PCS-structures and differential complexes

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- This talk reports on joint work with Tomás Salač (Prague), which is available in the arXiv preprints 1605.01161, 1605.01897 and 1701.01306.
- P(A)CS–structures are first order G–structures which have an underlying (almost) conformally symplectic structure. Any such structure admits a canonical connection, which provides a relation to special symplectic connections and thus to exotic affine homolonomies.
- In the PCS–case, there is a relation to quotients of parabolic contact structures by transverse infinitesimal automorphisms ("parabolic contactification").
- Using this, BGG sequences can be descended to sequences of invariant differential opertors which are complexes in the case of special symplectic connections. In important special cases, the cohomology of these complexes can be computed.

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Motivation

The orginal motivation for this project came from an article by M. Eastwood and H. Goldschmidt on integral geometry on $\mathbb{C}P^n$ (JDG, 2013). The main tool in that paper is a characterization of the image of the operator $D : \Gamma(S^k T^*\mathbb{C}P^n) \to \Gamma(S^{k+1} T^*\mathbb{C}P^n)$ defined by $D(\varphi_{b...c}) := \nabla_{(a}\varphi_{b...c)}$:

The image of D coincides with the kernel of a differential operator $\tilde{D}$ of order $k+1$ with values in a bundle associated to a specific irreducible representation of $Sp(2n,\mathbb{R})$ whose principal symbol is $Sp(2n,\mathbb{R})$–equivariant.

This is done by constructing a complex on $\mathbb{C}P^n$, which starts with the operators D and $\tilde{D}$ and showing that it has trivial cohomology in degree one. The result resembles a (contact) projective BGG complex in dimension $2n+1$.

(Almost) conformally symplectic structures

An almost conformally symplectic structure (ACS–structure) on an even dimensional manifold M is defined as a line–subbundle $\ell \subset \Lambda^2 T^*M$ such that each non–zero element of ℓ is non–degenerate. The structure is called *conformally symplectic* (or CS–structure) iff locally around each $x \in M$, there is $\tau \in \Gamma(\ell)$ with $\tau(x) \neq 0$ and $d\tau = 0$.

Equivalently, an ACS–structure is a G–structure corresponding to $CSp(2n, \mathbb{R}) \subset GL(2n, \mathbb{R})$. Such a G–structure has an intrinsic torsion, which vanishes if and only if the structure is CS.

Traditionally, such structures are often described using local sections of ℓ. For our purposes it will be much more convenient not to choose local sections.

Now let $\mathfrak{g}$ be a simple Lie algebra endowed with a *contact grading* $\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$. Thus $\mathfrak{g}_- = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ is a Heisenberg algebra on which $\mathfrak{g}_0$ acts by derivations.

Via restriction to $\mathfrak{g}_{-1}$, $\mathfrak{der}(\mathfrak{g}_-) \cong \mathfrak{csp}(\mathfrak{g}_{-1})$ and we obtain an injection $\mathfrak{g}_0 \hookrightarrow \mathfrak{csp}(\mathfrak{g}_{-1})$. If $\mathfrak{g}$ is of type C_n, this is an isomorphism. We'll exclude this case for now and return to it later.

Choosing a group G_0 with Lie algebra $\mathfrak{g}_0$, there is th notion of a G_0–structure on smooth manifolds M of dimension $\dim(\mathfrak{g}_{-1})$. Of course, this has an underlying ACS–structure, so we call these structures *PACS–structures* (parabolic ACS–structures) and *PCS–structures* if the underlying structure is CS.
Since contact gradings are well understood, one can derive explicity descriptions of all the PACS–structures, e.g. as almost quaternionic structures with a Hermitian ACS–structure.

If $\mathfrak{g}$ is not of type C_n, then $\mathfrak{g}_0 \subset \mathfrak{gl}(\mathfrak{g}_{-1})$ has vanishing first prolongation. There also is a natural normalization condition (constructed using the Kostant–codifferential), so any PACS–structure on M determins a canonical principal G_0–connection or a equivalently a linear connection on TM, whose torsion and curvature are basic invariants of the structure.

The torsion naturally splits as $T = T_i + T_h$, where T_i is the obstruction to the structure being PCS, while T_h is algebraically harmonic.

Remark: If $\mathfrak{g}$ is not of type A_n, then $\mathfrak{g}_0$ is a maximal subalgebra of $\mathfrak{csp}(\mathfrak{g}_{-1})$. In the A_n–case, there is a unique intermediate subalgebra, which is of infinite type.

Let $(M^\#, H)$ be a contact manifold. Then $\xi \in \mathfrak{X}(M^\#)$ is a transversal infinitesimal automorphism iff for all $\eta \in \Gamma(H)$ we get $[\xi, \eta] \in \Gamma(H)$ and for all $x \in M$, we have $\xi(x) \notin H_x$. (Equivalently, ξ is the Reeb field for some contact form.) Then ξ determines a one–dimensional foliation $\mathcal{F}$ of $M^\#$.

Any local leaf space M for $\mathcal{F}$ naturally inherits a CS–structure. Locally, any CS–structure can be realized in this way, unique up to local contactomorphism.

A contact grading on $\mathfrak{g}$ (not of type C_n), and an appropriate choice of group G_0 give rise to a *parabolic contact structure*. For a contact manifold $(M^\#, H)$ this is given by a reduction of structure group of $H \oplus (TM^\#/H)$ to G_0. This is a finite order structure, which determins a canonical Cartan connection of type (G, P), where P has Lie algebra $\mathfrak{g}_{\geq 0}$. Transversal infinitesimal automorphisms make sense in this setting but are much more rare.

The constructions of induced structures on quotients and of contactifications extend to the parabolic setting, relating a PCS–structure to the corresponding parabolic contact structures. In this setting one may also include structures corresponding to C_n–type Lie algebras. This then relates conformally Fedosov structures to contact projective structures.
We get a nice relation to special symplectic connections in the sense of Cahen–Schwachhöfer (Bochner–Kähler and –bi–Lagrangean, Ricci type, exotic symplectic holonomy)

Any special symplectic connection is the distinguished connection of a torsion–free PCS–structure. Concersely, the distinguished connection is special symplectic iff any local parabolic contactification is locally flat.

A representation of the group G_0 gives rise to a natural bundle $\mathcal{W}M \to M$ on PCS–structures and to a completely reducible natural bundle $\mathcal{W}M^\# \to M^\#$ on parabolic contact structures. For a PCS–quotient $q : M^\# \to M$, one can naturally identify $\Gamma(\mathcal{W}M)$ with the kernel of the action of the infinitesimal automorphism defining the quotient in $\Gamma(\mathcal{W}M^\#)$.

Using this locally, any invariant operator acting between sections of completely reducible natural bundles on some parabolic contact structure descends to a natural differential operator on the corresponding PCS–structure. Appying this to a differential complex, we obtain a descended complex.

Applying this to BGG complexes on locally flat parabolic contact structures, one obtains complexes associated to all special symplectic connections. The complexes of Eastwood–Goldschmidt are special cases for Ricci type connections.

Using the construction of BGG sequences from twisted de–Rham sequences, the cohomology of descended complexes can be computed. This works both locally and for global contactifcations like the Hopf–fibration $q : S^{2n+1} \to \mathbb{C}P^n$. These results vastly generlize what is needed for the applications to integral geometry.

The construction can also be applied to subcomplexes in curved BGG sequences and to relative BGG complexes (whose existence needs only weak assumptions). This gives rise to complexes on all Kähler and para–Kähler manifolds and generalizations of the latter, which remain to be explored.

Thank you for your attention!