Conformal Geometry and a Theory of Everything

Srní 2016

c̄ Gover, Bonezzi, Corradini, Latini, Glaros, Halbasch

Quantum Gravity

- relativistic particle
- conformal symmetry
- 2T physics
- tractors
- gravity
- GJMS algebras
- BV AKSZ
- quantum gravity models

AdS/CFT

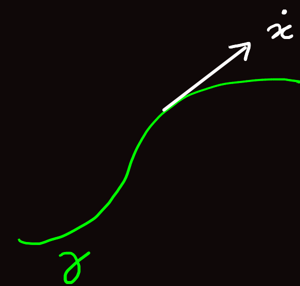
- calculus of scale
- Laplace–Robin operator
- solution generating algebra
- holographic formulae
- holographic renormalization
- wave equations
- Q curvature

Conformal Hypersurfaces

- entanglement entropy
- hypersurfaces & invariants
- conformal hypersurface invariants
- conformal infinity
- singular Yamabe problem
- Willmore invariant
- variational calculus
- higher Willmore energies

The Relativistic Particle

Riemannian arc length



$$S = \frac{1}{m}\int_\gamma \sqrt{g(\dot{x},\dot{x})}$$

Hamiltonian formulation

$$S = \int_\gamma \left[\theta - e\left(g^{-1}(p,p) + m^2 \right) \right]$$

Tautological 1-form $p_\mu dx^\mu$

$m^2 \to 0$ well-defined

Quantization

$$g^{-1}(p,p) \longmapsto -\Delta = H$$

Diffeos $\Rightarrow$ Ordering

Laplacian/d'Alembertian

Dirac: States

$$H\Psi = 0$$

Massless Klein-Gordon

Asymptotic one-particle collider states

Worldline diffeos $\Rightarrow$ constraint

Conformal Symmetry

Massless wave equation $SO(d,2)$ symmetry

Dirac $\Rightarrow$ Conformal wave equations in $\mathbb{R}^{4,2}$

Howe dual pair (maximal cocommutants)

$$sp(2(d+2)) \supset sp(2) \oplus so(d,2)$$

$sp(2) = so(1,2)$

worldline conformal $\rightarrow$ $so(1,2)$

spacetime conformal $\rightarrow$ $so(d,2)$

Marnelius $\Rightarrow$ $\mathbb{R}^{d,2}$ formulation of relativistic particle

Bars 2T physics $\rightarrow$ $\mathbb{R}^{d,2}$

Conformal $\mathbb{R}^{d,2}$ particle

$$S = \int_\gamma \Big[\underbrace{\oplus}_{P_M dX^M} - eH - \lambda D - \mu K \Big] \qquad \gamma : \mathbb{R} \to \mathbb{R}^{d,2}$$

$$sp(2) = \left\{ \begin{array}{lll} H = G^{-1}(P,P) & \text{translations} \\ D = X^M P_M & \text{dilations} \\ K = G(X,X) & \text{conformal boosts} \end{array} \right. \overset{\text{quantize}}{\longmapsto} \left\{ \begin{array}{c} \Delta \\ \nabla_X + \frac{D}{2} \\ X^2 \end{array} \right.$$

States $$\Delta \Psi = \left(\nabla_X + \frac{D}{2} \right) \Psi = X^2 \Psi = 0$$

"Singleton"

Tractors

Singleton is a tractor:

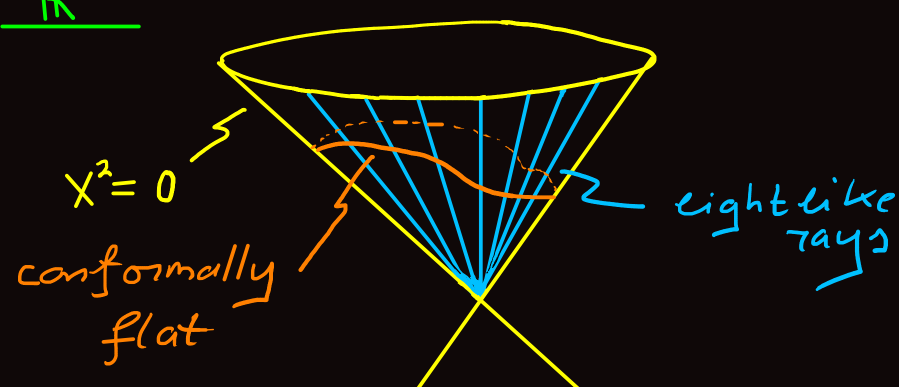
$$X^2 \Psi = 0 \quad \Rightarrow \quad \Psi = \delta(X^2)\,\psi \leftarrow \text{cone}$$

$$\therefore \quad \psi \sim \psi + X^2 \varphi$$

$$\nabla_X \Psi = -\frac{D}{2}\Psi \quad \Rightarrow \quad \nabla_X \psi = \left(1 - \frac{d}{2}\right)\psi \leftarrow \text{rays}$$

$$\therefore \quad \psi \;\; \text{Yamabe weight} \;\; w = 1 - \frac{d}{2}$$

$\mathbb{R}^{d,2}$



Tractor operators

Ambient space conformal group $SO(d+1,3)$

– momentum space representation

$$X^2 \Psi = 0$$ lightcone condition for massless excitations

Intertwine for "physical modes" $\psi \sim \psi + X^2 \varphi$

Translations	X^M	Canonical tractor
Lorentz	$X_M \nabla_N - X_N \nabla_M =: D_{MN}$	Double D-operator
Dilation	$\nabla_X =: w$	Weight
Conformal Boosts	$\nabla_M (d+2\nabla_X - 2) - X_M \Delta =: D_M$	Thomas-D

Curved Geometries

Fefferman–Graham metric

$$G_{MN} = \nabla_M X_N \overset{\Rightarrow}{=} \frac{1}{2}\nabla_M \nabla_N X^2$$

Ambient tractors (Čap–Gover)

$$\psi \sim \psi + X^2\varphi \,, \qquad \nabla_X \psi = \omega \psi$$

tensor spinor

weight

– Canonical tractor, double-D, weight, Thomas-D well defined

Deformed $SO(d+1,3)$ minimal representation (GW)

$$J(J+d) = 2$$

"Joseph Ideal"

Matrix of generators

Ex $D_M D^M = 0 = X_M X^M$

Tractor Bundle

Physics = Theory of densities $\longleftarrow$ $[g_{ab}; f] = [\Omega^2 g_{ab}; \Omega^w f]$
$\in \Gamma \mathcal{E}M[w]$

