Topological defects in CFT

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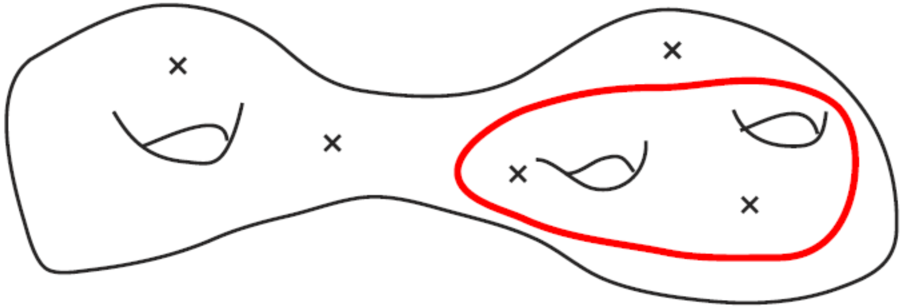
Lightning review of CFT

- 2d Conformal Field Theory is essentially a collection of maps $(\Sigma, \{V_i(z_i)\}) \rightarrow \mathbb{C}$



obeying certain well motivated and known axioms such as gluing conditions or

$$V_i(z)V_j(w) = \sum_k \frac{C_{ij}^k}{(z-w)^{h_i+h_j-h_k}} V_k(w) + \cdots$$

Lightning review of CFT

- Every CFT possesses a special holomorphic operator

$$T(z) = \sum_n \frac{L_n}{z^{n+2}}$$

$$[L_n, L_m] = \frac{c}{12} n(n^2-1)\delta_{n+m} + (n-m)L_{n+m}$$

and its antiholomorphic counterpart

$$\bar{T}(\bar{z}) = \sum_n \frac{\bar{L}_n}{\bar{z}^{n+2}}$$

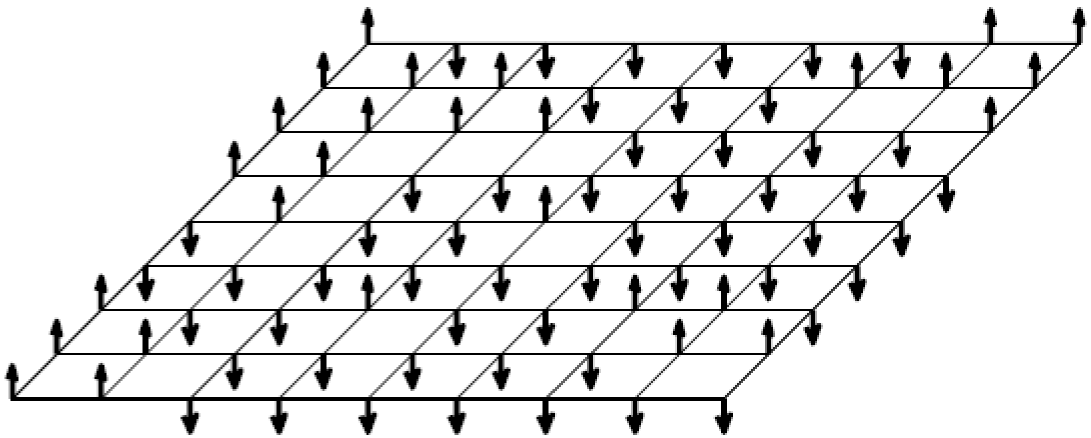
which obey

$$T(z)V(w,\bar{w}) = \cdots + \frac{h}{(z-w)^2} V(w,\bar{w}) + \frac{1}{z-w}\partial V(w,\bar{w}) + \cdots$$

$$T(z)T(w) = \frac{c}{2(z-w)^4} + \frac{h}{(z-w)^2} T(w) + \frac{1}{z-w}\partial T(w) + \cdots$$

