n))$-action will be much more complicated than in the $\mathfrak{gl}(V)$ case.
- The Baruer algebra $Br(k)$ is a cellular algebra in the sense of Graham-Lehrer. Its representation theory has been studied extensively in recent years.
- Results on $\mathfrak{gl}(V)$ and $\mathfrak{osp}(V)$ reported here have been generalised to the corresponding quantum supergroups [Lehrer-H.Zhang -R.Zhang] at generic q.

Sketch of a proof of the Key Lemma

Now we go back to the Key Lemma.

By identifying $\mathrm{End}(V_\Lambda)$ with $\mathcal{M}(m|n;\Lambda)$, we can reformulate the Key Lemma in terms of matrices.

Let $\mathcal{SA}(\Lambda)_{\bar{0}}$ be the set of the self-adjoint even matrices. Let

$$S = (s_{kl}) = \begin{array}{c} \\ m \\ 2n \end{array} \begin{array}{c} \begin{array}{cc} m & 2n \end{array} \\ \begin{pmatrix} (x_{ab}) & (z_{ji}) \\ (z_{ij}) & (y_{cd}) \end{pmatrix} \end{array}$$

be the "generic" self-adjoint even matrix. That is, x_{ab} and y_{cd} are variables taking values in $\Lambda_{\bar{0}}$, and z_{ji} taking values in $\Lambda_{\bar{1}}$.

Definition

1. Define the superalgebra $\mathcal{R}_{\mathbb{C}} := R \otimes_{\mathbb{C}} \wedge(\sum \mathbb{C} z_{ij})$, where $R := \mathbb{C}[x_{ab}, y_{cd}]$. Let $\widetilde{\Lambda} = \wedge(\sum \mathbb{C} z_{ij}) \otimes_{\mathbb{C}} \Lambda$.
2. $\Lambda_R := R \otimes \widetilde{\Lambda}$ may be identificed with the superalgebra $\mathcal{P}[\mathcal{SA}_{\bar{0}}]$ of polynomial functions on $\mathcal{SA}(\Lambda)_{\bar{0}}$.
3. K is the quotient field of R.
4. $L \supseteq K$ is the splitting field of the polynomial $\det(t^2 - (x_{ab}))_{1 \leq a,b \leq m}$ over K.
5. $\Lambda_K := K \otimes_R \Lambda_R = K \otimes_{\mathbb{C}} \widetilde{\Lambda}$,
   $\Lambda_L := L \otimes_K \Lambda_K = L \otimes_{\mathbb{C}} \widetilde{\Lambda}$.

If $\alpha : \Lambda_R \to \Lambda$ is a Λ-algebra homomorphism, we say that $\alpha(S) = (\alpha(s_{kl}))$ is a *specialisation* of S.

Lemma

Let $S = (s_{kl})$ be the 'generic matrix' in $\mathcal{SA}(\Lambda_R)_{\bar{0}} \subseteq \mathcal{SA}(\Lambda_L)_{\bar{0}}$. For any subring R' of Λ_L, write $Q : \mathcal{M}(m|2n; R') \longrightarrow \mathcal{SA}(R')$ for the quadratic map given by $Q(A) = A^{\dagger}A$.

1. *There is a matrix $X \in \mathcal{M}(m|2n; \Lambda_L)_{\bar{0}}$ such that $Q(X) = S$.*
2. *Given two such matrices X, X', there is an element $g \in \mathrm{OSp}(\Lambda_L)$ such that $X' = gX$.*

Remark. The field L is designed to make part (1) of the lemma hold.

Notation: $\mathrm{OSp}(\Lambda) = \mathrm{OSp}(V_\Lambda)$, $\mathrm{OSp}(\Lambda_L) = \mathrm{OSp}(V_{\Lambda_L})$ etc.

Suppose that $f : \mathcal{M}(m|2n;\Lambda)_{\bar{0}} \longrightarrow \Lambda$ is a polynomial function such that $f(gA) = f(A)$ for all $A \in \mathcal{M}(m|2n;\Lambda)_{\bar{0}}$ and $g \in \mathrm{OSp}(\Lambda)$.

Lemma

The function f has a unique extension $f_L : \mathcal{M}(m|2n;\Lambda_L)_{\bar{0}} \longrightarrow \Lambda_L$ such that for $A \in \mathcal{M}(m|2n;\Lambda_L)_{\bar{0}}$ and $g \in \mathrm{OSp}(\Lambda_L)$, $f_L(gA) = f_L(A)$.