~~AREA + LENGTH~~

Local choices of units is symmetry

$$g_{\mu\nu} \longmapsto \Omega^2 g_{\mu\nu}$$

Weyl symmetry

gauge-fixed when coupling to gravity

Tractor bundle – covariantizes w.r.t. Weyl

$$V^M \in TM \cong \mathcal{E}M[1] \oplus TM[-1] \oplus \mathcal{E}M[-1]$$

Example 0, 4-velocity, 4-acceleration. 4-velocity

$$(V^M)^{\Omega^2 g} = \begin{pmatrix} \Omega & 0 & 0 \\ \Upsilon^m & \mathbb{1} & 0 \\ -\frac{\Upsilon^2}{2\Omega} & -\frac{\Upsilon_n}{\Omega} & \frac{1}{\Omega} \end{pmatrix} \begin{pmatrix} V^+ \\ V^n \\ V^- \end{pmatrix}^g = \underbrace{U^M{}_N}_{SO(d,2)} V^N, \quad \Upsilon = \Omega^{-1} d\Omega$$

Thomas D-operator

$$D^M : \Gamma(\mathcal{T}M[\omega]) \longrightarrow \Gamma(\mathcal{T}M[\omega-1])$$

$$\begin{array}{ccc} \Psi & & \Psi \\ f & \longmapsto & \begin{pmatrix} (d+2\omega-2)\,\omega f \\ (d+2\omega-2)\nabla^m f \\ -(\Delta+\omega \underbrace{J}_{\frac{Sc}{2(d-1)}})f \end{pmatrix} \end{array}$$

Null $\quad D^M D_M = 0$

Extends to tensors

Yamabe $\quad D^M = -X^M \Box_Y \,,\ \omega = 1-\frac{d}{2}\,,\ \Box_Y := \Delta + (1-\frac{d}{2})J$

Leibniz' failure $\quad \left(\hat{D} = (d+2\omega-2)^{-1} D\right)$

$$\hat{D}_M(fg) = (\hat{D}_M f)g + f(\hat{D}_M g) - \frac{2X_M}{d+2\omega_f+2\omega_g-2}\,\hat{D}_N f \cdot \hat{D}^N g$$

Gravity

Bailey, Eastwood, Gover

$$G^{\sigma^{-2}g}_{\mu\nu} \propto g_{\mu\nu} \iff$$ TM admits parallel scale tractor $$\nabla^{T}_{\mu} I^{M} = 0$$

Tractor connection ($Ric = (d-2)P + gJ$)

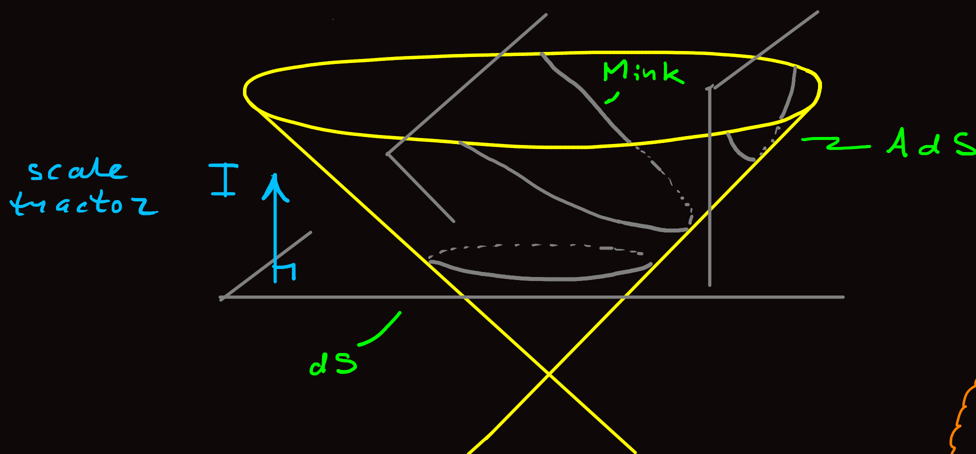
$$\nabla^{T}_{\mu} I^{M} = \nabla^{T}_{\mu} \begin{pmatrix} \sigma \\ n^{\nu} \\ \rho \end{pmatrix} := \begin{pmatrix} \nabla_{\mu}\sigma - n_{\mu} \\ \nabla_{\mu} n^{\nu} + P_{\mu}{}^{\nu}\sigma + \rho\delta_{\mu}^{\nu} \\ \nabla_{\mu}\rho - P_{\mu\nu} n^{\nu} \end{pmatrix}$$

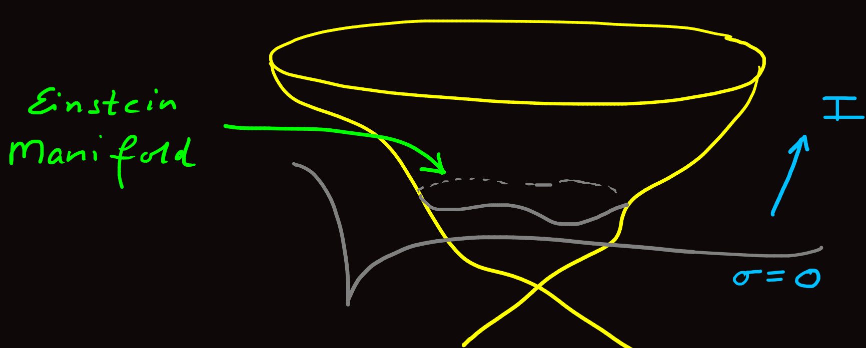
Cosmological constant $\quad I^2 = n^2 + 2\rho\sigma$

Einstein-Hilbert $\quad S = \int \frac{\sqrt{-g}}{\sigma^d} I^2$

Scale σ is dilaton

Curved cone cutting





Gravity is conformal geometry coupled ~~to~~ scale!

$$\nabla I = 0$$

GJMS algebra

Ambient metric

$$G_{MN} = \nabla_M X_N \Rightarrow \text{Curvy cone}$$

sp(2) algebra

$$Q_a := \begin{cases} K = X^2 \\ D = \nabla_X + \frac{D}{2} \\ H = \Delta \end{cases}$$

For any conformal geometry

GJMS : conformal Laplacian powers

$$P_2 = \Box_Y \;, \qquad P_4 = \Delta^4 + \text{l.o.t.} \;, \ldots$$

Quantum Gravity

Bars: study space of all GJMS algebras!

Problem: Given Hilbert space $\mathcal{H}$, find all operator triples (H, D, K) such that

$$\begin{cases} [D, H] = -2H \\ [H, K] = 4\left(D + \frac{d+2}{2}\right) \\ [D, K] = 2K \end{cases}$$

Solution predicts relativistic particle dynamics!