Example of an CFT: Ising model

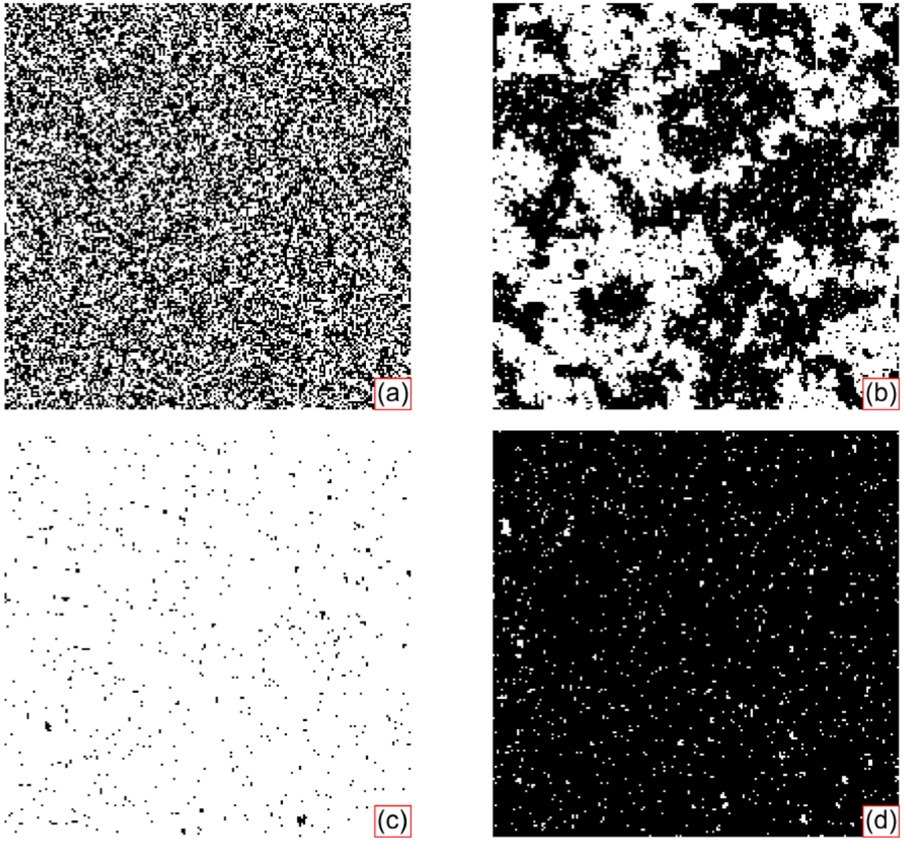
- One of the simplest systems is the Ising (more properly Lenz-Onsager) model of interacting spins



with Hamiltonian

$$\mathcal{H}(\{\sigma_i\}) = -J \sum_{(i,j)} \sigma_i \sigma_j - B \sum_i \sigma_i,$$

Example of an CFT: Ising model



- This model is described by the partition sum

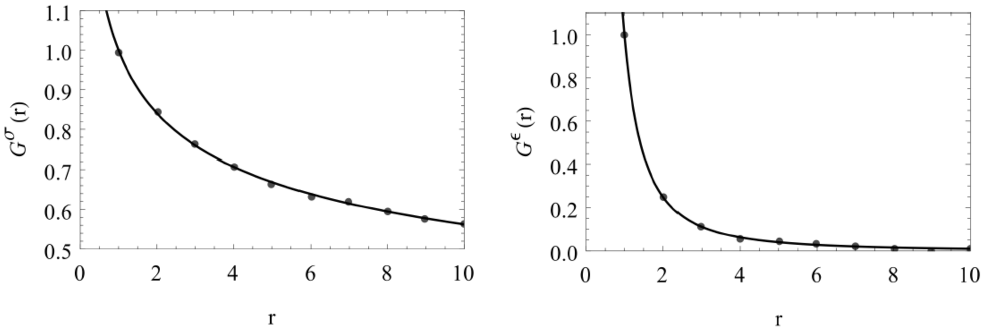
$$Z = \sum_{\{\sigma\}} e^{-\frac{H(\{\sigma\})}{kT}}$$

and shows very different behavior depending on the temperature

a) high temperature phase
b) critical temperature
c), d) low temperature phases

Example of an CFT: Ising model

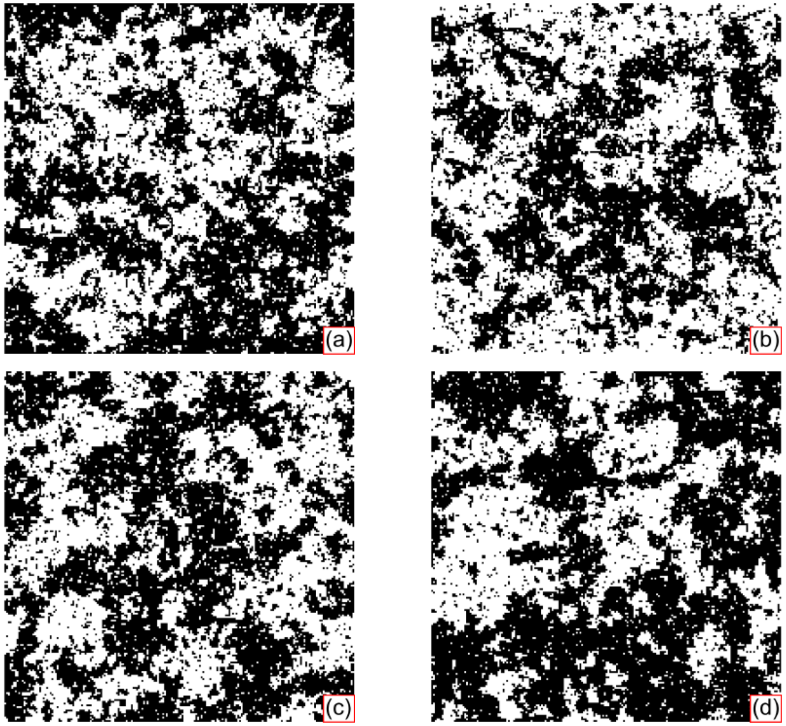
- At critical temperature it becomes the simplest 2D CFT with $c = \frac{1}{2}$ (free Majorana fermion)



