The (relative) BGG machinery
lecture 3

Andreas Čap[1]

University of Vienna
Faculty of Mathematics

Srni, January 2016

[1]supported by project P27072–N25 of the Austrian Science Fund (FWF)

Continuation of lecture 2

We have used the fundamental derivative on $\Gamma(\Lambda^k\mathcal{A}^*M \otimes \mathcal{V}M)$ and the Lie algebra cohomology differential for $\mathfrak{g}$ to construct an invariant differential operator $d^{\mathcal{V}} : \Omega^k(M, \mathcal{V}M) \to \Omega^{k+1}(M, \mathcal{V}M)$ on differential forms with values in a tractor bundle $\mathcal{V}M$.

The operator $d^{\mathcal{V}}$ is compressible, so one may apply the general procedure we have described to obtain a sequence of invariant differential operators $D^{\mathcal{V}} : \Gamma(\mathcal{H}_*^{\mathcal{V}}) \to \Gamma(\mathcal{H}_{*+1}^{\mathcal{V}})$. These are induced by polynomials in $\partial^* d^{\mathcal{V}}$ and have high order in general.

- In degree 0, $d^{\mathcal{V}}$ defines a linear connection $\nabla^{\mathcal{V}}$ on $\mathcal{V}M$, the *tractor connection*.
- An operator $\mathcal{D}$ is compressible iff $\mathcal{D} = d^{\mathcal{V}} + \mathcal{E}$ for some $\mathcal{E}$ raising homogeneous degree. In particular, this applies to d^{∇}.

Applications

The curvature $\kappa \in \Omega^2(M, \mathcal{A}M)$ of $\nabla^{\mathcal{A}}$ coincides with the Cartan curvature. The normalization condition on ω is exactly that $\kappa \in \Gamma(\ker(\partial^*))$. Hence we can form $\kappa_H := \pi_H(\kappa) \in \Gamma(\mathcal{H}_2^{\mathcal{A}})$, the *harmonic curvature*.

The Bianchi identity for linear connections reads as $d^{\nabla}\kappa = 0$, so denoting by S^{∇} the corresponding splitting operator, we get $\kappa = S^{\nabla}(\kappa_H)$. In particular, local flatness is equivalent to $\kappa_H \equiv 0$. More generally, this can be used efficiently to deduce restrictions on κ from restrictions on κ_H.

If $\kappa \equiv 0$, then each $\nabla^{\mathcal{V}}$ is flat, and $d^{\mathcal{V}} = d^{\nabla}$, so $(\Omega^*(M, \mathcal{V}M), d^{\mathcal{V}})$ is a resolution of the sheaf of parallel sections of $\mathcal{V}M$. This easily implies that $d^{\mathcal{V}} \circ S = S \circ D^{\mathcal{V}}$ and hence also $(\Gamma(\mathcal{H}_*^{\mathcal{V}}M), D^{\mathcal{V}})$ is a complex and S is a chain map. This induces an isomorphism in cohomology, so we obtain a *BGG resolution* in the locally flat case.

Applications 2

The adjoint BGG–sequence plays a special role in general:

There is a modification $\hat{\nabla}$ of $\nabla^{\mathcal{A}}$ such that the first steps in $(\Omega^*(M, \mathcal{A}M), d^{\hat{\nabla}})$ admit an interpretation in terms of infinitesimal automorphisms and deformations of the Cartan geometry. The corresponding BGG sequence converts this to the language of the underlying structure. In flat cases, and for some structures in semi–flat situations, one obtains deformation (sub–)complexes.

The connection $\hat{\nabla}$ has the property that its parallel sections are in bijective correspondence with the kernel of the first operator in the BGG sequence.
The construction of a natural modification of the tractor connection, whose parallel sections are in bijective correspondence with the kernel of the BGG operator has been generlized to arbitrary tractor bundles by Hammerl–Silhan–Somberg–Souček.

Applications 3

The first BGG–operator $D^{\mathcal{V}} : \Gamma(\mathcal{H}_0^{\mathcal{V}}) \to \Gamma(\mathcal{H}_1^{\mathcal{V}})$ has a kernel of dimension $\leq \dim(\mathbb{V})$, generically the kernel is trivial. If $s \in \Gamma(\mathcal{V}M)$ satisfies $\nabla^{\mathcal{V}} s = 0$, then $s = S(\pi_H(s))$ and $D(\pi_H(s)) = 0$. Hence parallel sections of $\mathcal{V}M$ correspond to special solutions of the first BGG operator ("normal solutions").

