General gauge dynamics and BRST cohomology

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The problem setting

Given:

Classical gauge theory defined by Equations of Motion.

Problems:

- Identify all the gauge symmetries, gauge identities and count physical degrees of freedom;
- Construct the classical BRST complex;
- Quantize the dynamics;
- Connect symmetries with characteristics (conservation laws).

A reminder about deriving gauge symmetry, BRST embedding and quantising Hamiltonian dynamics.

Given: Hamiltonian equations subject to primary constraints.
How the problems are solved:

- Find all the gauge symmetries of the system:
  Dirac-Bergmann algorithm of deriving of complete set of constraints and identifying the first and the second classes. It also provides counting of the physical degrees of freedom.
- Constructing the classical BRST complex for the dynamics:
  The BFV method does the job. It starts from Hamiltonian system with complete set of the first class constraints.
- Quantize the dynamics:
  The standard deformation quantisation of the ghost-extended phase space is sufficient to quantise.

A reminder about deriving gauge symmetry, connecting symmetries and conservation laws, and BRST quantising Lagrangian dynamics.

Given: Equations follow from action principle, $T_i(\phi) \equiv \partial_i S(\phi) = 0$

How the problems are solved:

- Find all the gauge symmetries of the system:
  It is sufficient to find the identities between the equations. Gauge symmetries are derived from the identities;
- Connecting global symmetries and conservation laws. *Noether theorem does the job for the symmetry of action $L_\Psi S \doteq 0$.*
- Constructing the classical BRST complex for the dynamics:
  Classical BV master equation. It involves the original action and complete set of gauge symmetry generators.
- Quantize the dynamics:
  BV covariant quantization *gives both the quantum master equation for probability amplitude, and it also provides an explicit path-integral construction.*

Examples of non-Lagrangian field equations.

The list of the best known examples includes:
Interacting massless higher spin field equations, (anti-)self-dual Yang-Mills and Donaldson-Ulenbeck-Yau equations, 5-branes.

Some other classical field equations, being quite reasonable as such, but non-variational, are sided away, because no perspectives are seen to quantise, and/or to apply Noether theorems.
Examples of this type: the gravity equations involving only irreducible components of the curvature tensor

$$R=\Lambda, \qquad \textbf{or} \qquad \tilde{R}_{\mu\nu}=0, \qquad \tilde{R}_{\mu\nu}\equiv R_{\mu\nu}-\frac{1}{d}g_{\mu\nu}R, \quad g^{\mu\nu}\tilde{R}_{\mu\nu}\equiv 0$$

The eq. $R=\Lambda$ probably defines topological theory in $d=4$.
The eqs $\tilde{R}_{\mu\nu}=0$ comprise all Einstein's solutions, with all the possible cosmological constants - noticed by Einstein.

General gauge dynamics: condensed notation.

General classical dynamics are defined by two principal constituents:

- A set of fields ϕ^i;
- A set of field equations $T_a(\phi)=0$.

The "condensed" indices i,a include the space-time point x^{μ}, and all the discrete indices labeling components of fields, or equations. The field equations $T(\phi)=0$ are PDE's in x^{μ}.
Functions of fields, $F(\phi)$ are understood as the local functionals, the derivatives ∂_i by fields ϕ^i are variational.
In Lagrangian theory, i and a coincide, in general they don't.
Lagrangian field equations read $T_i(\phi)\equiv\partial_i S(\phi)=0$

The Dynamics Bundle and the Shell

The set of all the field configurations $\mathcal{M}\ni\phi$ is considered a manifold, and the solutions to the field equations form a sub-manifold $\Sigma\subset\mathcal{M}$, called the ***shell***.

$$\Sigma=\{\phi\in\mathcal{M}|T(\phi)=0\}.$$

A vector bundle $\mathcal{E}\mapsto\mathcal{M}$ is assumed to exist such that the l.h.s. of the field equations $T_a(\phi)$ are the components of the certain section of this bundle

$$T=T_a(\phi)e^a\in\Gamma(\mathcal{E}).$$

We term $\mathcal{E}$ as the ***dynamics bundle***.

In Lagrangian theory $\mathcal{E}$ is identified with T^*M, and the field equations are just components of an exact one-form:

$$T\equiv dS(\phi)=\partial_i S(\phi)d\phi^i\in\Lambda^1(\mathcal{M}).$$

Regularity conditions in general dynamics.

Consider the Jacobi matrix $J_{ai} \equiv \partial_i T_a(\phi)$
The regularity implies that

$$\mathrm{rank} J|_{U_\Sigma} = const$$

The map defined by J,

$$\Gamma(T\mathcal{M}) \xrightarrow{J} \Gamma(\mathcal{E})$$

in general, is neither surjective, nor is it injective, and the same is true for the dual map defined by the transposed Jacobi matrix J^*
In Lagrangian theory, where $\mathcal{E} = T^*\mathcal{M}$, J is the symmetric Van Wleck matrix: $\partial_i T_j = \partial^2_{ij} S(\phi)$ whose on-shell kernel defines the gauge symmetry, and simultaneously, Noether identities.

Gauge algebra and Noether identities in general dynamics.

The rectangular Jacobi matrix $J_{ai}=\partial_i T_a(\phi)$ has different left and right on-shell kernels spanned by basis elements $R^i_\alpha(\phi)$ and $L^a_A(\phi)$:

$$J_{ai}R^i_\alpha\big|_\Sigma=0, \qquad L^a_A J_{ai}\big|_\Sigma=0.$$

Basis elements $R^i_\alpha(\phi)$ of the right kernel are understood as gauge symmetry generators. The left kernel basis elements $L^a_A(\phi)$ are understood as generators of "Noether"identities.