Matrix Model

Action principle (Hoppe, Kazakov, Kostov, Nekrasov, Bars)

$$S = \mathrm{tr}\left(Q_a Q^a + \frac{2}{3} \varepsilon^{abc} Q_a Q_b Q_c \right)$$

$sp(2) \cong so(2,1)$

$a = 1, 2, 3$

Extremum

$$[Q_a, Q_b] = \varepsilon_{abc} Q^c$$

Solutions ?

Quantization ?

Solutions

Gauge symmetry

$$Q_a \sim Q_a + [Q_a, \underset{\sim}{\varepsilon}]$$

any operator

Classical (Bars)
$$[A, B] \longmapsto \{A, B\}_{PB} := \frac{\partial A}{\partial P_M}\frac{\partial B}{\partial Y^M} - (A \leftrightarrow B)$$

ambient phase space $\quad T^*\tilde{M} = \{P_M, Y^M\}$

Hamiltonians $\quad H(X, P) \in C^\infty \tilde{M} \otimes \mathbb{R}[[P]]$

$$\begin{cases} K = X^M(Y)\, G_{MN}(Y)\, X^N(Y) \\ D = X^M(Y)(\nabla_M + A_M(Y)) \\ H = \Sigma_1 + G^{MN}(Y)(\nabla_M + A_M(Y))(\nabla_N + A_N(Y)) + \mathcal{H}(^{\geq 2}[\nabla + A] \end{cases}$$

Moduli $\quad G_{MN} = \nabla_M X_N, \quad X^M F_{MN}(A) = 0, \quad \mathcal{L}_X \Sigma_1 = -2\Sigma_1$

Quantum Solutions

Operators $Q_a \in C^\infty M \otimes \mathbb{R}[[\nabla + A]]$

Gauge away higher spin branch ($\Sigma_1 \neq 0$, Bonezzi, Latini, W)

$$K = X^2 \,, \qquad D = X\cdot(\nabla + A) + \frac{d+2}{2} \,, \qquad H = \Delta_A$$

Moduli: FG metric & $U(1)$ gauge field

State conditions

$$\begin{cases} X^2\,\Psi = 0 \\ \left(\nabla_X + X\cdot A + \frac{d+2}{2}\right)\Psi = 0 \\ \Delta_A\,\Psi = 0 \end{cases}$$

Triplet of Schrödinger equations

Gravity Redux

Action for Schrödinger equations

$$S[G_{MN}, A_N; \Psi, \Lambda, \Theta, \Omega] = \int_{\tilde{M}} \left(\Lambda H \Psi + \Theta D \Psi + \Omega K \Psi\right) = \int \Lambda^a Q_a \Psi$$

Heavily disguised Einstein-Hilbert

Gauge invariances

$$Q_a \sim Q_a + [Q_a, \varepsilon] \qquad \Psi \sim \Psi + \varepsilon \Psi$$

$$\Lambda^a \sim \Lambda^a + \left(\varepsilon \Lambda^a + \tfrac{1}{2} \varepsilon^{abc} Q_b \xi_c\right)^\dagger$$

Einstein-Hilbert via gauging
& integrating out auxiliaries

Residual gauge invariances of "quantum solutions"
= ambient diffeos + $U(1)$ $\quad A_M \sim A_M + \nabla_M \alpha$

Einstein Hilbert ActionS

Solve matrix model & fix "Fefferman-Graham" gauge:

Ignores backreaction; residual diffeos & Maxwell

Temporal gauge: $X^M A_M = \omega$ ~ partially fix Maxwell

Integrate out Θ, Ω: $\Psi = \delta(X^2)\,\psi$, $\nabla_X \psi = (\omega - \frac{d}{2} + 1)\,\psi$

cone

weights

"Tractorize": $\nabla_M \mapsto D_M$

parts $\Rightarrow$ bare δ

$$\underset{\substack{\mathbb{R} \\ c = [g_{\mu\nu}]}}{S(G, A, \varphi, \Omega)} = \underbrace{\int_{\tilde{M}} \delta(X^2)}_{\int_M \sqrt{-g}} \, \varphi \left(\frac{1}{\omega} A^M D_M - \frac{1}{d-2} (D_M A^M) + A^2 \right) \Omega$$

weight $-d$ tractor

Jona–Lasinio Gravity

Idea: $\int |\nabla_A \Phi|^2 \cong \int \mathcal{L}(\sigma)$ ~ σ-model for gauge invariant condensate φ

$A^2 + \dots$

integrate out A

SO(1,1) Gauge invariance:

$$A_M \sim A_M + \frac{1}{d-2} D_M \alpha$$

Remnant symplectic symmetry

$$\Omega \sim \Omega + \alpha\Omega \,, \qquad \Psi \sim \Psi - \alpha\Omega$$

Gauge condensate: $\sqrt{\Omega\Psi} =: \sigma^{1-\frac{d}{2}}$

σ-model:

scale tractor

$$S = \int \frac{\sqrt{-g}}{\sigma^d} \, I_M I^M \cong S_{EH} = \int \sqrt{-g}\, R$$

Quantum Gravity

Chern–Simons fantasy:

NOT $d=3$

$$S_{qu} = \int \mathrm{tr}\left(A\,\partial A + \tfrac{2}{3}A^3\right)$$

Dictionary:

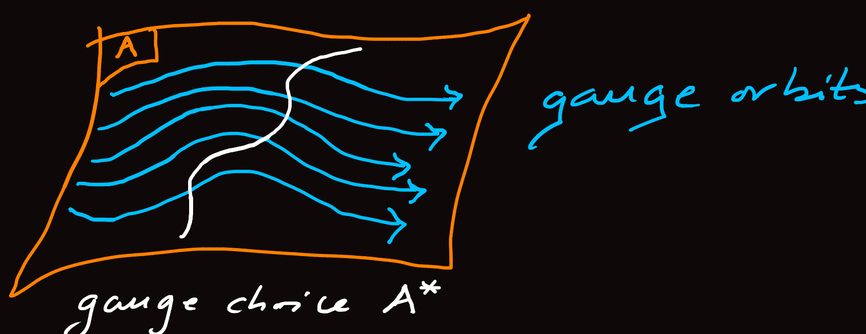
A	observable
tr	Hilbert space trace
$\int, \partial$	AKSZ – BV – BRST

c.f. Witten / Siegel's String field theory, matrix models

BV

Gauge theory: $\int [DA] \exp(-S[A])$ ill-defined

gauge fields



measure

Integral $\int [dA] = Vol_{gauge} \cdot \int [dA^*] \mu_{fix}$

ghost integral

Quantum action S_{qu}(gauge fields, ghosts) BRST

Q-manifold

BV field space

$$\Big\{ \underbrace{\overset{F}{\overset{\text{ghosts}}{C}} \qquad \overset{B}{\overset{\text{classical gauge fields}}{A}}}_{\text{Fields}} \qquad \underbrace{\overset{F}{A^*} \qquad \overset{B}{C^*}}_{\text{Anti-fields}} \Big\} = Z$$