- Magnetization and energy are described by local operators σ and ε with scale dimension $\frac{1}{16}$ and $\frac{1}{2}$.
  For instance 2-point functions of magnetization or energy density can be obtained very simply (critical exponents equal to -¼ and -2).

Example of an CFT: Ising model

- In practice we often need boundary conditions. Which of those preserve the conformal symmetry?



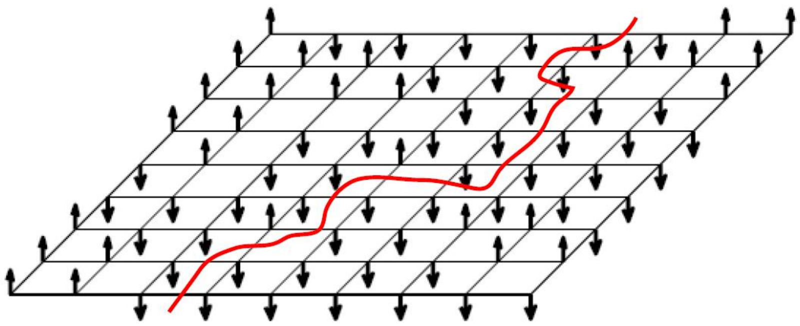
The figure depicts ++, --, free-free and periodic b.c..

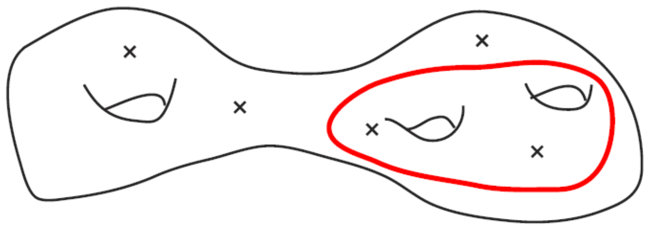
Interestingly the number of allowed conformal boundary conditions is the same as the number of local operators in a CFT (as shown by Cardy in 1980's)

In particular: $+ \leftrightarrow 1$,
$- \leftrightarrow \varepsilon$,
free $\leftrightarrow \sigma$

Defects in CFT

- In the underlining lattice realization defects are very simple objects (change of coupling along some bonds)





- Boundary conditions on the defect can be
  a) fully reflecting (same as pair of b.c.)
  b) fully transmitting (e.g. trivial defect)
  c) …arbitrary

Defects in CFT

There are two special families of defects:

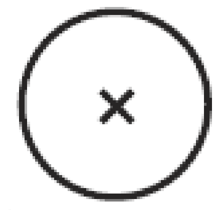
- Conformal defects (defects preserving energy in 1-dim picture) $T(z) - \bar{T}(\bar{z})$ is continuous; can be studied by folding trick

- Topological defects (preserve energy and momentum); $T(z), \bar{T}(\bar{z})$ are separately continuous, hence everything is invariant under small deformations. They are naturally associated to *homotopy* cycles.

Topological defects: Brief review

- Topological defects give rise to closed string operators obeying

$$[L_n, D] = [\tilde{L}_n, D] = 0, \qquad \forall n$$



- For diagonal minimal models they are labeled by the same index as primary fields. By Schur's lemma they are constant on every Verma module

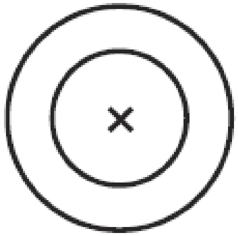
$$D_d = \sum_i X_{di} P^i$$

where P^i are projectors on the *i-th* Verma module

Topological defects: Brief review

- Bring two topological defects close to each other, the result will be a sum of elementary defects

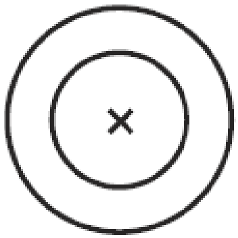
$$D_d D_c = \sum_{e \in d \times c} D_e = \sum_e N_{dc}{}^e D_e$$