Both sets of generators are defined modulo on-shell vanishing terms.

From the regularity of J_{ai} follows that the right kernel distribution is integrable on shell, and the left kernel is generated by Noether identity generators

$$R^j_\alpha\partial_j R^i_\beta-R^j_\beta\partial_j R^i_\alpha=U(\phi)^\gamma_{\alpha\beta}R^i_\gamma+W^{ia}_{\alpha\beta}T_a, \qquad L^a_A T_a\equiv 0,$$

In Lagrangian theory J is symmetric, and R^i_α and L^a_A coincide.

In general, they don't.

The condensed indices α,A labeling symmetries and identities can run the different sets.

1. Maxwell electrodynamics in the strength tensor formalism
Consider anti-symmetric rank 2 tensor subject to free Maxwell equations

$$T^{\nu} \equiv \partial_{\mu} F^{\mu\nu} = 0, \quad T_{\mu\nu\lambda} \equiv \partial_{[\mu} F_{\nu\lambda]} = 0. \tag{1}$$

There are no gauge symmetry for F, but the identities exist:

$$\partial_{\nu} T^{\nu} \equiv 0, \quad \partial_{[\rho} T_{\mu\nu\lambda]} \equiv 0. \tag{2}$$

2. Self-dual Yang-Mills fields
The (anti-)self-duality equations are invariant with respect to the usual gauge transformations of the Yang-Mills field A_{μ}. These equations are independent, however - no gauge identities at all. A similar phenomenon is observed with DUY equations.

Reducibility of gauge symmetries and identities in general dynamics.

Both gauge symmetry and Noether identity generators can be reducible, i.e. the "null-vectors" $\overset{(1)}{R}, \overset{(1)}{L}$ exist such that

$$\overset{(1)}{R}{}^{\alpha}_{\alpha_1}(\phi) R^i_{\alpha}(\phi)|_{\Sigma} = 0, \quad \overset{(1)}{L}{}^{A}_{A_1}(\phi) L^a_A(\phi)|_{\Sigma} = 0$$

The reducibility generators $\overset{(1)}{R}, \overset{(1)}{L}$ can be reducible in their own turn, so we have a sequence of the "null-vectors" $\overset{(k)}{R}, \overset{(l)}{L}, [k] = m, [l] = n$. In Lagrangian theory $m = n$, and the reducibility generators coincide for Noether identities and gauge symmetries. In general, these are different.

Reducibility of (m,n)-type.

The reducibility generators are supposed to define morphism of certain bundles, such that

$$0\longleftarrow\Gamma(\mathcal{F}^*_m)\overset{\overset{(m-1)}{R}{}^*}{\longleftarrow}\cdots\Gamma(\mathcal{F}^*_1)\overset{R^*}{\longleftarrow}\Gamma(T^*\mathcal{M})\overset{J^*}{\longleftarrow}\Gamma(\mathcal{E}^*)\overset{L}{\longleftarrow}\Gamma(\mathcal{G}_1)\overset{\overset{(1)}{L}}{\longleftarrow}\cdots\Gamma(\mathcal{G}_n)\longleftarrow 0$$

This sequence is on-shell exact as

$$\mathrm{Im}\overset{(k)}{R}=\mathrm{Ker}\overset{(k-1)}{R},\quad \mathrm{Im}R=\mathrm{Ker}J,\quad \mathrm{Im}\overset{(k)}{L}=\mathrm{Ker}\overset{(k-1)}{L},\quad \mathrm{Im}L=\mathrm{Ker}J^*$$

In Lagrangian theory $m=n$, $\mathcal{F}_k=\mathcal{G}_k,\forall k$, $\mathcal{E}=T^*M$, $J=J^*,R=L$,and the "wings"outside the central segment, defined by J^*, can be identified just by taking dual and transposed map.

In general, none of these coincidences occurs, and the theory is termed (m,n)-reducible.

Gauge identities and symmetries in linear field theories - syzygies.

Consider fields $\phi^i(x)$ subject to the linear PDE system

$$T_a(\phi) \equiv J_{ai}(\partial)\phi^i(x) = 0 \ , \qquad (3)$$

where $J_{ai}(\partial)$ is a rectangular matrix, whose entries are polynomials in partial derivatives $\partial_\mu = \frac{\partial}{\partial x^\mu}$.
Once J is a square matrix, which is Hermitian in the sense that $J_{ij}(\partial) = J_{ji}(-\partial)$, the equations (3) admit Lagrangian:

$$S[\phi(x)] = \frac{1}{2}\int dx \phi^i(x) J_{ij}(\partial)\phi^j(x)$$

Gauge symmetry generators $R^i_\alpha(\partial)$ (also polynomials in ∂) define (over)complete basis for the right kernel of J, i.e.

$$\forall R^i: \quad J_{ai}(\partial)R^i(\partial) = 0 \quad \Leftrightarrow \quad \exists r^\alpha(\partial): \ R^i(\partial) = r^\alpha(\partial)R^i_\alpha(\partial) \qquad (4)$$

The gauge symmetry transformations read

$$\delta_\epsilon \phi^i = R^i_\alpha(\partial)\epsilon^\alpha(x), \quad \delta_\epsilon T_a(\phi, \partial\phi, \partial^2\phi \dots) \equiv 0, \quad \forall \epsilon^\alpha(x)$$

Gauge identities and symmetries in linear field theories - syzygies.