Odd symplectic manifold: $\{ Z^\alpha , Z^\beta] = J^{\alpha\beta}$

Require nilpotent odd, "Hamiltian", vector field:

$$Q = \{ \underbrace{S}_{\text{BV action}} , \cdot \,] \,, \qquad Q^2 = 0$$

Quantum action

Idea:

Lagrangian physics:

$$S = \int_\gamma \Big(\underbrace{\theta}_{\text{symplectic current}} - H\,dt\Big)\Big|_{\mathcal{L}}$$

Any $\mathcal{L}$

BV quantization:

$$Z = \int_{\mathcal{L}} e^{-S}$$

BV action

Any Lagrangian submanifold of Q-manifold

AKSZ

Example:

$$S_{cl} = \int_{M_3} tr\Big(\underbrace{A}_{\mathfrak{g}\text{-valued 1 form}} dA + \frac{2}{3}A^3\Big)$$

BV fields: $\mathcal{A} \in \Lambda M_3 \otimes \mathfrak{g}$

0-form	+	1-form	+	2-form	+	3-form
F		B		F		B
C		A		A^*		C^*

BV action:

$$S = \int_{M_3} tr\left(\mathcal{A}\, d\mathcal{A} + \frac{2}{3}\mathcal{A}^3\right)$$

~ "S_{cl} + BRST"

Chern-Simons!

Quantum Gravity quantum action

$$Q^*_{ab} \qquad C^*_{abc}$$
$$\cong \qquad \cong$$

Field content: operators C, Q_a, Q^{*a}, C^*

ghosts

classical gauge

antifields

"Worldline ghosts": $c^a \sim$ "dx^a" Grassman

"BV superfield": $\mathcal{A} := C + c^a Q_a + c^a c^b Q^*_{ab} + c^a c^b c^c C^*_{abc}$

BV differential: $\mathcal{Q} := \frac{1}{2} \varepsilon_{abc}\, c^a c^b \frac{\partial}{\partial c_c}$, $\mathcal{Q}^2 = 0$

Eilenberg-Chevalley differential for Lie algebra cohomology $H^*(\mathfrak{g}, 1)$

or

BRST operator for $S_{cl} = \int_\gamma (\oplus - e_a Q^a)$ ambient relativistic particle

Coupling to scale

Quantization of GJMS/causal structures:

$$S = \int \mathrm{tr}\left(\mathcal{A}\,\partial\,\mathcal{A} + \tfrac{2}{3}\mathcal{A}^3\right)$$

$\int d^3c$

no $*$ product

Gravity: Couple to scale!

Minimal prescription: $\mathcal{A}(c) \xmapsto[\text{susy''}]{\text{``}\mathcal{N}=2} \mathcal{A}(c, \gamma, \bar{\gamma})$

BV quantum gravity action:

$$\boxed{S = \int \mathrm{str}\left(\mathcal{A}\,\partial\,\mathcal{A} + \tfrac{2}{3}\mathcal{A}^3\right)}$$

$\int d^2\gamma \,.\, \mathrm{tr}$

Open questions

~ cf. primordial string theory

Know spectrum has graviton — previous computation in pure state limit

Finiteness ?

Matrix regularization

Tachyon-free ?

Possibly not — cf. bosonic string

Anomalies ?

Model building ?

Quantizing Dirac Operators

"Square root of ambient Δ" $\quad \not{\nabla} = \Gamma^M \nabla_M$ ~ acts on ambient spinors

$$\{\Gamma_M, \Gamma_N\} = 2 g_{MN}$$

Tractor Dirac equation (Branson)

$$\begin{cases} \not{\nabla} \Psi = 0 \\ \not{X} \Psi = 0 \end{cases}$$

~ remember massless Dirac equation $\gamma^\mu \nabla_\mu \psi = 0$ is conformal

super GJMS algebra (Holland–Sparling)

$$osp(1|2) = \{ \underset{X^2}{\overset{\shortparallel}{Q^{++}}}, \underset{\nabla_X + \frac{D}{2}}{\overset{\shortparallel}{Q^{+-}}}, \underset{\Delta + \frac{R}{4}}{\overset{\shortparallel}{Q^{--}}} ; \underset{\not{X}}{\overset{\shortparallel}{S^+}}, \underset{\not{\nabla}}{\overset{\shortparallel}{S^-}} \}$$

$$(S^+)^2 = Q^{++}, \quad \{S^+, S^-\} = 2Q^{+-}, \quad (S^-)^2 = Q^{--}$$

Matrix Model

Qa composite

Classical gauge fields: S^+, S^-

Grassmann-valued operators

Equations of motion:

$$S^-S^+S^+ - S^+S^+S^- = 2S^+$$

$$S^-S^-S^+ - S^+S^-S^- = 2S^-$$

Action:

$$S = \mathrm{tr}\left(S^+S^- + \tfrac{1}{2}S^+S^+S^-S^-\right)$$

Gauge invariance: $S^\pm \sim S^\pm + [S^\pm, \varepsilon]$

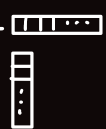
BV superfield

BV action: $\mathrm{tr}\int A\,dA + \frac{2}{3}A^3$

detour quantized worldline BRST

Open questions

Linearized spectrum

E  $\supset$ $h_M{}^A$ vielbein fluctuations

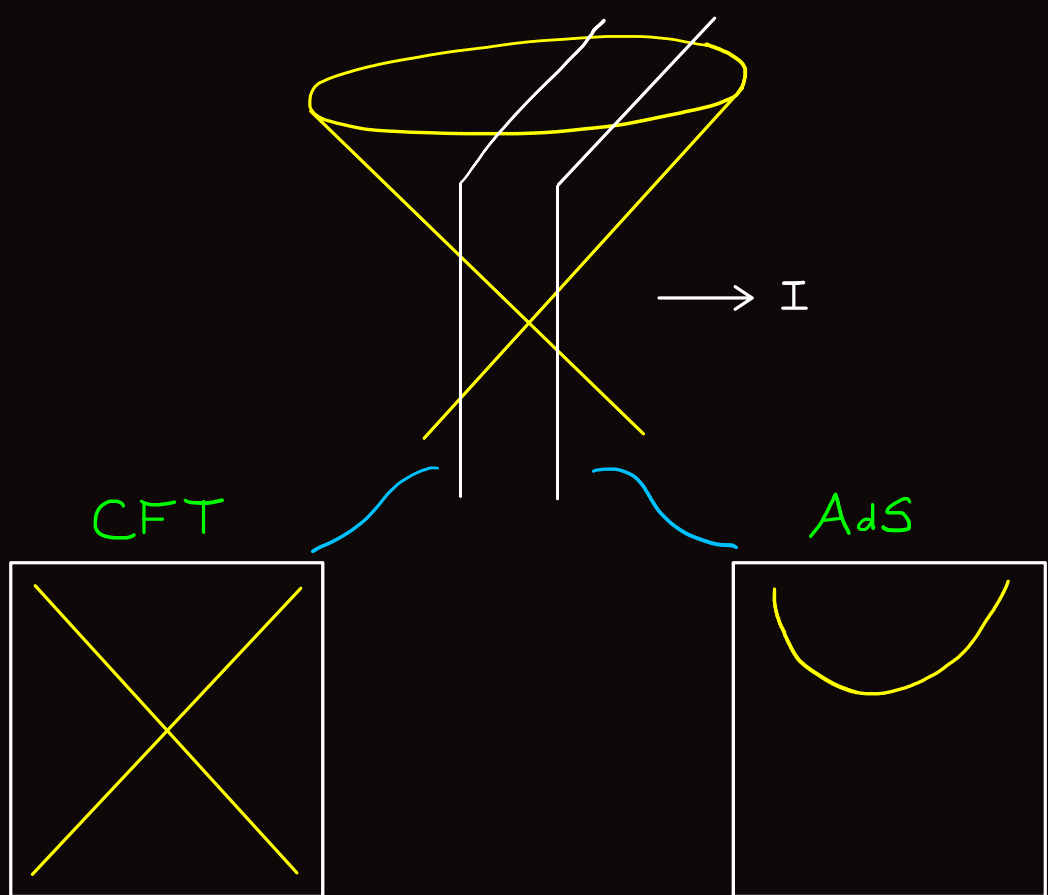
Infinite tower of fields subject to coupled conformal equations

Vasiliev-like theory?

Coupling to scale & gravity?

+ same laundry list as before

AdS/CFT



Defining density

Weight w conformal density:

$$[g_{ab}; \sigma] = [\Omega^2 g_{ab}; \Omega^w \sigma]$$

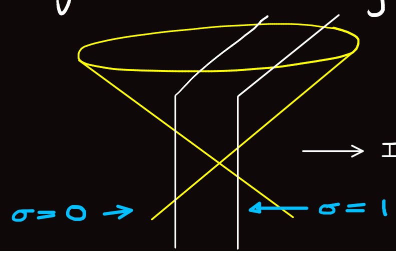
True scale: weight 1, nowhere vanishing density

$$[g_{ab}; \tau] = [\mathring{g}_{ab}; 1]$$ defines Riemannian geometry $\mathring{g}_{ab}$

When zero locus $Z(\sigma)$ of a weight 1 density σ is a hypersurface / boundary, call σ a defining density

$\Downarrow$

Almost Riemannian geometry



Scale Tractor

Data: c – conformal class of metrics

σ – scale (defining / true)

Thomas

$$\text{Scale tractor: } \quad I^M := \frac{1}{d} D^M \sigma \underset{gec}{=} \begin{pmatrix} \sigma \\ \nabla\sigma \\ -\frac{\Delta\sigma + J\sigma}{d} \end{pmatrix} =: \begin{pmatrix} \sigma \\ n \\ \rho \end{pmatrix}$$

Tractor connexion

Recall $\nabla I = 0$ determine σ s.t. $\sigma^{-2} g_{ab}$ is Einstein

Along Σ: I^M "conformal normal vector"

Bulk: I^M generates matter evolution

Laplace–Robin operator

Want analog of ambient ∇_I

$$I \cdot D = I^M D_M \overset{gec}{=} (d+2w-2)(\nabla_n + w\rho) - \frac{\sigma}{d}(\Delta + wJ)$$

– Tractor coupled ∇

– $\Gamma(\mathcal{T}M[w]) \longrightarrow \Gamma(\mathcal{T}M[w-1])$

Bulk wave operator:

Physics wave equations $I.D\,\Phi = 0$ & Transversality

$$m^2 = \frac{2J}{d}\left[\left(\frac{d-1}{2}\right)^2 - \left(w + \frac{d-1}{2}\right)^2\right]$$

BF bound

Mass–Weyl weight relationship

Boundary Robin: $\quad I.D\big|_\Sigma \propto \nabla_n + w\rho$

Solution-generating algebra

Lemma: For any conformal structure c & scale σ

$$[I\cdot D, \sigma] = I^2(d+2w)$$

Proof: direct computation acting on general tractors

An $sl(2)$, suppose $I^2 \neq 0$

$$x := \sigma, \qquad h := d+2w, \qquad y := -\frac{1}{I^2} I\cdot D$$

$$sl(2) = \{x, h, y\}$$

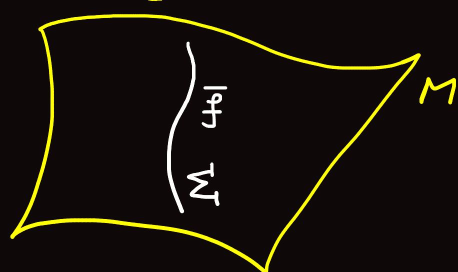
$sl(2)$ factoid

$$[y^k, x] = -y^{k-1}k(h-k+1)$$

Applications: bulk boundary propagators, holography

Holographic Formulae

Idea:



$$\bar{f} = f_{ext}\big|_{\Sigma}$$

gauge invariance / equivalence

$$f_{ext} \sim f_{ext} + \sigma k$$

smooth

$\Sigma = Z(\sigma)$

Tangential operators:

$$\mathcal{O}\sigma = \sigma\tilde{\mathcal{O}}$$

"1st class", $\mathcal{O}/_{\sim}$ well-defined

Theorem: $P_k : \Gamma\left(\mathcal{T}^{\Phi}M\left[\frac{k-d+1}{2}\right]\right) \longrightarrow \Gamma\left(\mathcal{T}^{\Phi}M\left[\frac{k-d+1}{2} - k\right]\right)$

$=$ $\quad P_k = \left(-\frac{1}{I^2} I \cdot D\right)^k \qquad (I^2 \neq 0)$

is tangential.