Defects obey so called fusion algebra

Topological defects: Brief review

- For minimal models it is readily verified



$$
\begin{aligned}
D_a D_b &= \left(\sum_i \frac{S_{ai}}{S_{0i}} P^i\right)\left(\sum_j \frac{S_{bj}}{S_{0j}} P^j\right) \\
&= \sum_i \frac{S_{ai}}{S_{0i}} \frac{S_{bi}}{S_{0i}} P^i \\
&= \sum_i \sum_c N_{ab}{}^c \frac{S_{ci}}{S_{0i}} P^i \\
&= \sum_c N_{ab}{}^c D_c
\end{aligned}
$$

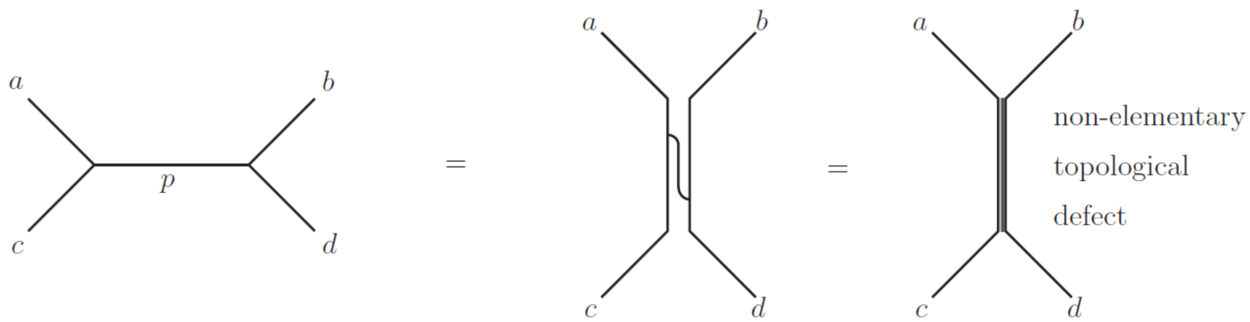
←Verlinde formula used

$$[\phi_a] \times [\phi_b] = \sum_c N_{ab}{}^c [\phi_c]$$

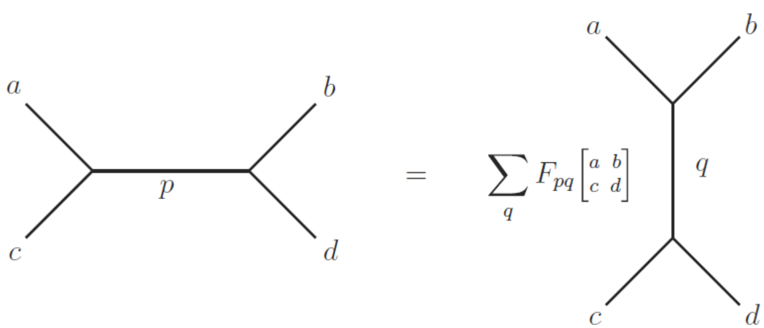
Petkova, Zuber 2000

Defect networks

- Let us consider defects ending on defects



and hence



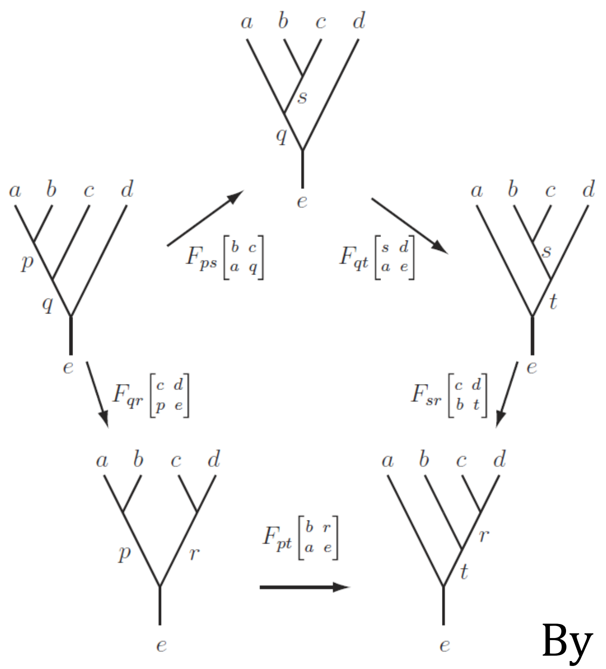
NB: This looks just like transformation properties of 4pt-conformal blocks.