The image of matrix $J^*(\partial)$ can be considered as a module over the ring of polynomials in ∂_μ, $\mu{=}0,1,\ldots,d{-}1$. In terms of algebra, the kernel R of J is the *first syzygy* of the module. The gauge symmetry generators $R^i_\alpha(\partial)$ are generatings of the module.
The generators can be *reducible* (in physics language), or there can be a *second syzygy* (in algebraic language):

$$\overset{(1)}{R}{}^{\alpha}_{\alpha_1}(\partial)R^i_\alpha(\partial)\equiv 0$$

Then, there will be gauge symmetry of gauge symmetry, i.e. the gauge parameters ϵ^α can be transformed without any impact on the gauge transformation of original fields

$$\delta_{\overset{(1)}{\epsilon}}\epsilon^\alpha = \overset{(1)}{R}{}^{\alpha}_{\alpha_1}(\partial)\overset{(1)}{\epsilon}{}^{\alpha_1}\ , \quad \delta_{\overset{(1)}{\epsilon}}\delta_\epsilon\phi^i\equiv 0$$

Gauge Identities, and their reducibility.

The image of $J(\partial)$ is a module over the ring of polynomials in ∂_μ. The kernel of J is the first syzygy of this module. The generators of gauge identities $L^a_A(\partial)$ constitute a generating set in the kernel

$$L^a(\partial)J_{ai}(\partial)=0 \;\Leftrightarrow\; \exists l^A(\partial): \;\; L^a=l^A L^a_A$$

The kernel is a module by itself, and it can have it's own syzygy, i.e. $\overset{(1)}{L}$ can exist such that

$$\overset{(1)}{L}{}^A_{A_1}(\partial)\; L^a_A(\partial)\equiv 0$$

There can be further syzugies/reducibilities of syzygies/reducibilities both for gauge symmetries and identities. The sequence of syzygies can be always chosen in the way to terminate in the finite number of steps, not exceeding d. This means, the gauge symmetry can't be infinitely reducible in linear systems.

Normal form of evolutionary equations, and pre-requisites for deformation quantization.

- The equations of motion are to be of the form:

$$\dot{x}^i = v^i(x) + Z^i_\alpha(x)\boldsymbol{\lambda}^\alpha, T_a(x)=0; \quad ZT \sim T, [Z,Z] \sim Z+T, [v,Z] \sim Z+T$$

- Weak Poisson bi-vector P should exist such that:

$$[Z,P] \sim Z+T, \quad [v,P] \sim Z+T, \quad [P,T] \sim Z+T, \quad [P,P] \sim Z+T$$

Bi-victor P turns the variety of on-shell gauge invariants into Poisson algebra. The gauge generators and time drift differentiate this algebra, not being Hamiltonian vector fields. Upon BRST embedding this is turned into the Poisson algebra of the cohomology classes. Quantisation results in associative $*$ in the cohomology. Quantum BRST operator and drift are the differentiations of $*$, although not interior.

Hamiltonian constrained system:

$$P^{ij} = \{x^i, x^j\}, \quad [P,P]=0, \quad v^i = \{x^i, H(x)\}, \quad Z^i_\alpha = \{x^i, T_\alpha\}, \quad \alpha \equiv a$$

Primary normal form of evolutionary equations.

Any ODE system can be depressed to the first order in the form of inhomogeneous constrained Pfaffian system:

$$\boldsymbol{\theta}_{Ji}(x)\dot{x}^i = V_J(x), \quad T_a(x) = 0$$

Let the vectors $Z^i_{\boldsymbol{\alpha}}(x)$ span the on-shell kernel of the Pfaff one-forms $Z^i_{\boldsymbol{\alpha}}(x)\boldsymbol{\theta}_{Ji}(x) \sim T(x)$, then the equations can be rewritten in the *primary normal form*:

$$\dot{x}^i = v^i(x) + Z^i_{\boldsymbol{\alpha}}(x)\boldsymbol{\lambda}^{\boldsymbol{\alpha}}, \quad T_a(x) = 0.$$

The vector field v is called a *primary drift*, and the vector distribution $\mathcal{Z} = \mathrm{span}\{Z_{\boldsymbol{\alpha}}\}$ is called a *primary characteristic distribution*. $\mathcal{Z}$ is not necessarily integrable, nor is it necessarily tangential to the primary constraint surface.

Extension of Dirac-Bergmann algorithm to general evolutionary eqs: key steps

1.Derivation of compatibility conditions for primary eqs:

- Checking the conservation of primary constraints.

$$\dot{T}_a(x) \approx v^i(x)\partial_i T_a + \boldsymbol{\lambda}^\alpha Z^i_\alpha(x)\partial_i T_a = 0$$

  Results can be three-fold:
  (i) some primary constraints can conserve identically;
  (ii) determining some of the multipliers $\boldsymbol{\lambda}_\perp$ as functions of x;
  (iii) appearance of the secondary constraints $T^{(2)}$.

- $\dot{T}^{(2)} \approx 0 \Rightarrow$ further secondary ('tetriary') constraints $T^{(3)}$, more fixed multipliers, identical conservation. $\dot{T}^{(3)} \approx 0 \Rightarrow \ldots$

- The iterative procedure ends when the new constraints stop appearing and/or all the multipliers are determined.