Proof: use the factoid

GJMS redux

$\mathcal{O}$ tangential $\Rightarrow$ $\bar{\mathcal{O}}\,\bar{f} := (\mathcal{O} f_{ext})|_{\Sigma}$ well defined

Theorem: Let c be almost Einstein, i.e. $\nabla I_\sigma = 0$, k even then $\bar{P}_{k<d}$ is $[(-)^k (k-1)!!]^2$ times the order k GJMS operator

$$\bar{\Delta}^{\frac{k}{2}} + \text{l.o.t.}$$

Proof: relate ambient space $I.D$ & Δ powers

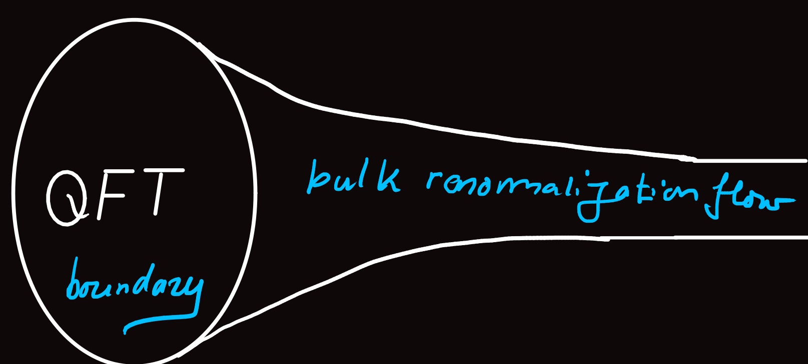
Expect relation to holographic anomalies

Holographic renormalization

$$QFT_\infty \longrightarrow QFT_{\varepsilon,\tau} \longrightarrow QFT^{ren}_{\tau} \xrightarrow{\text{expt}} QFT^{phys}_{\tau}$$

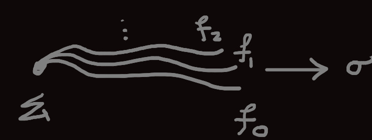
infinite	regulate	renormalize	renormalization point
	Laurent(ε) + Anomaly $\cdot \log \varepsilon$	Taylor(ε)	

Equivalent geometric problem:



Wave equations

Problem: Given $\bar{f}$ along Σ_1, solve



$$I\cdot D f = 0$$

with $f = f_0 + \sigma f_1 + \sigma^2 f_2 + \cdots$ & $f\big|_{\Sigma_1} = f_0$

"solution-generating algebra"

Solution: Recall $I\cdot D = -I^2 y$, $\sigma = x$, $[y,x] = -h$

$$y f = (y f_0 - h f_1) + \sigma\left(y f_1 - 2[h+1] f_2\right) + \cdots$$

$$\Rightarrow f = \left(1 - \frac{\sigma}{d+2w-2}\,\frac{1}{I^2}\,I\cdot D + \frac{\sigma^2}{2(d+2w-2)(d+2w-3)}\left(\frac{1}{I^2}I\cdot D\right)^2 + \cdots\right) f_0$$

Normal ordering: $f = \,:K(z):\, f_0$, $z = xy$, $:(xy)^k: = x^k y^k$

Effective equation: $z K'' - (d+2w)K' + K = 0 \Rightarrow K$ Bessel

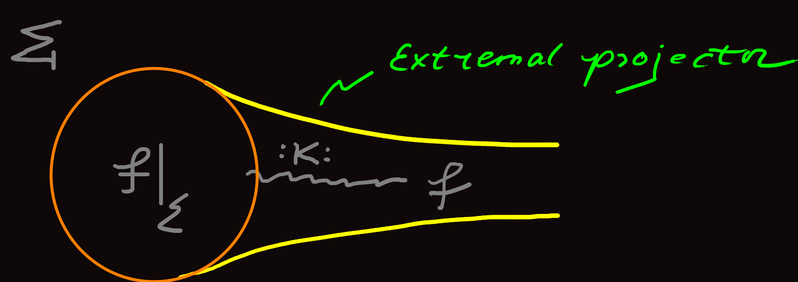
Bulk boundary propagator

The operator $:K:$ is tangential

$$:K(z):\sigma = 0$$

Proposition: $:K(z): \; : \Gamma(\mathcal{T}^{\Phi}\Sigma[w]) \longrightarrow \Gamma(\mathcal{T}^{\Phi}M[w]) \cap \ker(I\cdot D)$

Proof: $:K:(f_0 + \sigma g) = :K: f_0$ depends only on $f_0|_\Sigma$.



Propagates boundary data into bulk for arbitrarily curved structures & tensor types!

log Solutions

Critical weights: $d + 2\omega = 2, 3, 4, \ldots$

recursion breaks down

Frobenius:

$$f = f_0 + \sigma f_1 + \sigma^2 f_2 + \cdots$$
$$+ \sigma^{d+2\omega-1} (\log \sigma - \log \tau)(\tilde{f}_0 + \sigma \tilde{f}_1 + \sigma^2 \tilde{f}_2 + \cdots)$$

log density: $[g_{ab}; \log \sigma] = [e^{2\omega} g_{ab}; \log \sigma + \omega]$

True scale τ $\Rightarrow$ $\log \sigma - \log \tau$ weight 0 density

Second solution: $z^{d+2\omega-1} \tilde{K}(z)$ generates $\tilde{f}_i$

Anomalies

log coefficent:

$$\tilde{f}_0 = -\frac{1}{(d+2w_0-1)!\,(d+2w_0-2)!}\underbrace{\left(\frac{-1}{I^2}\, I.D\right)^{d+2w_0-1}}_{y^k} f_0$$

obstruction: $y^k f_0$ obstructs smoothness!

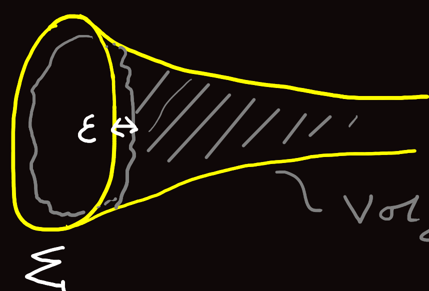
QFT Anomaly!

