Singular BGG resolutions for type A

Pavle Pandžić, University of Zagreb

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This is joint work with Vladimír Souček.

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Rafael Mrđen has similar results for type C (talk on Friday)

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P has a Levi subgroup $L \cong GL(k) \times GL(l)$ consisting of block-diagonal matrices.

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In particular, crossing all vertices we get the Borel subalgebra of upper triangular matrices.

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Then M decomposes into a direct sum of modules with generalized infinitesimal character. So it is enough to study modules with (fixed) generalized infinitesimal character.

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The module $V_\mathfrak{p}(\lambda)$ has a unique maximal submodule. The quotient of $V_\mathfrak{p}(\lambda)$ by this submodule is irreducible. In this way one gets all irreducible objects of $\mathcal{O}_\mathfrak{p}$.

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Geometrically, elements of W parametrize orbits of B on G/B, and $w \leq w'$ means the closure of $Bw'B$ contains BwB.

Algebraically, w' is an immediate successor of w if $l(w') = l(w) + 1$ and $w' = s_\alpha w$ for some root α. (α is not necessarily simple.)

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For a good choice of scalars (signs!), get a resolution of F_λ.

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The Hasse diagram of $W^{\mathfrak{p}}$ can be obtained from that of W: immediate successors are those that are closest w.r.t. W.

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The BGG resolutions of finite-dimensional modules in $\mathcal{O}_\mathfrak{p}$ are now constructed as for the case $P = B$.

(For the morphisms we take the standard morhisms, obtained by composing corresponding morphisms in $\mathcal{O}_\mathfrak{b}$.)

Enright-Shelton equivalence

The case we consider, $G = GL(n+1, \mathbb{C})$ and P maximal, with Levi $L = GL(k) \times GL(l)$, corresponds to the Hermitian real form $G_{\mathbb{R}} = U(k, l)$, $K_{\mathbb{R}} = U(k) \times U(l)$, and θ-stable parabolic $\mathfrak{k} \oplus \mathfrak{s}^+$.

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If λ is an integral parameter which is singular for $\mathfrak{g}$ but regular for $\mathfrak{l}$, then it has a certain number of pairs of repeated coordinates.

Deleting all these repeated coordinates, we get a regular parameter for a smaller pair $(\mathfrak{g}', \mathfrak{p}')$.

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Our approach is to construct the resolutions more directly and more explicitly, using the Penrose transform, in the dual setting of homogeneous bundles and differential operators. In particular, we do not use Enright-Shelton equivalences.

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(By "more explicitly", I mean that the differential operators in the bundle setting are very explicit, which is useful for applications in parabolic geometry and makes it possible to pass to the "curved case".)

Passing to G-equivariant bundles

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In particular, if V is the irreducible finite-dimensional representation F_λ^*, we denote the sheaf of holomorphic sections of the corresponding bundle by $\mathcal{O}_\mathfrak{p}(\lambda)$.

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In particular, if V is the irreducible finite-dimensional representation F_λ^*, we denote the sheaf of holomorphic sections of the corresponding bundle by $\mathcal{O}_\mathfrak{p}(\lambda)$.

One shows that the G-invariant differential operators between $\mathcal{O}_\mathfrak{p}(\lambda)$ and $\mathcal{O}_\mathfrak{p}(\mu)$ are in one-to-one correspondence with $\mathfrak{g}$-maps between Verma modules $V_\mathfrak{p}(\mu)$ and $V_\mathfrak{p}(\lambda)$.

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The arrows now go in the opposite direction, i.e., in the same direction as the arrows in the Hasse diagram.

Penrose transform

R.J.Baston, M.G.Eastwood, *The Penrose transform: its interaction with representation theory*, Oxford Mathematical Monographs, Clarendon Press, Oxford 1989.

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Then transfer information from G/Q to G/P, by pulling up to $G/(P \cap Q)$, then pushing down to the other side.

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Let X be the big cell in G/P, or a ball in the big cell. Let $Y = \tau^{-1}(X)$ and let $Z = \eta(Y)$.

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Furthermore, the fibers of $\eta : Y \to Z$ are contractible.

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In this situation, one can start with an $\mathcal{O}$-module $\mathcal{S}$ on Z, and pull it back to Y (sheaf-theoretic, not $\mathcal{O}$-module pullback).

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Now one resolves $\eta^{-1}\mathcal{S}$ by locally free sheaves Δ^p, and pushes this resolution down to X by the derived direct image.

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Then there is a "hypercohomology spectral sequence"

$$E_1^{pq} = \Gamma(X, \tau_*^q \Delta^p) \Rightarrow H^{p+q}(Z, \mathcal{S}).$$

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Since $\mathfrak{l}$ is the product of two $\mathfrak{gl}$ factors (of sizes $k \leq l$), the coordinates of λ can be repeated at most twice. Assume that there are j repeated coordinates ($j \leq k$) and s non-repeated coordinates ($2j + s = n + 1$).