2. Complete normal form of the local dynamics.

After excluding determined multipliers and finding all the secondary constraints, the equations take the *complete normal form*:

$$\dot{x}^i = \tilde{v}^i(x) + \boldsymbol{\lambda}_{\|}^{\alpha} Z^i_{\alpha\|}, \qquad \tilde{T}(x)=0, \quad \tilde{T} = (T, T^{(2)}, T^{(3)}, \ldots)$$

The primary distribution is decomposed into tangential and transverse sub-distributions w.r.t. the complete constraint surface:

$$\boldsymbol{\mathcal{Z}} = \boldsymbol{\mathcal{Z}}_{\perp} \oplus \boldsymbol{\mathcal{Z}}_{\|}, \quad \boldsymbol{\mathcal{Z}}_{\|} \tilde{T}(x) \approx 0, \quad \dim \boldsymbol{\mathcal{Z}}_{\perp} = rank Z_{\alpha} \tilde{T}_a$$

The complete constraint set is also decomposed into transverse and tangential subsets w.r.t. to $\boldsymbol{\mathcal{Z}}$:

$$\tilde{T} = (T_{\perp}, T_{\|}) \quad Z T_{\|} \approx 0, \quad Z_{\perp} T_{\perp} = D, \quad \det D \neq 0$$

Complete drift $\tilde{v}^i = v^i - v^j \partial_j T_{\perp a} (D^{-1})^{ab} Z^i_{b\perp}$ is tangential to the complete constraint surface, $\tilde{v}\tilde{T} \approx 0$. Conservation of the transverse constraints determines all the multipliers corresponding to the transverse sub-distribution: $\boldsymbol{\lambda}^a_{\perp} = -(D^{-1})^{ab} v T_{\perp b}$.

3. Gauge symmetry of the complete normal form.

Given the equations in the complete normal form

$$\dot{x}^i - v^i(x) - \boldsymbol{\lambda}^{\boldsymbol{\alpha}} Z^i_{\boldsymbol{\alpha}}(x) = 0, \quad T_a(x) = 0; \qquad ZT \approx 0, \quad vT \approx 0 \qquad (5)$$

they are fully consistent, having no further consequences.
Let us find all the infinitesimal local gauge transformations for (5):

$$\boldsymbol{\delta}_{\epsilon} x^i = \sum_{n=0}^{p} R^i_{(p-n)}(x, \boldsymbol{\lambda}, \dot{\boldsymbol{\lambda}}, \ddot{\boldsymbol{\lambda}}, \ldots) \overset{(n)}{\varepsilon}, \quad \boldsymbol{\delta}_{\varepsilon} \boldsymbol{\lambda}^{\boldsymbol{\alpha}} = \sum_{n=0}^{p+1} U^{\boldsymbol{\alpha}}_{(p+1-n)}(x, \boldsymbol{\lambda}, \dot{\boldsymbol{\lambda}}, \ddot{\boldsymbol{\lambda}}, \ldots) \overset{(n)}{\varepsilon},$$

such that the equations are left invariant in the sense that their variations vanish on shell with ε being arbitrary function of time.
The first fact we find about the transformations is that the number of the independent parameters coincides to the dimension of $\mathcal{Z}$,
and the choice is always possible $\boldsymbol{\delta}_{\epsilon} x^i = Z^i_{\boldsymbol{\alpha}} \overset{(p)}{\varepsilon}{}^{\boldsymbol{\alpha}} + \cdots, \quad \boldsymbol{\delta}_{\varepsilon} \boldsymbol{\lambda}^{\boldsymbol{\alpha}} = \overset{(p+1)}{\varepsilon}{}^{\boldsymbol{\alpha}}$
where $\cdots$ stand for the lower order derivatives of the parameter.

4. Gauge distribution.

The lower order terms can be iteratively found for the gauge transformation, and their general structure is as follows.

- The derivatives of all the orders from the parameters are involved in the transformation without gaps, and with linear independent coefficients;
- The coefficients at the derivatives $\overset{(n)}{\varepsilon}{}^{\alpha}$ in the transformation span and are spanned by the *gauge distribution* $span\{R_{(0)}\}\cup\cdots\cup span\{R_{(p)}\}=\mathcal{Z}_V$
- The gauge distribution is a closure of the primary characteristic distribution

$$\mathcal{Z}_V=\mathcal{Z}\cup[\mathcal{Z},\mathcal{Z}]\cup[\mathcal{Z},v]\cup\cdots,$$

where $\cdots$ mean higher iterated commutators $\mathcal{Z}$ and v

5. Local gauge invariants.

The physical observables are the on-shell gauge invariants:

$$\delta_\varepsilon O(x,\boldsymbol{\lambda},\dot{\boldsymbol{\lambda}},\ddot{\boldsymbol{\lambda}},\ldots)\approx 0$$

As $\delta_\varepsilon \boldsymbol{\lambda} = \overset{(p+1)}{\varepsilon} + \ldots$, the local physical observables are defined as the phase space on-shell invariants of the gauge distribution:

$$ZO(x)_{|T(x)=0}=0, \quad \forall Z\in\mathcal{Z}_V \quad \Leftrightarrow \quad \delta_\varepsilon O(x)_{|T(x)=0}=0$$

The observables are considered equivalent if their difference vanishes on shell,

$$O_1\sim O_2 \quad \Leftrightarrow \quad (O_1-O_2)_{T(x)=0}=0.$$

The time evolution of the equivalence classes is consistent with the invariance, and only the invariants evolve causally:

$$\dot{O}=vO+\boldsymbol{\lambda}^\alpha Z_\alpha O, \quad T(x)=0; \qquad \delta_\varepsilon \dot{O}\approx 0 \quad \Leftrightarrow \quad Z_V O\approx 0$$

6. The involutive normal form of the local dynamics.

The complete normal form is sufficient for classical BRST embedding and covariant quantisation. But it is insufficient for the deformation quantization. Introduce the *involutive normal form*

$$\dot{x}^i = v^i(x) + Z^i_{V\boldsymbol{\alpha}(x)}\boldsymbol{\lambda}^{\boldsymbol{\alpha}}, \qquad T_a(x) = 0$$

where independent $\boldsymbol{\lambda}$'s are included entire gauge distribution $\mathcal{Z}_V$. These equations involve more variables than the complete normal equations, and even for the original variables, they have different gauge symmetry transformations:

$$\boldsymbol{\delta}_{\varepsilon} x^i = Z_{V\boldsymbol{\alpha}}\varepsilon^{\boldsymbol{\alpha}}$$

These transformations involve more parameters, but without time derivatives. The involutive normal form is equivalent to the complete normal form in the sense that the gauge invariants remain the same, and have the same time evolution.