Tangential: $y^k f_0 = y^k(f_0 + \sigma g)$

$\Rightarrow$ y^k is boundary GJMS operator

Q-curvature

Renormalized volume problem (Graham et al)



Branson Q-curvature

$$\mathrm{Vol}_{\varepsilon}(M) = \text{Laurent}(\varepsilon) + \log\varepsilon \cdot \underbrace{\int Q}$$

Boundary trace anomaly (Skenderis/Henningson)

Formulas

$$Q_2 = J \;, \quad Q_4 \overset{\text{Branson}}{=} P_{ab}P^{ab} - J^2 \;, \quad Q_6 \overset{\text{Sken/Henn}}{=} P_{ab}B^{ab} + \text{"}3P^3 \text{ terms"}$$

$Q_8 =$ 1 page ... Gover/Peterson

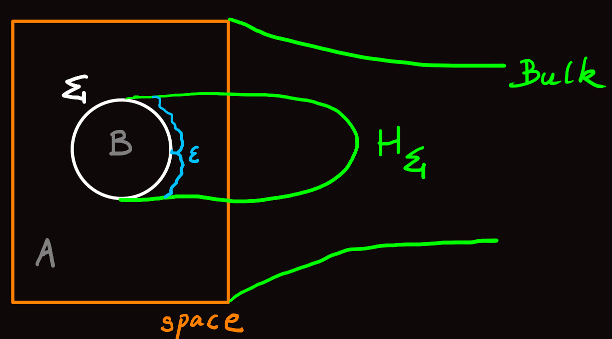
Theorem:

$$Q_{n\text{-even}} = \left(-\tfrac{1}{I^2} I.D\right)^n \log \tau \Big|_{\Sigma}$$

Holographic formula, Constructive

Conformal hypersurfaces

Holographic entanglement entropy (Ryu-Takanayagi)



Minimal surface in AAdS (static) bulk H_{Σ}

$$\partial H_{\Sigma} = \Sigma$$

Renormalized area:

Conformal hypersurface invariant

$$S_{A|B} = \frac{\mathrm{Area}(\Sigma)}{\varepsilon^2} + \log\varepsilon \cdot \text{Willmore Energy}(\Sigma)$$

Entanglement entropy

$\dim\Sigma = 2$

Hypersurfaces

zero locus

Embedded hypersurface: $\Sigma = Z(\sigma)$

defining function

Hypersurface invariants:

diff invariant

Hypersurface preinvariant: $P(g_{ab}, \sigma)$

such that $$P(g_{ab}, \sigma)\big|_{\Sigma} = P(g_{ab}, v\sigma)\big|_{\Sigma} =: P(g_{ab}, \Sigma)$$

smooth non-zero

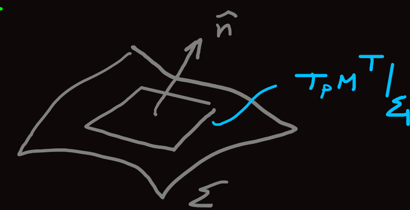
Hypersurface invariant

Basic hypersurface invariants

Unit normal:

preinvariant

$$\hat{n}_a := \frac{\nabla_a \sigma}{|\nabla_a \sigma|}\Big|_\Sigma$$



First fundamental form:

induced metric

$$I_a := \left(g_{ab} - \frac{\nabla_a \sigma}{|\nabla \sigma|}\frac{\nabla_b \sigma}{|\nabla \sigma|}\right)\Big|_\Sigma = \bar{g}_{ab}$$

$TM^\top|_\Sigma \cong T\Sigma$

Mean curvature:

$$H := \nabla_a\left(\frac{\nabla^a \sigma}{|\nabla \sigma|}\right)\Big|_\Sigma$$

Second fundamental form:

$$II_{ab} := \left(\nabla_a - \left(\frac{\nabla_a \sigma}{|\nabla \sigma|}\right)\left(\frac{\nabla^c \sigma}{|\nabla \sigma|}\right)\nabla_c\right)\left(\frac{\nabla_b \sigma}{|\nabla \sigma|}\right)\Big|_\Sigma$$

Gauß formula:

$$(\nabla - \bar{\nabla})\big|_\Sigma = n\, II^\#$$

shape operator

Conformal hypersurface invariants

Data: (M, c, Σ)

Density hypersurface invariant:

$$P(\Omega^2 g_{ab}, \Sigma) = \Omega^w P(g_{ab}, \Sigma)$$

$\Rightarrow$ Conformal hypersurface invariants

$$P(c, \Sigma) := [\, g_{ab}, P(g_{ab}, \Sigma)\,]$$

Examples: $\hat{n}_a := [g_{ab}, \hat{n}_a]$ unit normal

$$\mathring{II}_{ab} := [\, g_{ab}, II_{ab} - H\, I_{ab}\,]$$

trace-free second fundamental form

Fialkow tensor: $F_{ab} := [g_{ab}, P^\top_{ab} - \bar{P}_{ab} + H\, \mathring{II}_{ab} - \frac{1}{2} I_{ab} H^2]$

$\nabla^\top|_\Sigma - \bar{\nabla}^\top$

Conformal Infinity

Data $(M, c, \underset{\text{defining density}}{\sigma}) \Rightarrow$ conformal infinity

Coordinate definition:

choose $\sigma = x$ & $ds^2 = dx^2 + h(x)$, $\Sigma = M|_{x=0}$

$$[\, dx^2 + h(x), x \,] \underset{x \neq 0}{=} [\, \underbrace{\frac{dx^2 + h(x)}{x^2}}_{ds^2} , 1 \,]$$

Say Σ is a conformal infinity of ds^2 because

$ds^2|_\Sigma$ ill-defined but

$x^2 \Omega^2 ds^2|_\Sigma \cong [h] = c_\Sigma$ gives boundary conformal geometry.

Idea: Treat hypersurfaces as conformal infinities

Bulk problem

Normal tractor: $N^M = \begin{pmatrix} 0 \\ \hat{n} \\ -H \end{pmatrix}$ tractor-valued hypersurface invariant (BEG)

$$N^2 = \hat{n}^2 = 1$$

Bulk geometry hint:

Theorem: (Gover) If $I_\sigma^2 = 1 + \mathcal{O}(\sigma)$ $\left(\Rightarrow I^2\big|_\Sigma = 1\right)$,

then $$I^M\big|_\Sigma = N^M$$

Singular Yamabe Problem

Yamabe: Is every metric conformal to a metric of constant scalar curvature? (Trudinger, Aubin, Schoen)

Weak version of Einstein:

$$\cancel{\nabla I^M = 0} \Rightarrow I^2 \overset{\checkmark}{=} \text{constant}$$

Singular Yamabe: Given $I^2|_\Sigma = 1$, solve $I^2 = 1$.

Note: $$\underbrace{I^2 =}_{\text{choose } g\in c} |\nabla\sigma|^2 - \frac{2\sigma}{d}(\Delta\sigma + J^g\sigma) \underset{\substack{\text{away}\\\text{from}\\\Sigma}}{\overset{\sigma=1}{\sim}} \boxed{-\frac{2J^{g_0}}{d} \overset{?}{=} 1}$$

Non-Linear Loewner–Nirenberg PDE with $\sigma = \rho^{\frac{-2}{d-2}}$

Can always solve $|\nabla\sigma|_g^2 = 1$, "unit defining function"

Obstruction density

Theorem: Let σ be a defining density (w.l.o.g. $I_\sigma^2\big|_\Sigma = 1$).

