The (relative) BGG machinery lecture 2

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Review of lecture 1

- Representation theory imposes strong restrictions on existence of conformally invariant differential operators. This extends in a similar fashion to operators acting between irreducbile bundles for other parabolic geometries.
- For a parabolic geometries of type (G, P) representations of P induce natural bundles and P–equivariant maps give rise to natural bundle maps.
- This gives rise to "new" geometric objects like the adjoint tractor bundle $\mathcal{A}M$. Such bundles often come with natural filtrations or other additional structures.
- The Cartan connection gives rise to the fundamental derivative, $D : \Gamma(\mathcal{A}M) \times \Gamma(E) \to \Gamma(E)$ for any natural bundle E, which is similar to a covariant derivative.
- We will now use this to construct invariant operators on irreducible bundles.

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This is a fairly technical result in pure representation theory, and I'll start by explaining its relvance for our puropses. Even more than the result itself, we will need the algebraic Hodge theory that Kostant introduced for proving it. Let us recall the neccesary setup:

Choosing a parabolic subalgebra $\mathfrak{p} \subset \mathfrak{g}$, we obtain a nilpotent ideal $\mathfrak{p}_+ \subset \mathfrak{p}$ (the nilradical of $\mathfrak{p}$). In particular representations of $\mathfrak{g}$ and $\mathfrak{p}$ can be restricted to $\mathfrak{p}_+$.

Given a representation $\mathbb{W}$ of $\mathfrak{p}_+$, one defines the *Lie algebra homology* $H_*(\mathfrak{p}_+, \mathbb{W})$ via a standard complex $(C_*(\mathfrak{p}_+, \mathbb{W}), \partial^*)$ as:

$$C_k(\mathfrak{p}_+, \mathbb{W}) = \Lambda^k \mathfrak{p}_+ \otimes \mathbb{W} \qquad \partial^* : C_k(\mathfrak{p}_+, \mathbb{W}) \to C_{k-1}(\mathfrak{p}_+, \mathbb{W})$$

$$\partial^*(Z_1 \wedge \cdots \wedge Z_k \otimes w) := \textstyle\sum_i (-1)^i Z_1 \wedge \cdots \widehat{Z_i} \cdots \wedge Z_k \otimes Z_i \cdot w$$

$$+ \textstyle\sum_{i<j} (-1)^{i+j} [Z_i, Z_j] \wedge Z_1 \wedge \cdots \widehat{Z_i} \cdots \widehat{Z_j} \cdots \wedge Z_k \otimes w.$$

If $\mathbb{W}$ is a representation of P, then each $C_*(\mathfrak{p}_+, \mathbb{W})$ is a P–module and the maps ∂^* are P–equivariant. Hence the homology $H_*(\mathfrak{p}_+, \mathbb{W})$ carries a natural representation of P. A simple computation shows that $\mathfrak{p}_+$ acts trivially on the homoglogy, so this is a completely reducible representation of P. Kostant's theorem describes the representation $H_*(\mathfrak{p}_+, \mathbb{V})$ for an irreducible representation $\mathbb{V}$ of $\mathfrak{g}$. Phrased for our purposes, this reads as:

- $H_*(\mathfrak{p}_+, \mathbb{V})$ splits into a direct sum of different irreducible representations of P.
- The representations showing up in that sum correspond to those weights in the affine Weyl orbit of the weight determined by $\mathbb{V}$, which can be realized by finite dimensional representations of $\mathfrak{p}$.
- The homology degree in which a representation occurs is given by the length of the corresponding Weyl group element.

Before continuing, let us convert this to geometry. Given $(p : \mathcal{G} \to M, \omega)$, $\mathbb{V}$ defines a tractor bundle $\mathcal{V}M \to M$. Since $\mathfrak{p}_+$ is dual to $\mathfrak{g}/\mathfrak{p}$ via the Killing form, $\mathcal{G} \times_P \mathfrak{p}_+ \cong T^*M$. So $C_k(\mathfrak{p}_+, \mathbb{V})$ induces the bundle $\Lambda^k T^*M \otimes \mathcal{V}M$ of $\mathcal{V}M$–valued k–forms.