- Existence of solutions and normal solutions gives interesting geometric conditions generalizing conformal Killing equations on all kinds of tensors and the twistor equation on spinors.
- Via the associated parallel tractors, normal solutions of first BGG equations give rise to holonomy reductions of parabolic geometries with associated curved orbit decompositions.
- Working on manifolds with boundary, there are weakenings of some of the holonomy conditions, which define nice classes of geometric compactifications (c.f. conformal compactness).

Contents

Here we consider two nested parabolic subalgebras $\mathfrak{q} \subset \mathfrak{p} \subset \mathfrak{g}$ and groups $Q \subset P \subset G$. We will be interested in parabolic geometries of type (G, Q) while P is an additional input (and $P = G$ produces usual BGGs).

Since $\mathfrak{q} \subset \mathfrak{p}$, we get $\mathfrak{p}_+ \subset \mathfrak{q}_+$. Hence get a a refinement of the $\mathfrak{q}$–invariant filtration $\mathfrak{g} \supset \mathfrak{q} \supset \mathfrak{q}_+$ to $\mathfrak{g} \supset \mathfrak{p} \supset \mathfrak{q} \supset \mathfrak{q}_+ \supset \mathfrak{p}_+$. In particular, $\mathfrak{p}_+$ is an ideal in $\mathfrak{q}_+$, so $\mathfrak{q}_+/\mathfrak{p}_+$ naturally is a Lie algebra.

Relative natural bundles are associated to representations of Q, on which $\mathfrak{p}_+$ acts trivially. Starting from representations of P on which $\mathfrak{p}_+$ acts trivially, one obtaines *relative tractor bundles*.
Basic examples for relative bundles for $(p : \mathcal{G} \to M, \omega)$:

- relative tangent bundle $T_\rho M = \mathcal{G} \times_Q (\mathfrak{p}/\mathfrak{q}) \subset TM$
- relative cotangent bundle $T^*_\rho M = \mathcal{G} \times_Q (\mathfrak{q}_+/\mathfrak{p}_+)$
- relative adjoint tractor bundle $\mathcal{A}_\rho M = \mathcal{G} \times_Q (\mathfrak{p}/\mathfrak{p}_+)$.

Let $\mathbb{V}$ be a representation of P, on which $\mathfrak{p}_+$ acts trivially. By restricting, we obtain a representation of $\mathfrak{q}_+/\mathfrak{p}_+$ on $\mathbb{V}$. Hence there is the standard complex $(C_*(\mathfrak{q}_+/\mathfrak{p}_+, \mathbb{V}), \partial^*_\rho)$ computing the Lie algebra homology $H_*(\mathfrak{q}_+/\mathfrak{p}_+, \mathbb{V})$.

The maps ∂^*_ρ are Q–equivariant, so they induce natural bundle maps on the bundles $\Lambda^k T^*_\rho M \otimes \mathcal{V}M$ of relative forms with values in the relative tractor bundle induced by $\mathbb{V}$. Hence there are natural subbundles $\operatorname{im}(\partial^*_\rho) \subset \ker(\partial^*_\rho)$ and their quotient $\mathcal{H}^{\mathcal{V},\rho}_k$ is a completely reducible bundle.

Writing the grading of $\mathfrak{g}$ induced by $\mathfrak{q}$ as $\mathfrak{g} = \mathfrak{q}_- \oplus \mathfrak{q}_0 \oplus \mathfrak{q}_+$ we can decompose $\mathfrak{q}_- = \mathfrak{p}_- \oplus (\mathfrak{q}_- \cap \mathfrak{p}_0)$ and likewise for $\mathfrak{q}_+$. This shows that $(\mathfrak{q}_+/\mathfrak{p}_+)^* \cong \mathfrak{g}/\mathfrak{p}$ is, as a $\mathfrak{q}_0$–module, isomorphic to the Lie subalgebra $\mathfrak{q}_- \cap \mathfrak{p}_0$ of $\mathfrak{g}$.

Using these ingredients, one gets an anlog of Kostant's algebraic Hodge theory with $\mathfrak{q}_0$–homomorphisms ∂_ρ and $\Box_\rho$. These can again be interpreted as Q–homomorphisms on the associated graded with respect to a natural Q–invariant filtration.

One also obtains a relative version of Kostant's theorem, describing $H_*(\mathfrak{q}_+/\mathfrak{p}_+, \mathbb{V})$ as a direct sum of different irreducibles with weights determined by the action of a subset of the Weyl group on the weight of $\mathbb{V}$.

Translating this to geometry, we obtain ∂_ρ^* on the bundles of $\mathcal{V}M$–valued relative forms. On the associated graded bundles, we get $\underline{\partial}_\rho^*$, ∂_ρ and $\Box_\rho$, and a point–wise Hodge decomposition. Hence we can define compressible operators (filtration–preserving and inducing ∂_ρ on the associated graded) and apply the same machniery decribed before to construct operators on the bundles $\mathcal{H}_*^{\mathcal{V},\rho}$.