GW, cf also Anderson, Chruściel, Friedrich

Then $\exists_1^{*}$ $\quad \bar\sigma = \sigma(1 + \alpha_1\sigma + \cdots + \alpha_n\sigma^n)$

such that $\quad I^2_{\bar\sigma} = 1 + \sigma^d \mathcal{B}$

Proof: constructive, uses solution generating algebra

Remarks: $B := \mathcal{B}\big|_\Sigma$ is a conformal hypersurface invariant called the obstruction density $\to$ Yamabe analog of FG obstruction tensor (Bach in $d=4$)

"conformal unit defining scale"

$\bar\sigma$ jets $\Rightarrow$ conformal hypersurface calculus

$$D^M\bar\sigma\big|_\Sigma \qquad D^M D^N \bar\sigma\big|_\Sigma \qquad I_{\bar\sigma}\cdot D\, D^M D^N \bar\sigma\big|_\Sigma \qquad \cdots^{*}$$

Normal tractor $\qquad$ Tractor II $\qquad$ Tractor Fialkow

* up to order $\bar\sigma$ determined

The Willmore Invariant

Minimal surfaces minimize $\mathrm{Area}_\Sigma = \int_\Sigma dA^{\bar{g}}$, $\delta \mathrm{Area} = H$

Willmore energy $E = \int_\Sigma dA^{\bar{g}} H^2$ — Rigid string (Polyakov)

Functional gradient $\delta E = \underbrace{L_{ab}}_{\text{extrinsic BGG}} \mathring{\mathrm{II}}^{ab} \overset{g\epsilon c}{=} (\bar{\nabla}_a \bar{\nabla}_b + P_{ab} + H \mathring{\mathrm{II}}_{ab}) \mathring{\mathrm{II}}^{ab} = \underbrace{B_2}_{\text{obstruction density}}$

- Willmore energy is anomaly term in entanglement entropy.
- Higher Willmore invariants B_n variational (Graham)

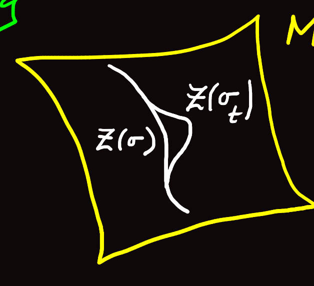
Variational Calculus

Holographic formulae $\Rightarrow$ Hypersurface preinvariants

Lemma: Let $P(\Sigma, g) = P(\sigma, g_{ab})$, then

$$\int_\Sigma dA^{\bar g}\, P(\Sigma, g) = \int_M dV^g\, \delta(\sigma)\, |\nabla\sigma|\, P(\sigma, g_{ab})$$

Remark: Conformal generalization $\int_\Sigma \underbrace{P(\Sigma, c)}_{\text{weight } -d \text{ density}} = \int_M \delta(\sigma)\, |I|\, P(\sigma, c)$



Variational formula:

$$\delta \int_\Sigma dA^{\bar g}\, P(\Sigma, g_{ab}) = \delta P - \delta_R P\big|_\Sigma$$

Robin $= I.D|_\Sigma$
= conformal Lie derivative $\mathcal{L}_{\hat n}$

Extrinsic conformal Laplacian powers

evenness **NOT** required

$$P_k^{ext} = \left(-\frac{1}{I^2_{\bar\sigma}}\, I_{\bar\sigma}\cdot D\right)^k \,, \qquad k \leqslant n$$

conformal unit defining scale

tangential at weight $\frac{k-n}{2}$

$$\Rightarrow \quad \bar P_k^{ext} : \Gamma\left(\mathcal{T}^{\Phi}\Sigma\left[\tfrac{k-n}{2}\right]\right) \longrightarrow \Gamma\left(\mathcal{T}^{\Phi}\Sigma\left[\tfrac{k-n}{2} - k\right]\right)$$

k even $\quad \bar\Delta^{k/2} + (\text{extrinsic \& intrinsic curvatures})$

k odd $\quad \mathring{I\!I}$ plays role of metric, ex $\bar P_3^{ext} = \mathring{I\!I}^{ab}\,\nabla_a \nabla_b + (\text{curvatures})$

Embedding data $\Rightarrow$ all order extrinsic GJMS operators

Holographic Formula

Theorem:

$$B_n = \frac{2}{n!\,(n+1)!}\,\bar{D}_M\Big[\Sigma^M_N\big(\bar{P}^{ext}_n N^N\big) - (-)^n\big(I_{\bar\sigma}\cdot D^n[X^N K]\big)\Big]$$

Tractor first fundamental form

Rigidity density

$= (D_R I_S)(D^R I^S) \overset{\Sigma}{=} \mathring{I\!I}_{ab}\mathring{I\!I}^{ab}$

Proof: Leibniz failure & solution generating algebra on steroids.
— explicit recursion for $\bar\sigma$.

Example: Four manifold obstruction ~ obstructs sol^ns to Einstein for 4 dim spacetime

$$B_3 = \frac{1}{6}\Big[L^{ab}\big(3\mathring{I\!I}^2_{(ab)\circ} - \hat{W}_{ab}\big) - \mathring{I\!I}^{ab}B_{ab} + K^2 - 7\hat{W}^{ab}\mathring{I\!I}^2_{ab} + 2\hat{W}_{ab}\hat{W}^{ab} + \mathring{I\!I}^{ab}\mathring{I\!I}^{cd}W_{abcd} + \hat{W}^{\top}_{abc}\hat{W}^{abc}\Big]$$

Hypersurface Bach $\hat{C}^{\top}_{ab} + \cdots$

B_n for explicit metrics easily generated

Energy functionals & Higher Willmores

Renormalized volume expansion for singular Yamabe (Graham)

$\Rightarrow$ Obstruction density is variation of anomaly

Higher Willmore energies "Q-curvature"-like

Conjecturally invariant piece is

$$E_n = \int_\Sigma dA^{\bar g}\, N_M P_n N^M \underset{n\text{ even}}{\overset{\text{linearize}}{\Rightarrow}} \int_\Sigma dA^{\bar g}\, H\, \bar\Delta_{\bar g}^{\frac{n}{2}-1} H$$

Linearized functional gradient $\underset{n\text{ even}}{=}$ linearized Willmore invariant $= \bar\Delta^{\frac{n}{2}} H$

Examples $E_2 = \int_\Sigma dA^{\bar g}\, \mathring{I\!I}_{ab} \mathring{I\!I}^{ab}$, $\quad E_3 = \int_\Sigma dA^{\bar g}\, \mathring{I\!I}_{ab} \mathcal{F}^{ab}$ ~ Fialkow

$\delta E_2 = B_2 \qquad \delta E_3 = B_3$