Non–involutive symmetries on parabolic contact geometries

Jan Gregorovič

(joint work with L. Zalabova)

Parabolic contact geometry

Let G be simple Lie group and P parabolic subgroup corresponding to contact grading $\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$. Parabolic contact geometry is Cartan geometry $(p : \mathcal{G} \to M, \omega)$ of type (G, P).

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Equivalently, a parabolic geometry is a contact distribution $\mathcal{D} \subset TM$ together with an additional structure depending on the type (G, P). Automorphisms of the parabolic contact geometry are diffeomorphisms of M preserving the contact distribution and the additional structure.

Symmetries of parabolic contact geometries

Definition

We say that automorphism $f : M \to M$ of the parabolic contact geometry $(p : \mathcal{G} \to M, \omega)$ is a symmetry at $x \in M$ if

1. $f(x) = x$
2. $T_x f|_{\mathcal{D}} = -id_{\mathcal{D}}$

We say that symmetry is involutive if $f^2 = \mathrm{id}_M$. We say that parabolic contact geometry is symmetric if there is symmetry at each $x \in M$.

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If there is symmetry at x, then harmonic torsion vanishes at x. There can be symmetry at a point with non–zero harmonic curvature only of parabolic contact geometries of (para)–CR type or contact projective type.

Description of symmetries of parabolic contact geometries

Theorem

Let $\phi : \mathcal{G} \to \mathcal{G}$ be the P–bundle morphism covering symmetry f at x. Then there are $u \in \mathcal{G}$ in the fiber over x, inner automorphism of $\mathfrak{g}$ given by element $s \in G_0$ acting as $(-1)^i$ on $\mathfrak{g}_i$ and $Z \in \mathfrak{g}_2$ such that

$$\phi(u) = us\exp(Z).$$

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If $\phi(u) = us\exp(Z)$, then the symmetry is involutive if and only if $Z = 0$.
Thus involutive symmetries are linear in normal coordinates at u.
However, non–involutive symmetries are not linear in any normal coordinates at u.

Involutive symmetries in normal coordinates

We denote $Fl_t^{\omega^{-1}(X)}$ the flow of the constant vector field $\omega^{-1}(X)$ for $X \in \mathfrak{g}_-$.
Then $f(p \circ Fl_t^{\omega^{-1}(X)}(u))$ is normal coordinate system given by $u \in \mathcal{G}$ with coordinates (in some neighborhood of 0) in $\mathfrak{g}_-$.

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Suppose $\phi(u) = us$ holds. Then the underlying involutive symmetry f of ϕ has form

$$f(p \circ Fl_t^{\omega^{-1}(X)}(u)) = p \circ Fl_t^{\omega^{-1}(\mathrm{Ad}(s)X)}(u)$$

and thus
$f(p \circ Fl_t^{\omega^{-1}(X)}(u)) = p \circ Fl_{-t}^{\omega^{-1}(X)}(u)$ holds for $X \in \mathfrak{g}_{-1}$
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Comparison with model

Theorem (Frances, Melnick 2010)

Let φ be an automorphism of the parabolic geometry $(\mathcal{G} \to M, \omega)$ of type (G, P). Assume:

1. $\varphi(u) = up$ *for* $u \in \mathcal{G}$ *and* $p \in P$,
2. $\mathrm{Fl}_t^{\omega^{-1}(X)}(u)$ *is defined for* t *in an interval* I *around* 0 *and for some* $X \in \mathfrak{g}$
3. $pe^{tX} = e^{c(t)X} r(t)$ *in* G*, where* $r(t) : I \to P$ *with* $r(0) = p$ *and* $C : I \to I'$ *is a diffeomorphism fixing* 0.

Then $\mathrm{Fl}_t^{\omega^{-1}(X)}(u)$ *is also defined on* I' *and*

$$\varphi \circ \mathrm{Fl}_t^{\omega^{-1}(X)}(u) = Fl_{c(t)}^{\omega^{-1}(X)}(u) r(t).$$

Non-involutive symmetries in normal coordinates

If $\phi(u) = us\exp(Z)$ for $0 \neq Z \in \mathfrak{g}_2$, then the comparison with the model provides the following results:
There is unique $X_{-2} \in \mathfrak{g}_{-2}$ such that $(X_{-2}, E = [X_{-2}, Z], Z)$ is an sl_2–triple, where E is the grading element, and

$$s\exp(Z)e^{tX_{-2}} = e^{\frac{t}{1+t}X_{-2}}s\exp(\log(1+2kt)E)\exp(\frac{1}{1+t}Z),$$

thus $f(p \circ Fl_t^{\omega^{-1}(X_{-2})}(u)) = p \circ Fl_{\frac{t}{1+t}}^{\omega^{-1}(X_{-2})}(u)$ holds.

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thus $f(p \circ Fl_t^{\omega^{-1}(X_{-2})}(u)) = p \circ Fl_{\frac{t}{1+t}}^{\omega^{-1}(X_{-2})}(u)$ holds.
There are $X \in \mathfrak{g}_{-1}$ (they form open subset of projectivization of $\mathfrak{g}_{-1}$) such that $[X, [X, [X, Z]]] = 0$ and

$$s\exp(Z)e^{tX} = e^{-tX}s\exp(Z - [tX, Z] + \frac{1}{2}[tX, [tX, Z]]),$$

thus $f(p \circ Fl_t^{\omega^{-1}(X)}(u)) = p \circ Fl_{-t}^{\omega^{-1}(X)}(u)$ holds for such X.

Powers of non–involutive symmetries

If $\phi(u) = us\exp(Z)$ for $0 \neq Z \in \mathfrak{g}_2$, then $\phi^2(u) = u\exp(2Z)$, $\phi^{2k}(u) = u\exp(2kZ)$ and

$$\exp(2kZ)e^{tX_{-2}} = e^{\frac{t}{1+2tk}X_{-2}}\exp(\log(1+2tk)E)\exp(\frac{2k}{1+2tk}Z).$$

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Thus from comparison with model the following holds for harmonic curvature κ_H of homogeneity $h \geq 1$

$$(1+2tk)^h\kappa_H(p \circ Fl_t^{\omega^{-1}(X_{-2})}(u)) = \kappa_H(p \circ Fl_{\frac{t}{1+2tk}}^{\omega^{-1}(X_{-2})}(u)).$$

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For $t \neq 0$ and $k \to \infty$, $\kappa_H(p\circ Fl_{\frac{t}{1+2tk}}^{\omega^{-1}(X_{-2})}(u))$ converges to $\kappa_H(x)$ and $(1+2tk)^h$ converges to $\pm\infty$. This implies that $\kappa_H(p\circ Fl_t^{\omega^{-1}(X_{-2})}(u)) = 0$ for all $t \in I$.

Consequences of existence of non–involutive symmetry

The same arguments using chains as in [Cap, Melnick 2012] allow us to prove.

Theorem

On parabolic contact geometry, suppose there is non–involutive symmetry at x, then there is open neighborhood of x, where the parabolic contact geometry is flat.

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Theorem

On parabolic contact geometry, suppose there is non–involutive symmetry at x, then there is open neighborhood of x, where the parabolic contact geometry is flat.

This to together with our results for involutive symmetries prove.

Theorem

A symmetric parabolic contact geometry is either flat or a one dimensional fiber bundle over a symmetric space with conformaly symplectic structure invariant w.r.t. to the symmetries.

References

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