Let $E \to M$ be a relative natural bundle. For $\sigma \in \Gamma(E)$ we can form $D_s\sigma$ for sections s of $\mathcal{G} \times_Q \mathfrak{p} \subset \mathcal{A}M$. This vanishes for $s \in \Gamma(\mathcal{G} \times_Q \mathfrak{p}_+)$, so we obtain the *relative fundamental derivative* $D^\rho : \Gamma(E) \to \Gamma(\mathcal{A}^*_\rho M \otimes E)$.

Using this we define $d^{\mathcal{V}}_\rho$ as in the absolute case and prove that it is a compressible operator. Hence the BGG machinery produces operators $D^{\mathcal{V},\rho}$ on the bundles $\mathcal{H}^{\mathcal{V},\rho}_*$ ("relative BGG operators").

In degree 0, $d^{\mathcal{V}}_\rho$ defines a partial connnection $\nabla^{\mathcal{V},\rho}$ (differentiation only in $T_\rho M$–directions). In general, a partial connection cannot be coupled to the exterior derivative.
This works only if the distribution $T_\rho M \subset TM$ is involutive. In this case, there is a relative torsion $\tau_\rho \in \Gamma(\Lambda^2 T^*_\rho M \otimes T_\rho M)$. The operators $d^{\mathcal{V}}_\rho$ and d^{∇}_ρ differ by a term involving insertion of τ_ρ.

Involutivity of $T_\rho M$ is easily seen to be equivalent to the fact that the Cartan curvature $\kappa \in \Omega^2(M, \mathcal{A}M)$ maps $\Lambda^2 T^*_\rho M$ to the subbundle $\mathcal{G} \times_Q \mathfrak{p} \subset \mathcal{A}M$. Assuming thas this is the case, one can project it to the *relative curvature* $\kappa_\rho \in \Omega^2_\rho(M, \mathcal{A}_\rho M)$.

It $T_\rho M$ is involutive and κ_ρ vanishes identically, $d^{\mathcal{V}}_\rho$ coincides with d^∇_ρ, and $(\Omega^*_\rho(M, \mathcal{V}M), d^{\mathcal{V}}_\rho)$ is a fine resolution of the sheaf $\ker(\nabla^{\mathcal{V},\rho})$. Hence the relative BGG sequence is a complex, which resolves the same sheaf.

Vanishing of κ_ρ can be well analyzed via the harmonic curvature and is a rather weak condition. It is always satisfied for correspondence space associated to geometries of type (G, P), in which case $\ker(\nabla^{\mathcal{V},\rho})$ is identified with a sheaf of smooht sections of the natural bundle on the underlying geometry induced by $\mathbb{V}$. In general, the resolved sheafs locally descend to leaf spaces for the involutive distribution $T_\rho M$.

Relative vs. absolute BGG sequences

Let $\tilde{\mathbb{V}}$ be a representation of G. Then Kostant's theorem shows that each $\mathbb{V}_j := H_j(\mathfrak{p}_+, \tilde{\mathbb{V}})$ is a completely reducible representation of P, and hence determines a relative BGG sequence on geometries of type (G, Q). Combining Kostant's theorem and its relative analog, one obtains

$H_k(\mathfrak{q}_+, \tilde{\mathbb{V}}) = \oplus_{i+j=k} H_i(\mathfrak{q}_+/\mathfrak{p}_+, H_j(\mathfrak{p}_+, \tilde{\mathbb{V}}))$, so the bundles showing up in the aboslute BGG sequence determined by $\tilde{\mathbb{V}}$ are exactly the bundles which show up in the relative BGG sequences determined by the representations $\mathbb{V}_j$ for $j = 0, \dots, \dim(\mathfrak{p}_+)$.

Constructing BGGs from $d^{\tilde{\mathcal{V}}}$ and $d_\rho^{\mathcal{V}_j}$, respectively, the absolute and relative constructions produce the same operators between the bundles contained in one relative sequence.

For non–flat geometries with $\kappa_\rho \equiv 0$, this proves existence of many subcomplexes in each BGG sequence.

On the other hand, there are many irreducible representations of P, which cannot occur in representations of the form $H_i(\mathfrak{p}_+, \tilde{\mathbb{V}})$. In terms of weights, this either happens in singular infinitesimal character or in the case that the weight associated to $\mathbb{V}$ is non–integral. (This can be caused e.g. by non–integral density weights.)

In this case, one obtains "new" operators for which no general constructions were known before.

Thanks for your attention!