It can be also justified by the TFT approach of Felder, Fröhlich, Fuchs, Runkel and Schweigert

Defect networks

- Key consistency relation (pentagon identity)

$$\sum_s F_{ps}\begin{bmatrix} b & c \\ a & q \end{bmatrix} F_{qt}\begin{bmatrix} s & d \\ a & e \end{bmatrix} F_{sr}\begin{bmatrix} c & d \\ b & t \end{bmatrix} = F_{qr}\begin{bmatrix} c & d \\ p & e \end{bmatrix} F_{pt}\begin{bmatrix} b & r \\ a & e \end{bmatrix}$$



axiom of monoidal (tensor) category or Elliott-Biedenharn identity for 6j symbols

By Mac Lane coherence theorem guarantees consistency

Defect networks

- Various properties

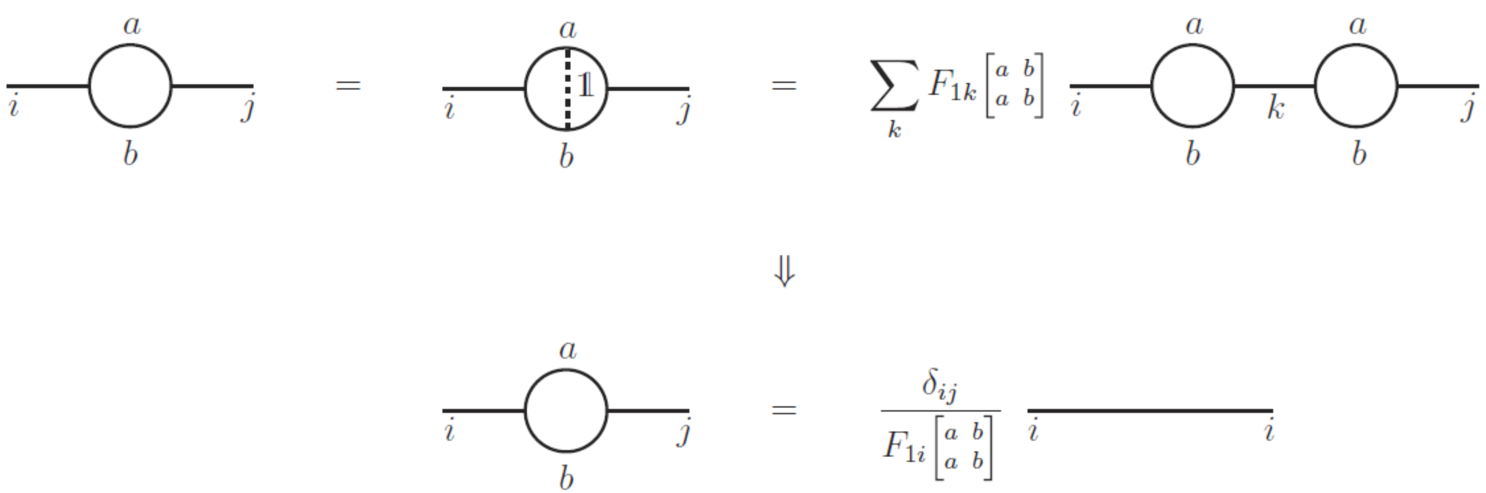
$$F_{pq}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = 1 \qquad \text{whenever } 1 \in \{a, b, c, d\}$$

$$\sum_s F_{ps}\begin{bmatrix} b & c \\ a & d \end{bmatrix} F_{sr}\begin{bmatrix} c & d \\ b & a \end{bmatrix} = \delta_{pr}.$$

- In special cases (parity invariant defect in parity invariant theory) also symmetry under permutation of rows and columns

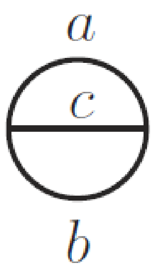
Defect networks

- In the absence of obstructions one can get rid of defect loops



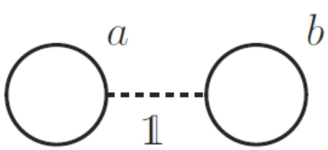
Defect networks

- From that the sunset diagram follows

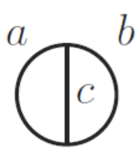


$$= \frac{1}{F_{11}\begin{bmatrix} a & a \\ a & a \end{bmatrix} F_{1a}\begin{bmatrix} b & c \\ b & c \end{bmatrix}} = \Delta(a,b,c)$$

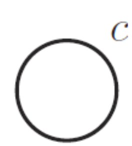
- Another curious relation (part of Verlinde formula):



$$= \sum_{c \in a \times b} F_{1c}\begin{bmatrix} a & b \\ a & b \end{bmatrix}$$



$$= \sum_{c \in a \times b}$$



$$\Downarrow$$

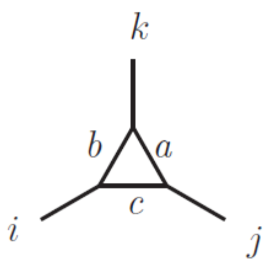
$$\frac{1}{F_{11}\begin{bmatrix} a & a \\ a & a \end{bmatrix}} \times \frac{1}{F_{11}\begin{bmatrix} b & b \\ b & b \end{bmatrix}} = \sum_{c} N_{ab}{}^{c} \frac{1}{F_{11}\begin{bmatrix} c & c \\ c & c \end{bmatrix}}$$