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We now choose Q with Levi $GL(j) \times GL(j + s)$. Then there is exactly one permutation $\bar{\lambda}$ of λ which is dominant and regular for the Levi of $\mathfrak{q}$; the first group of coordinates must contain each of the repeated coordinates once.

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We pull this restriction back to Y and resolve it using the relative BGG resolution $\Delta^p(\lambda)$ with respect to the fiber. The part of λ corresponding to the fiber is regular, so this is essentially the BGG resolution of a finite-dimensional module.

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The direct images of $\Delta^p(\lambda)$ can be computed explicitly using the Bott-Borel-Weil theorem. We show that the nonzero sheaves obtained in this way can be organized into a BGG resolution on G/P, corresponding to our singular λ.

Example

Consider the dominant weight (55432210). The relative BGG resolution has the form

$$
\begin{array}{ccccc}
(52|54|3210) & (52|53|4210) & (52|52|4310) & (52|51|4320) & (52|50|4321) \\
 & (52|43|5210) & (52|42|5310) & (52|41|5320) & (52|40|5321) \\
 & & (52|32|5410) & (52|31|5420) & (52|30|5421) \\
 & & & (52|21|5430) & (52|20|5431) \\
 & & & & (52|10|5432)
\end{array}
$$

(The arrows go right and down.)

Example

Now erase the first bar, and use Bott-Borel-Weil to obtain the direct images:

$$\begin{array}{ccccc} x & x & x & x & x \\ & 2 & x & 1 & 1 \\ & & x & 1 & 1. \\ & & & x & x \\ & & & & 0 \end{array}$$

(The numbers denote cohomology degree.)

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At every point of the Hasse diagram corresponding to the relative BGG resolution, d_1 has components $d_1^{(r)}$ in the directions of the arrows in the Hasse diagram emanating from that point.

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Then we define $d_i^r[a]$ by diagram chasing corresponding to the following picture.

Definition of the higher differentials

$$\begin{tikzcd}
a_0 \arrow[r] & a_1 & & & \\
& a_2 \arrow[u] \arrow[r] & a_3 & & \\
& & \cdots \arrow[u] & & \\
& & {} \arrow[r] & a_{2i-3} & \\
& & & a_{2i-2} \arrow[u] \arrow[r] & a_{2i-1},
\end{tikzcd}$$

Vertical arrows: d_0; horizontal arrows: $d_1^{(r)}$. So

$$d_1^{(r)} a_{2k} = a_{2k+1} = d_0 a_{2k+2}. \tag{1}$$

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$$d_i[a] = \bigoplus_r d_i^{(r)}[a].$$

Lemma

The above definition of d_i is good, i.e., independent of all the choices.

Proposition

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Then d is a differential, i.e., $d^2 = 0$.

Proof

It is enough to prove that for any two directions r, s at a point of the Hasse diagram of the fiber, and any $i, j \geq 1$,

$$d_i^{(r)} d_j^{(s)} + d_j^{(s)} d_i^{(r)} = 0.$$

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Similarly, $d_i^{(r)} d_j^{(s)} a$ is constructed using a sequence on the left face, followed by a sequence on the back face.

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Similarly, $d_i^{(r)} d_j^{(s)} a$ is constructed using a sequence on the left face, followed by a sequence on the back face.

We fix the sequences on the front and on the left face.

Proof - continued

One shows by diagram chasing that the sequence on the front face of the box can be pushed through the box in a zig-zag motion to the back face of the box, with all the sequences in between satisfying similar properties, and with the left ends following our chosen sequence on the left face.

Proof - continued

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The sequence obtained on the back face, together with our sequence on the left face, computes $d_i^{(r)} d_j^{(s)} a$. The sequence of the right ends of the intermediate sequences, together with our sequence on the front face, computes $d_j^{(s)} d_i^{(r)} a$.

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It follows from the construction that $d_i^{(r)} d_j^{(s)} a$ equals $d_j^{(s)} d_i^{(r)} a$ except for a minus sign. This proves the proposition.

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Recall also the hypercohomology spectral sequence:

$$E_1^{pq} = \Gamma(X, \tau_*^q \Delta^p(\lambda)) \Rightarrow E_\infty^{pq} = H^{p+q}(Z, \mathcal{O}_\mathfrak{p}(\lambda)).$$

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Since X is Stein, taking global sections commutes with taking cohomology of a complex. So it is enough to see that E_∞^{pq} vanishes in the degrees above $j(k - j)$. (Recall that the Levi of P is $GL(k) \times GL(l)$, while the Levi of Q is $GL(j) \times GL(j + s)$.)

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We prove the last statement by covering $\mathrm{Gr}(j,k)$ by $j(k-j)+1$ open Stein sets.