Definitions and terminology.

- The order of eq. is the maximal order of derivatives involved; The order of system is the maximal order of the eqs involved;
- A system of order n is said *involutive* if any differential consequence of the order less than or equal to n is already contained in the system.
- Any regular system can be brought to involution by inclusion of the lower order differential consequences. Then, it is said to be the *involutive closure* of the original system.
- The maximal order of derivative of gauge parameter ϵ^α is said the order of gauge symmetry generator R^i_α;
- The order of gauge identity is a sum maximal order of the identity generator L^a_A and the order of the eq. T_a it acts on.

Identification of gauge algebra for general field theory. Involutive closure and implicit gauge identities.

Remark 1. If the system is not involutive, it is equivalent to its involutive closure. The involutive closure has the same gauge symmetry, while it may have extra *implicit* gauge identities.

Remark 2. The involutive closure of Lagrangian system is not necessarily Lagrangian.

Example of involutive closure and implicit identity: Proca.

$$T_\mu \equiv (\eta_{\mu\nu}\,\Box - \partial_\mu\partial_\nu - m^2\,\eta_{\mu\nu})A^\nu = 0\,, \qquad \mathrm{ord}(T_\mu) = 2\,. \tag{6}$$

Involutive closure is got by inclusion of the first order consequence:

$$T_\perp \equiv \partial_\mu A^\mu = 0\,, \qquad \mathrm{ord}(T_\perp) = 1. \tag{7}$$

The involutive closure has the third order gauge identity:

$$L^a T_a \equiv 0\,, \quad a = (\mu, \perp), \quad L = (\partial^\mu, m^2)\,, \quad \mathrm{ord}(L) = 3 \tag{8}$$

Involutive closure and covariant degree of freedom count

The number of physical degrees of freedom $\mathcal{N}$ is understood as the number of independent Cauchy data modulo gauge transformations.

Given the involutive system with gauge symmetries and identities, $\mathcal{N}$ reads:

$$\mathcal{N}=\sum_{k=0}^{\infty}k(t_k-l_k-r_k). \qquad (9)$$

- t_k is a number of equations of order k;
- l_k is the number of gauge identities of k-th order;
- r_k is the number of gauge symmetries of kth order

Example - Proca: $t_2=4$, $t_1=1$, $l_3=1$, hence $\mathcal{N}=2{\cdot}4+1{\cdot}1-3{\cdot}1=6$, that corresponds to 3 polarizations of massive spin 1 in $d=4$.

The problem of consistent inclusion of interactions.

Given: Free/linear EoM's $T_a^{(0)}(x){=}0$, or quadratic action $S_{(0)}(x)$.

Find: $T_a(x){=}T_a^{(0)}(x){+}T_a^{int}(x)$ or $S(x){=}S_{(0)}(x){+}S_{int}(x)$ such that the number of degrees of freedom does not change.

Perturbative solution – Noether procedure:

The idea is to deform action and gauge symmetry order by order

$$S{=}S_{(0)} + gS_{(1)} + g^2 S_{(2)} + \cdots ; \quad R{=}R_{(0)} + gR_{(1)} + g^2 R_{(2)} + \cdots ; \tag{10}$$

$$R_{(0)}{}^i_\alpha \partial_i S_{(1)} + R_{(1)}{}^i_\alpha \partial_i S_{(0)}{=}0 ; \tag{11}$$

$$R_{(1)}{}^i_\alpha \partial_i S_{(1)} + R_{(2)}{}^i_\alpha \partial_i S_{(0)} + R_{(0)}{}^i_\alpha \partial_i S_{(2)}{=}0 \tag{12}$$

$$\cdot \quad \cdot \quad \cdot$$

The procedure controls the mere fact that number of gauge symmetries does not change. This is insufficient to ensure consistency. As we will demonstrate, it is even unnecessary.

Involution and perturbative inclusion of consistent interactions

Given the *free involutive gauge system*,

$$T_a^{(0)}{=}0\ ,\qquad L_A^{(0)a}T_a^{(0)}{\equiv}0\ ,\qquad R_{\boldsymbol{\alpha}}^{(0)i}\partial_i T_a^{(0)}{\equiv}0\ ,\qquad (13)$$

perturbative inclusion of interaction is a deformation of the equations, identities and gauge symmetries by nonlinear terms,

$$T_a^{(0)}\ \rightarrow\ T_a = T_a^{(0)} + gT_a^{(1)} + g^2 T_a^{(2)}\ + \dots\ ,\qquad (14)$$

$$R_{\boldsymbol{\alpha}}^{(0)i}\ \rightarrow\ R_{\boldsymbol{\alpha}}^{i}{=}R_{\boldsymbol{\alpha}}^{(0)i} + gR_{\boldsymbol{\alpha}}^{(1)i} + g^2 R_{\boldsymbol{\alpha}}^{(2)i} + \dots\ ,\qquad (15)$$

$$L_A^{(0)a}\ \rightarrow\ L_A^{a}{=}L_A^{(0)a} + gL_A^{(1)a} + g^2 L_A^{(2)a} + \dots\ .\qquad (16)$$

Here g is a coupling constant, generators $L_A^{(1)a}$ and $R_{\boldsymbol{\alpha}}^{(1)i}$ are linear in fields; $T_a^{(1)}$, $L_A^{(2)a}$, and $R_{\boldsymbol{\alpha}}^{(2)i}$ are bi-linear, etc.
Notice that in each order of the deformation, the orders of equations, identities and symmetries can never decrease.