Ring of quantum dimensions

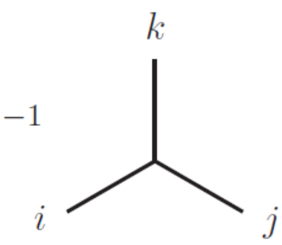
$$g'_a g'_b = \sum_c N_{ab}{}^{c} g'_c$$

Defect networks

- An important relation is



$$= F_{ck}\begin{bmatrix} b & a \\ i & j \end{bmatrix} \times \left(F_{1k}\begin{bmatrix} a & b \\ a & b \end{bmatrix} \right)^{-1}$$



which is actually S_3 symmetric (in the parity symmetric case), thanks to nontrivial relations for the fusion matrices.

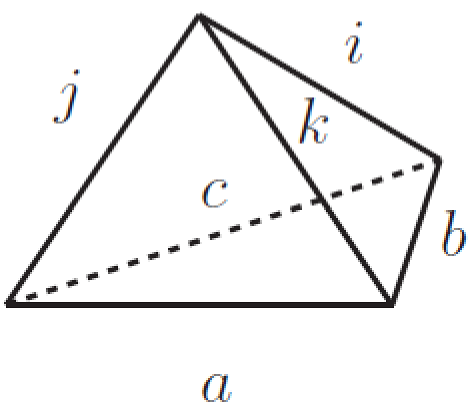
- This prefactor is 1 when one of the internal line is the identity defect. When one of the external defects is 1, we get simple, but nontrivial normalization for the bubble.

Defect networks

- Another relation of this sort are the tetrahedral identities for the "6j-symbol"

$$\begin{bmatrix} c, a, b \\ k, i, j \end{bmatrix} = \frac{F_{ck}\begin{bmatrix} b & a \\ i & j \end{bmatrix}}{F_{1k}\begin{bmatrix} a & b \\ a & b \end{bmatrix} F_{1i}\begin{bmatrix} j & k \\ j & k \end{bmatrix} F_{11}\begin{bmatrix} i & i \\ i & i \end{bmatrix}}$$

which one gets from defect tetrahedron



Defect action on boundary states

- Action on Cardy boundary states is straightforward

$$
\begin{aligned}
D_a\|B_b\rangle\rangle &= \sum_i \frac{S_{ai}}{S_{0i}} P^i \left(\sum_j \frac{S_{bj}}{\sqrt{S_{0j}}} |j\rangle\rangle \right) \\
&= \sum_i \frac{S_{ai}}{S_{0i}} \frac{S_{bi}}{\sqrt{S_{0i}}} |i\rangle\rangle \\
&= \sum_i \sum_c N_{ab}{}^c \frac{S_{ci}}{\sqrt{S_{0i}}} |i\rangle\rangle \\
&= \sum_c N_{ab}{}^c \|B_c\rangle\rangle
\end{aligned}
$$

←Verlinde formula used

Graham, Watts 2003

Topological defects in OSFT

- Let us try to define defect action on boundary fields

$$D^d : \mathcal{H}_{closed} \rightarrow \mathcal{H}_{closed}$$

$$\mathcal{D}^d : \mathcal{H}^{(ab)} \rightarrow \bigoplus_{\substack{a' \in d \times a \\ b' \in d \times b}} \mathcal{H}^{(a'b')}$$

- Since the defect is topological and we demand $[Q, \mathcal{D}] = 0$ we expect that $[L_n, \mathcal{D}] = 0$ and hence by Schur's lemma

$$\mathcal{D}^d \, \psi_i^{(ab)} = \sum_{\substack{a' \in d \times a \\ b' \in d \times b}} X^{dab}_{ia'b'} \, \psi_i^{(a'b')}$$

Topological defects in OSFT

- To satisfy constraint $\mathcal{D}(\phi * \chi) \;=\; (\mathcal{D}\phi) * (\mathcal{D}\chi) \qquad \forall \phi, \chi$
  it is enough to require for primary fields

$$\mathcal{D}^d \left(\phi_i^{(ab)}(x) \phi_j^{(bc)}(y) \right) = \left(\mathcal{D}^d \phi_i^{(ab)}(x) \right) \left(\mathcal{D}^d \phi_j^{(bc)}(y) \right)$$

from which it follows

$$X^{dac}_{ka'c'} C^{(abc)k}_{ij} = \sum_{b' \in d \times b} C^{(a'b'c')k}_{ij} X^{dab}_{ka'b'} X^{dbc}_{kb'c'}$$