The differentials ∂^* induce natural bundle maps, and one obtains natural subbundles $\operatorname{im}(\partial^*) \subset \ker(\partial^*) \subset \Lambda^k T^*M \otimes \mathcal{V}M$.
These give rise to natural subquotients $\mathcal{H}_k^{\mathcal{V}} = \ker(\partial^*)/\operatorname{im}(\partial^*)$, which split into direct sums of irreducible bundles.
These summands (for all k) exhaust the bundles in one of the patterns discussed in lecture 1. This works exactly for affine Weyl orbits which contain a dominant integral weight.

So we have a conceptual way to construct all bundles in one pattern from forms with values in a tractor bundle. The hope then is to construct the operators in the pattern from analogs of the exterior derivative.

Kostant's algebraic Hodge theory

It turns out that there is a grading on $\mathfrak{g}$ of the form $\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_k$, which is compatible with the Lie bracket, such that $\mathfrak{p} = \oplus_{i \geq 0} \mathfrak{g}_i$ and $\mathfrak{p}_+ = \oplus_{i \geq 1} \mathfrak{g}_i$. In particular, for each ℓ, $\mathfrak{g}^\ell := \oplus_{i \geq \ell} \mathfrak{g}_i$ is a $\mathfrak{p}$–invariant subspace of $\mathfrak{g}$.

As a representation of $\mathfrak{g}_0$, we have $\mathfrak{g}/\mathfrak{p} \cong \mathfrak{g}_- = \oplus_{i<0} \mathfrak{g}_i$. Hence, over $\mathfrak{g}_0$, we have $C_k(\mathfrak{p}_+, \mathbb{V}) \cong C^k(\mathfrak{g}_-, \mathbb{V})$, and one obtains Lie algebra cohomology differentials $\partial : C_*(\mathfrak{p}_+, \mathbb{V}) \to C_{*+1}(\mathfrak{p}_+, \mathbb{V})$ such that $\partial \circ \partial = 0$.

It turns out that ∂ and ∂^* are adjoint with respect to a certain inner product, which implies that for $\Box := \partial^* \partial + \partial \partial^*$, one obtains a Hodge decomposition $C_k(\mathfrak{p}_+, \mathbb{V}) = \operatorname{im}(\partial^*) \oplus \ker(\Box) \oplus \operatorname{im}(\partial)$ with the first two summands adding up to $\ker(\partial^*)$ and the last two summands adding up to $\ker(\partial)$.

Since ∂ and $\Box$ are only invariant for the action of $\mathfrak{g}_0$ (respectively a corresponding subgroup $G_0 \subset P$), they do not directly admit geomtric counterparts. But there this a conceptual way to obtian such counterparts:

- Simiarly to $\mathfrak{g}$, $\mathbb{V}$ admits a grading such that $\mathfrak{g}_i \cdot \mathbb{V}_j \subset \mathfrak{g}_{i+j}$. This induces a P–invariant filtration on $\mathbb{V}$.
- Hence there are G_0–invariant gradings and P–invariant filtrations on the spaces $C_k(\mathfrak{p}_+, \mathbb{V})$ (by homogeneity), which both are preserved by ∂^*, ∂ and $\Box$.
- One can pass to the associated graded P–modules $gr(C_k(\mathfrak{p}_+, \mathbb{V}))$, which are isomorphic to $C_k(\mathfrak{p}_+, \mathbb{V})$ as G_0–modules but have trivial P_+–action.
- Then ∂^*, ∂ and $\Box$ can be viewed as P–equivariant maps on the associated graded spaces, thus allowing geometric counterparts.

The bundles $\Lambda^k T^*M \otimes \mathcal{V}M$ are filtered by smooth subbundles, and this filtration is preserved by ∂^*, hence inducing iltrations on the subbundles $\mathrm{im}(\partial^*) \subset \ker(\partial^*)$. Hence there are the associated graded bundles $\mathrm{gr}(\Lambda^k T^*M \otimes \mathcal{V}M)$ and ∂^* induces a bundle map $\underline{\partial^*}$ on them.

By construction, $\mathrm{gr}(\Lambda^k T^*M \otimes \mathcal{V}M)$ is induced by $\mathrm{gr}(C_k(\mathfrak{p}_+, \mathbb{V}))$, so there we have natural bundle maps ∂ and $\Box$ and a (point–wise) algebraic Hodge–decomposition.