Involution and perturbative inclusion of interactions

The perturbative consistency implies that deformed EoM's posses deformed gauge symmetries and identities in every order in g:

$$L_A^a T_a \equiv 0 \,, \quad R_\alpha^i \partial_i T_a = U_\alpha^a T_a \tag{17}$$

The expansion in g reads:

$$\begin{gathered} R_\alpha^{(0)i} \partial_i T_a^{(1)} = U_{\alpha a}^{(1)b} T_b^{(0)} - R_\alpha^{(1)i} \partial_i T_a^{(0)}, \\ L_A^{(0)a} T_a^{(1)} + L_A^{(1)a} T_a^{(0)} = 0. \end{gathered} \tag{18}$$

$$\begin{gathered} R_\alpha^{(0)i} \partial_i T_a^{(2)} + R_\alpha^{(1)i} \partial_i T_a^{(1)} + R_\alpha^{(2)i} \partial_i T_a^{(0)} = U_{\alpha a}^{(1)b} T_b^{(1)} + U_{\alpha a}^{(2)b} T_b^{(0)}, \\ L_A^{(0)a} T_a^{(2)} + L_A^{(1)a} T_a^{(1)} + L_A^{(2)a} T_a^{(0)} = 0 \,, \end{gathered} \tag{19}$$

$$. \quad . \quad .$$

The relations (18), (19) impose restrictions on interaction even if there is no gauge symmetry. Resolving the relations above order by order one constructs all the consistent interactions. If any obstruction arise in some order, it is a no-go theorem.

Summary of the procedure for perturbative inclusion of interactions

1. The free system is brought to the involutive form.
2. All the gauge symmetries and identities are identified.
3. The interaction vertices are iteratively included to comply with three basic requirements in every order of coupling constant:

- The field equations have to remain involutive;
- The gauge algebra of the involutive system can be deformed, though the number of gauge symmetry and gauge identity generators remains the same as it has been in the free theory;
- The number of physical degrees of freedom, being defined by n_k,l_k,r_k, cannot change, while all the these numbers can.

This procedure ensures finding all the consistent interaction vertices, for any regular system of free field equations.

An example of by-passing the perturbative no-go theorem for consistent interactions.

Consider the following action in $2d$ Minkowski space:

$$S[\phi,A]=\int d^2x\phi\left(\partial_\mu A^\mu+\frac{g}{2}A_\mu A^\mu\right) . \tag{20}$$

The field equations read

$$\partial_\mu A^\mu+\frac{g}{2}A_\mu A^\mu=0 \ , \qquad D^-_\mu\phi=0, \tag{21}$$

where $D^\pm_\mu=\partial_\mu\pm gA_\mu$, and $\epsilon^{\mu\nu}D^-_\mu D^-_\nu=g\epsilon^{\mu\nu}\partial_\mu A_\nu\equiv gF$.
Unless $F\neq 0$, it is a topological theory, as there is a consequence $\phi=0$, while the two components of A_μ are subject to a single equation, so they are pure gauge.
In the free limit $g\to 0$, ϕ is still fixed, while A_μ should be pure gauge for the same reason as with $g\neq 0$.
However, the Noether procedure leads to the no-go theorem.

An example of by-passing the perturbative no-go theorem for consistent interactions.

Free Lagrangian $L = \phi \partial_\mu A^\mu$ has an *irreducible gauge symmetry*:

$$\delta_\varrho \phi = 0, \qquad \delta_\varrho A^\mu = \epsilon^{\mu\nu} \partial_\nu \varrho, \tag{22}$$

that gauges out A^μ, while $\phi = 0$ shell.The free model is topological. The cubic vertex ϕA^2 is *not invariant* w.r.t. (22) even modulo a total divergence and the free equations,

$$\delta_\varrho \int d^2x \phi A^2 = -2 \int d^2x \phi F \varrho \neq 0 \ . \tag{23}$$

It is a standard no-go theorem for the cubic interaction.
The interaction is consistent, however, in the sense that it does not change the degree of freedom number.
The explanation is that the interacting theory has the *reducible gauge symmetry* with a smooth limit that differs from (22)

Multiple choice of gauge symmetries and consistency of interactions.

Lagrangian $L = \phi(\partial_\mu A^\mu + \frac{g}{2} A^2)$ enjoys gauge symmetry

$$\delta_\varepsilon \phi = 0 \,, \qquad \delta_\varepsilon A^\mu = g\varepsilon^\mu - \epsilon^{\mu\nu} D^+{}_\nu (F^{-1} D^+_\lambda \varepsilon^\lambda) \,, \tag{24}$$

where ε^μ is gauge parameter. The gauge-for-gauge transform reads

$$\delta_\varkappa \varepsilon^\mu = \epsilon^{\mu\nu} D^+_\nu \varkappa \,, \tag{25}$$

The free limit of gauge transformations (24), (25) reads

$$\delta_\varepsilon \phi = 0, \quad \delta_\varepsilon A^\mu = -\epsilon^{\mu\nu} \partial_\nu (F^{-1} \partial_\lambda \varepsilon^\lambda), \quad \delta_\varkappa \varepsilon^\lambda = \epsilon^{\lambda\nu} \partial_\nu \varkappa. \tag{26}$$

These transformations reproduce the irreducible free transformation $\delta_\varrho A^\mu = \epsilon^{\mu\nu} \partial_\nu \varrho$ with $\varrho = -F^{-1} \partial_\lambda \varepsilon^\lambda$.
At the free level the reducible and irreducible transformations are equivalent, as each of them spans the on-shell kernel of the $d^2 S$.
The reducible symmetry ***is compatible*** *with interaction, while the irreducible one* ***is not.***

Conclusions on the involutive closure of gauge systems and consistent inclusion of interactions.