For minimal models Runkel found the boundary structure constants

$$C^{(abc)k}_{ij} = F_{bk} \begin{bmatrix} a & c \\ i & j \end{bmatrix}$$

(for the A-series)

Topological defects in OSFT

- Inserting this explicit solution into the constraint we found a general solution

$$X^{dab}_{ia'b'} = \frac{N(d,a,a')}{N(d,b,b')} F_{b'a}\begin{bmatrix} i & b \\ a' & d \end{bmatrix}$$

Generalizes result by Graham and Watts (2003)

thanks to the *pentagon identity* of rational CFT.

- Further demanding twist symmetry fixes the form

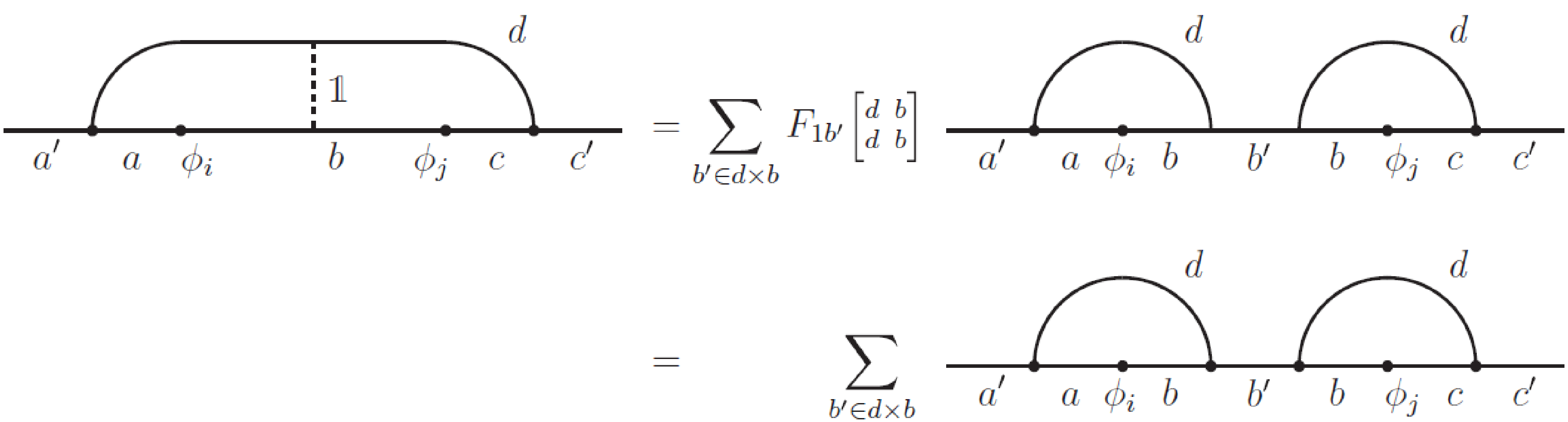
$$X^{dab}_{ia'b'} = F_{di}\begin{bmatrix} a & b \\ a' & b' \end{bmatrix} \frac{\sqrt{F_{1a'}\begin{bmatrix} a & d \\ a & d \end{bmatrix} F_{1b'}\begin{bmatrix} b & d \\ b & d \end{bmatrix}}}{F_{1i}\begin{bmatrix} a & b \\ a & b \end{bmatrix}}$$

Topological defects in OSFT

- Distributivity

$$\mathcal{D}^d \left(\phi_i^{(ab)}(x) \phi_j^{(bc)}(y) \right) = \left(\mathcal{D}^d \phi_i^{(ab)}(x) \right) \left(\mathcal{D}^d \phi_j^{(bc)}(y) \right)$$

can be nicely understood graphically:



Topological defects in OSFT

- Interestingly it turns out that

$$\mathcal{D}^d\mathcal{D}^c \neq \bigoplus_e N_{dc}{}^e \mathcal{D}_e$$

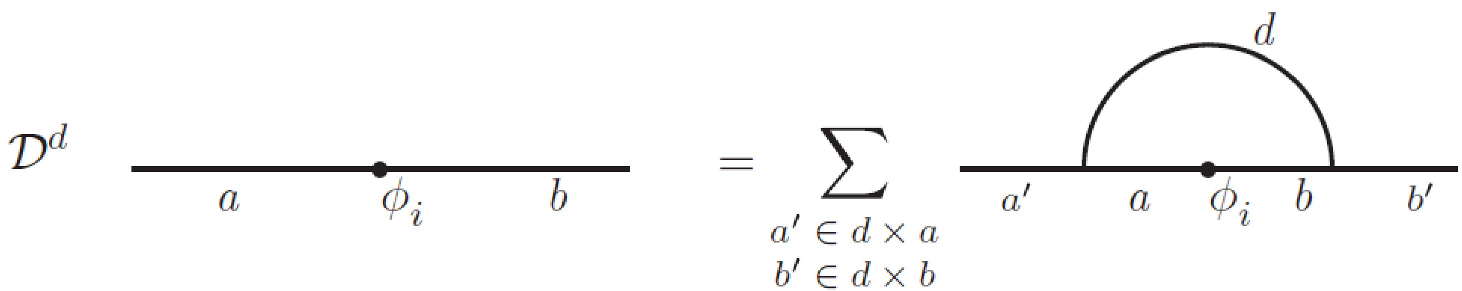
but fortunately at least

$$\mathcal{D}^d\mathcal{D}^c = U\left(\bigoplus_e N_{dc}{}^e \mathcal{D}_e\right)U^{-1}$$

is true! The matrices *U* are simply given by the fusion matrix (Racah symbols), they square to 1, but most importantly they do not contribute to bulk observables.

Topological defects in OSFT

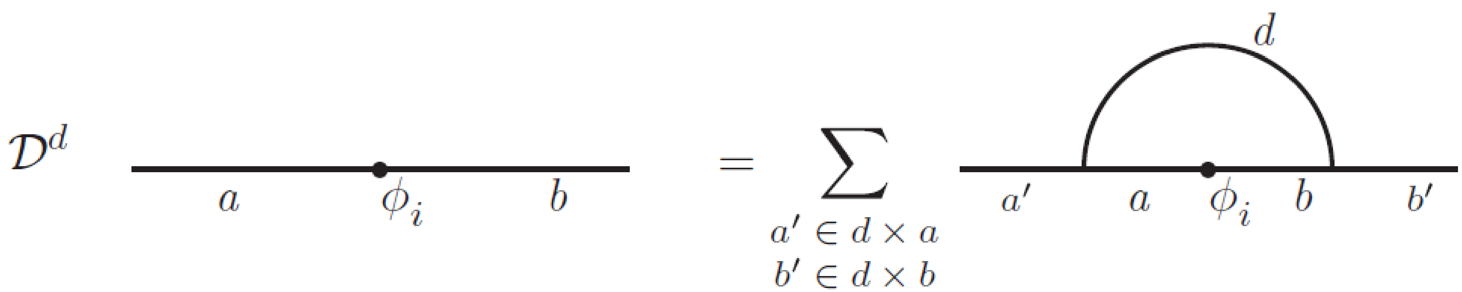
- To understand these extra factors it is convenient to develop a geometric formalism. Moving topological defect towards boundary, we get new boundary conditions. When we want to understand the action on boundary operators we need to fuse it only partway:



- We have to understand the CFT with defects ending on boundaries and possible operator insertions at junctions

Topological defects in OSFT

- Let us assume that the original boundary conditions *a* and *b* arise from the action of defect on the identity boundary condition (if it exists). Then one can reinterpret the same diagram



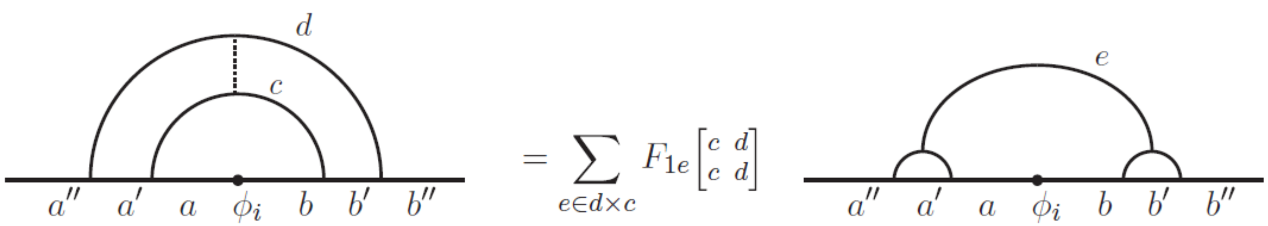
as one for *defect action* on *defect changing operators*

Topological defects in OSFT

- So finally the extra factors in

$$\mathcal{D}^d \mathcal{D}^c = U \left(\bigoplus_e N_{dc}{}^e \mathcal{D}_e \right) U^{-1}$$

can be deduced for example from



or better yet by refusing on *a'* and *b'* defect

Summary

- Topological defects are a fundamental ingredient in 2d CFTs and they lead to a lot of exciting mathematics

- The topological defects can also be used to relate different solutions in various theories, but one has to be careful when extending their action on open string fields