We also get an induced filtration on $\Omega^k(M, \mathcal{V}M)$ and we assume that $\mathcal{D} : \Omega^k(M, \mathcal{V}M) \to \Omega^{k+1}(M, \mathcal{V}M)$ is a linear operator preserving this filtration.If φ is homogenous of degree $\geq \ell$, so is $\mathcal{D}(\varphi)$ and $\mathrm{gr}_\ell(\mathcal{D}(\varphi))$ depends only on $\mathrm{gr}_\ell(\varphi)$.Hence there is an induced linear operator
$\mathrm{gr}_0(\mathcal{D}) : \Gamma(\mathrm{gr}(\Lambda^k T^*M \otimes \mathcal{V}M)) \to \Gamma(\mathrm{gr}(\Lambda^{k+1} T^*M \otimes \mathcal{V}M))$.

Now suppose that $\mathcal{D}$ is *compressible*, i.e. filtration preserving and such that $\mathrm{gr}_0(\mathcal{D}) = \partial$. Then the restriction of $\partial^*\mathcal{D}$ defines an operator $\Gamma(\ker(\partial^*)) \to \Gamma(\mathrm{im}(\partial^*))$. Denoting by π_H the tensorial projection $\Gamma(\ker(\partial^*)) \to \Gamma(\mathcal{H}_k^{\mathcal{V}})$, we get:

1. π_H restricts to a linear isomorphism $\ker(\partial^*\mathcal{D}) \to \Gamma(\mathcal{H}_k^{\mathcal{V}})$.
2. The inverse of this isomorphism is induced by a an operator S which can be written as a (universal) polynomial in $\partial^* D$.
3. One obtains an induced operator $D : \Gamma(\mathcal{H}_k^{\mathcal{V}}) \to \Gamma(\mathcal{H}_{k+1}^{\mathcal{V}})$ by putting $D(\alpha) := \pi_H(\mathcal{D}(S(\alpha)))$.

Idea of proof for 1: Show $\ker(\partial^*\mathcal{D}) \cap \Gamma(\mathrm{im}(\partial^*)) = \{0\}$. Suppose that $\partial^*\mathcal{D}\partial^*\psi = 0$ and $\partial^*\psi$ is homogeneous of degree $\geq \ell$. Apply gr_ℓ to conclude that $0 = \underline{\partial^*}\partial(\mathrm{gr}_\ell(\partial^*\psi))$. But on $\Gamma(\mathrm{im}(\underline{\partial}^*))$, $\underline{\partial^*}\partial$ coincides with $\Box$ and thus is injective by the Hodge decomposition. So $\mathrm{gr}_\ell(\partial^*\psi) = 0$ and $\partial^*\psi$ is homogenous of one higher degree.

The twisted exterior derivative

Observe that $\Omega^k(M, \mathcal{V}M) \subset \Gamma(\Lambda^k \mathcal{A}^* M \otimes \mathcal{V}M)$. Start constructing a compressible operator on the latter bundle. For a section φ, form $D\varphi$ and alternate to obtain a section of $\Lambda^{k+1}\mathcal{A}^*M \otimes \mathcal{V}M$. The Lie algebra cohomology differential for $\mathfrak{g}$ acts on the spaces $\Lambda^*\mathfrak{g}^* \otimes \mathbb{V}$, thus inducing natural bundle maps $\partial_{\mathfrak{g}}$ between the same bundles.

Theorem

$\varphi \mapsto \mathrm{Alt}(D\varphi) + \partial_{\mathfrak{g}}\varphi$ maps $\Omega^k(M, \mathcal{V}M)$ to $\Omega^{k+1}(M, \mathcal{V}M)$, and defines a compressible differential operator $d^{\mathcal{V}}$ of order 1.

- Applying the BGG construction to $d^{\mathcal{V}}$, one obtains a sequence $D^{\mathcal{V}} : \Gamma(\mathcal{H}^{\mathcal{V}}_*) \to \Gamma(\mathcal{H}^{\mathcal{V}}_{*+1})$ of invariant differential operators.
- In degree 0, $d^{\mathcal{V}}$ defines a linear connection $\nabla^{\mathcal{V}}$ on $\mathcal{V}M$, the *tractor connection*.
- An operator $\mathcal{D}$ is compressible iff $\mathcal{D} = d^{\mathcal{V}} + \mathcal{E}$ for some $\mathcal{E}$ raising homogeneous degree. In particular, this applies to d^{∇}.