Algorithm for inclusion of interactions:

- The free system is brought to the involutive form;
- The generating set is chosen (the choice isn't unique) for gauge symmetries and identities of the involutive system;
- The deformations are iterated for EoM's, identities and symmetries in a consistent way with the DoF count relation.
- If one generating set of symmetries and identities obstructs interaction, another resolution can by-pass the obstruction.

Advantages against the Noether procedure

- It controls DoF number, not just gauge symmetry. All the vertices are identified once they comply with the DoF number;
- It applies to Lagrangian and non-Lagrangian systems;
- It allows one to by-pass the no-go theorems for certain generating set by switching to another generating set.

BRST complex for gauge theory of (1,1)-type: introducing ghosts

The bundle $\mathcal{F} \mapsto M$ that "hosts" the gauge symmetries, is termed the Gauge Algebra Bundle.

The bundle $\mathcal{G} \mapsto M$ "hosts" the generators of gauge identities. It is termed the Noether Identity Bundle.

Ghosts for general (1,1) dynamics

Consider $Z_2 \bigotimes Z$-graded bundle

$$\mathcal{L} \mapsto M : \mathcal{L} = \Pi(\mathcal{F}[1]) \oplus \Pi(\mathcal{E}[-1]) \oplus (\mathcal{G}[-2])$$

The coordinates are denoted correspondingly:

$$C^\alpha, \quad \eta_a, \quad \xi_A, \qquad gh(C) = 1, \quad gh(\eta) = -1, \quad gh(\xi) = -2.$$

In Lagrangian case, η_a would be the anti-field ϕ^*_i to the original field ϕ^i; C^α - the gauge ghost, and ξ_A identified as anti-field to C.

Classical BRST embedding of general (1,1) dynamics

The BRST-differential $Q, gh(Q)=1$ is sought for in the form

$$Q \equiv Q^I(\varphi)\frac{\partial}{\partial\varphi^I} = T_a\frac{\partial}{\partial\eta_a} \quad + \quad \eta_a Z^a_A\frac{\partial}{\partial\xi_A} \quad + \quad C^\alpha R^i_\alpha\frac{\partial}{\partial\phi^i} \quad + \quad \cdots,$$

carrying all the information about the classical system $(\mathcal{E}, T)$ as such. Evaluating the condition $Q^2=0$ in the lowest order in r-degree, $|\xi|_r=2, \quad |\eta|_r=1$, one immediately comes to the relations $Z^a_A T_a \equiv 0$, $R^i_\alpha \partial_i T_b = U^a_{\alpha b}(\phi) T_a$ characterizing $T_a(x)=0$ as a set of gauge invariant and linearly dependent equations of motion, with R and Z being the generators of gauge transformations and Noether identities, respectively.

Lagrange anchor - a preliminary observation.

Consider first (0,0) type dynamics with $\mathcal{E}=T^*\mathcal{M}$, so the left hand sides of dynamical equations are the components of one-form:

$$T_i(\phi)d\phi^i=T\in\Lambda T^*\mathcal{M}, \qquad \Sigma=\{\phi\in\mathcal{M}\,|\,T_i(\phi)=0\} \qquad J_{ij}=\partial_i T_j$$

The fact that the dynamics are Lagrangian means that $dT=0$, or that the Jacobi matrix is symmetric, $J^*=J$, i.e. the following diagram commutes:

$$\begin{tikzcd}
\Gamma(T\mathcal{M}) \arrow[r, "J"] & \Gamma(T^*\mathcal{M}) \\
\Gamma(T\mathcal{M}) \arrow[r, "J^*"] \arrow[u, "id"] & \Gamma(T^*\mathcal{M}) \arrow[u, "id"]
\end{tikzcd}$$

Lagrange anchor: a diagrammatic definition.

The Lagrange anchor V defines a bundle homomorphism $V\colon\mathcal{E}^*\to TM$ such that the diagram

$$\begin{tikzcd} \Gamma(TM) \arrow[r, "J"] & \Gamma(\mathcal{E}) \\ \Gamma(\mathcal{E}^*) \arrow[u, "V"] \arrow[r, "J^*"] & \Gamma(T^*M) \arrow[u, "V^*"'] \end{tikzcd} \tag{27}$$

commutes on the shell. Off shell this explicitly reads

$$V_a^i\partial_i T_b - V_b^i\partial_i T_a = C_{ab}^c T_c$$

If the anchor was invertible, V^{-1} would be an integrating multiplier for the inverse problem of variational calculus, i.e. $\exists S(\phi)\colon\partial S_i=(V^{-1})_i^a T_a$.

Lagrange anchor for general (1,1) dynamics.