Proof.

The function f clearly has an extension to $\mathcal{M}(m|2n;\Lambda_L)_{\bar{0}}$. Consider $d : \mathrm{OSp}(\Lambda_L) \times \mathcal{M}(m|2n;\Lambda_L)_{\bar{0}} \longrightarrow \Lambda_L$, given by $d(g, A) = f_L(gA) - f_L(A)$. Then d vanishes on the Λ-points of the domain, and this implies that $d = 0$. $\square$

Lemma

If $X \in \mathcal{M}(m|2n;\Lambda_L)_{\bar{0}}$ is such that $Q(X) = S$, then $f_L(X) \in \Lambda_K$.

Proof.

The Galois group $\mathrm{Gal}(L/K)$ acts on $\Lambda_L = L \otimes_{\mathbb{C}} \widetilde{\Lambda}$ by application to the first factor (L), and has fixed subalgebra Λ_K.

Applying $\sigma \in \mathrm{Gal}(L/K)$ to the equation $X^\dagger X = S$, we see that $X' = \sigma(X)$ also satisfies $(X')^\dagger X' = S$, whence by earlier lemma, we have $X' = gX$ for some $g \in \mathrm{OSp}(\Lambda_L)$.

Therefore, $f_L(\sigma(X)) = f_L(gX) = f_L(X)$. That is, for each $\sigma \in \mathrm{Gal}(L/K)$, $\sigma(f_L(X)) = f_L(X)$, and so $f_L(X) \in \Lambda_K$. $\square$

Proof of Key Lemma

It follows from the last Lemma that $f_L(X)$ is a finite sum of terms of the form $\frac{f_1^\theta}{f_2^\theta} \otimes \theta$, where θ runs over a set of linearly independent elements of $\widetilde{\Lambda}_{\bar{1}}$, and $f_i^\theta \in R = \mathbb{C}[x_{ab}, y_{cd}]$.

Now let $\alpha : \Lambda_R \to \Lambda$ be a Λ-algebra homomorphism such that $\alpha(x_{ab})$ and $\alpha(y_{cd}) \in \Lambda_{\bar{0}}$, $\alpha(z_{ij}) \in \Lambda_{\bar{1}}$ and satisfies other appropriate conditions, in particular, the matrix X explicitly constructed from S earlier has a specialisation $X_0 := \alpha(X)$. All such specialisations of X define a Zariski-dense subset of $\mathcal{SA}(\Lambda)_{\bar{0}}$. Write $S_0 = \alpha(S) \in \mathcal{SA}(\Lambda)_{\bar{0}}$.

Thus if $A \in \mathcal{M}(m|2n; \Lambda)_{\bar{0}}$ is such that $Q(A) = \alpha(S) = S_0$, then $A = gX_0$ for some $g \in \mathrm{OSp}(\Lambda)$, whence $f(A) = f(X_0) = \sum_\theta \frac{f_1^\theta(X_0)}{f_2^\theta(X_0)} \otimes \alpha(\theta)$.

Fix a $\theta_0 \in \widetilde{\Lambda}_{\bar{1}}$. Since $f(X_0)f_2^{\theta_0}(X_0) = \sum_\theta \frac{f_1^\theta(X_0)f_2^{\theta_0}(X_0)}{f_2^\theta(X_0)} \otimes \alpha(\theta)$, we see that $f_2^{\theta_0}(X_0) = 0$ implies that $f_1^{\theta_0}(X_0) = 0$. As this is true for a Zariski-dense set of X_0, this shows that $\frac{f_1^\theta}{f_2^\theta} \in R$.

We have now shown that for A in a Zariski-dense subset of $\mathcal{M}(m|2n;\Lambda)_{\bar{0}}$, the $\mathrm{OSp}(\Lambda)$ invariant function $f(A)$ is a polynomial in the entries of $Q(A)$. It follows from density that this is true for all $A \in \mathcal{M}(m|2n;\Lambda)_{\bar{0}}$.

This completes the proof of the Key Lemma.

Remark. We also have a more down to earth proof of the Key Lemma, which uses only a generalised Gram-Schmidt process for orthosymplectic superspaces and some density arguments.

Thank you for your attention!