Consider now the case of (1,1) dynamics, and denote gauge algebra bundle $\mathcal{F}$, and Noether identity bundle $\mathcal{G}$. Then, the regularity of the (1,1) dynamics is formulated in terms of the following exact sequence of homomorphisms

$$0 \longrightarrow \Gamma(\mathcal{F}) \xrightarrow{R} \Gamma(TM) \xrightarrow{J} \Gamma(\mathcal{E}) \xrightarrow{L^*} \Gamma(\mathcal{G}^*) \longrightarrow 0$$

Its transpose reads:

$$0 \longleftarrow \Gamma(\mathcal{F}^*) \xleftarrow{R^*} \Gamma(T^*M) \xleftarrow{J^*} \Gamma(\mathcal{E}^*) \xleftarrow{Z} \Gamma(\mathcal{G}) \longleftarrow 0$$

Upon restriction to Σ these sequences make cochain complexes; the properties $L^*\circ J|_\Sigma = 0$ and $J^*\circ L|_\Sigma = 0$ follow from the gauge identity $L^a T_a = 0$.

Lagrange anchor for general (1,1) dynamics.

Given the Lagrange anchor, V, the previous two diagrams can be combined into the following unified one:

$$
\begin{tikzcd}
0 \arrow[r] & \Gamma(\mathcal{F}) \arrow[r, "R"] & \Gamma(TM) \arrow[r, "J"] & \Gamma(\mathcal{E}) \arrow[r, "Z^*"] & \Gamma(\mathcal{G}^*) \arrow[r] & 0 \\
0 \arrow[r] & \Gamma(\mathcal{G}) \arrow[r, "Z"] \arrow[u, "W"'] & \Gamma(\mathcal{E}^*) \arrow[r, "J^*"] \arrow[u, "V"'] & \Gamma(T^*M) \arrow[r, "R^*"] \arrow[u, "V^*"'] & \Gamma(\mathcal{F}^*) \arrow[r] \arrow[u, "W^*"'] & 0
\end{tikzcd}
$$

We know that the horizontal arrows of this diagram make cochain complexes upon restriction to the shell. Then, the on-shell commutativity of the squares implies that the upward arrows define a co-chain map. It is sufficient to have only V providing commutativity of the central block, then the map W can always be constructed.

The Lagrange anchor: coordinate free definition

Lagrange anchor is a linear map $d_{\mathcal{E}}:\Gamma(\wedge^n\mathcal{E})\to\Gamma(\wedge^{n+1}\mathcal{E})$ such that:

(i) $d_{\mathcal{E}}T=0$,

(ii) $d_{\mathcal{E}}$ is a derivation of degree 1, i.e.

$$d_{\mathcal{E}}(A\wedge B)=d_{\mathcal{E}}A\wedge B+(-1)^nA\wedge d_{\mathcal{E}}B, \qquad \forall A\in\Gamma(\wedge^n\mathcal{E}),\forall B\in\Gamma(\wedge^\bullet\mathcal{E}).$$

Here $\Gamma(\wedge^0\mathcal{E})\equiv C^\infty(M)$. Due to (ii), the operator $d_{\mathcal{E}}$ is defined by its action on coordinate functions ϕ^i and basis sections e^a of $\mathcal{E}$:

$$d_{\mathcal{E}}\phi^i=V^i_a(\phi)e^a, \qquad d_{\mathcal{E}}e^a=-\frac{1}{2}C^a_{bc}(\phi)e^b\wedge e^c. \tag{28}$$

Applying $d_{\mathcal{E}}$ to the section $T=T_ae^a$, one can derive the definition of the anchor in terms of coordinates from (i):

$$0=d_{\mathcal{E}}T=\frac{1}{2}(V^i_a\partial_iT_b-V^i_b\partial_iT_a-C^c_{ab}T_c)e^a\wedge e^b.$$

In the particular case where $d_{\mathcal{E}}{}^2=0$, T is nothing but a closed 1-$\mathcal{E}$-form associated to the Lie algebroid with the anchor V.

A brief preview of BRST quantization algorithm for not necessarily Lagrangian dynamics.

1. The classical BRST differential Q is constructed on the bundle $\mathcal{L}\mapsto M$: $\mathcal{L}=\Pi(\mathcal{F})[1]\oplus\Pi(\mathcal{E})[-1]\oplus\mathcal{G}[-2]$.
2. Given the Lagrange anchor, the classical BRST differential Q is promoted to a BRST charge Ω, being a function(al) on a bundle $T^*\mathcal{L}$. Q defines the first order of Ω in the momenta in $\mathcal{L}$. The second order is defined by the Lagrange anchor and the higher orders are sought from the equation $\{\Omega,\Omega\}=0$.
3. Upon quantisation, Ω turns into the quantum BRST operator $\hat{Q}=\Omega(\phi,\bar{\phi}\mapsto i\hbar\partial)=Q_{classical}+i\hbar(\cdots)$, $\hat{Q}^2=0$. The operator $\hat{Q}$ defines the probability amplitude by the equation $\hat{Q}\Psi(x)=0$.

Identification for Lagrangian system:

$a\equiv i,\quad T_i=\partial_i s(x),\quad V_a^j=\delta_a^j,\quad Q=(\cdot,S),\quad \hat{Q}=Q+i\hbar\Delta,\quad \Psi=e^{\frac{i}{\hbar}S}$

What is the main impact of the Lagrange anchor existence for general dynamics?

The Lagrange anchor, being found for the system of classical field equations, allows one to solve the following problems:

- Covariantly quantize dynamics in three different ways:
  1. Construct the quantum BV (or Schwinger-Dyson) equation for the amplitude;
  2. Convert not necessarily Lagrangian model in d into an equivalent topological Lagrangian theory in $d+1$ dimensions;
  3. Embed any field theory model into an *augmented* Lagrangian theory that allows to derive the quantum correlators for original fields.
- Connect conservation laws with symmetries;
- Equip the variety of conserved currents with the structure of Poisson algebra (in Lagrangian case that reduces to Gelfand-Dickey algebra).