# Skyrmions, multi-instantons and $SU(\infty)$-Toda equation

The 37th Winter School Geometry and Physics
Srni, Czech Republic, 16 January 2017

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# Introduction

- Construction of Skyrmions from gravitational instantons

  (Dunajski 2013)

- Motivation: Atiyah, Manton and Schroers (AMS) model
  - elementary particles $\longleftrightarrow$ gravitational instantons

    electron $\leftrightarrow$ Taub-NUT, proton $\leftrightarrow$ Atiyah-Hitchin
  - quantum numbers $\longleftrightarrow$ topological invariants

- Multi-Taub-NUT as a system of electrons

  (Franchetti-Manton 2013)

- Skyrme model for baryons

- Dunajski (2013): Skyrmions from Taub-NUT and Atiyah-Hitchin instantons

- **Our work**: explore Skyrmions construction of Dunajski and related integrable system

  1. Skyrmions from multi-Taub-NUT instantons
  2. Associated metrics governed by solutions of $SU(\infty)$-Toda equation

# Skyrmion construction

- Skyrme model for baryons (Skyrme 1962):

$$U : \mathbb{R}^3 \to SU(2), \qquad U(\mathbf{x}) \to \mathbf{1} \quad \text{as} \quad \mathbf{x} \to \infty$$

  - degree $\pi_3(SU(2)) \quad \longleftrightarrow \quad$ baryon numbers
  - Skyrmions are fields with minimum energy.

- Atiyah-Manton (1989): $SU(2)$ Yang-Mills field on $\mathbb{R}^4$
  - holonomy along lines in one direction: $U : \mathbb{R}^3 \to SU(2)$
  - degree of $U$ equal to instanton number

# Yang-Mills field from spin connection

- Charap and Duff (1977):
  - Ricci-flat metric $g$ on 4-manifold $M$

    connection one-form $\leftrightarrow$ $O(4)$ Yang-Mills potential
  - $O(4) = SU(2) \times SU(2)$

    $\rightarrow$ $SU(2)$ self-dual Y-M field on $g$ background

- $SU(2)$ self-dual Yang-Mills instanton in Taub-NUT background (Pope and Yuille 1978)

# Multi-Taub-NUT background

- Multi-Taub-NUT metric (Gibbons-Hawking 1978)

  $$g = V(d\rho^2 + \rho^2 d\phi^2 + dz^2) + V^{-1}(d\psi + \alpha)^2, \quad d\alpha = *_3 dV$$

  - $V = 1 + \sum_{n=1}^{N} \frac{1}{\sqrt{\rho^2 + (z - z_n)^2}}$ axially symmetric

  $\psi \in [0, 4\pi)$

- $g = e_0 \odot e_0 + e_i \odot e_i, \quad i = 1, 2, 3$

  self-dual two-forms $\Sigma_i = e_0 \wedge e_i + \frac{1}{2}\varepsilon_{ijk} e_j \wedge e_k$

  self-dual spin connection $\gamma_{ij}$ : $d\Sigma_i + \gamma_{ij} \wedge \Sigma_j = 0$

- self-dual Yang-Mills potential

  $$A = \frac{1}{2}\varepsilon_{ijk}\, \gamma_{jk} \otimes \mathbf{t}_i, \qquad [\mathbf{t}_i, \mathbf{t}_j] = -\varepsilon_{ijk}\mathbf{t}_k$$

# Multi-Taub-NUT Skyrmions

- holonomy along $S^1$-orbits $\Gamma$ of $\dfrac{\partial}{\partial\phi}$

$$U(r,\theta,\psi) = \exp\left(-i\,\pi\,\gamma_j\,\tau_j\right)$$

$$\gamma_1 = -\cos\psi\left(\sin\theta + \frac{1}{r}\left(\frac{\hat{\alpha}}{V}\right)_\theta\right) \qquad V = 1 + \sum_{n=1}^{N} \frac{1}{||\mathbf{x}-\mathbf{x}_n||}$$

$$\gamma_2 = -\sin\psi\left(\sin\theta + \frac{1}{r}\left(\frac{\hat{\alpha}}{V}\right)_\theta\right) \qquad *_3 dV = d\alpha,\ \ \alpha = \hat{\alpha}(r,\theta)\,d\phi$$

$$\gamma_3 = \cos\theta - \left(\frac{\hat{\alpha}}{V}\right)_r \qquad (TN:\ \ V = 1 + r^{-1},\ \hat{\alpha} = \cos\theta)$$

- Need to choose a preferred gauge.
  - Taub-NUT and Atiyah-Hitchin: a preferred gauge is fixed by $SU(2)$ symmetry.
  - No obvious preferred gauge for Multi-Taub-NUT.

# $SU(\infty)$-Toda equation

- $\mathcal{B}$ has induced metric from multi-TN metric.
- Multi-TN is hyperKähler, $\frac{\partial}{\partial\phi}$ preserving a Kähler form.

(LeBrun 1991)

$$g = Wh + \frac{1}{W}(d\phi + \lambda)^2, \quad h = e^u(dx^2 + dy^2) + dt^2$$

$$u_{xx} + u_{yy} + (e^u)_{tt} = 0 \quad SU(\infty)\text{ Toda equation}$$

- $$x = -z + \ln\left(\frac{\rho^N}{\prod_{n=1}^N\left(z - z_n + \sqrt{\rho^2 + (z - z_n)^2}\right)}\right),$$

$$y = \psi, \qquad t = \frac{\rho^2}{2} + \sum_{n=1}^N \sqrt{\rho^2 + (z - z_n)^2}.$$

$$u(x,t) = \ln(\rho^2) \qquad V = 1 + \sum_{n=1}^N \frac{1}{\sqrt{\rho^2 + (z - z_n)^2}}$$

# Limit $N \to \infty$

- $V = 1 + \sum_{n=1}^{N} \frac{1}{\sqrt{\rho^2 + (z - z_n)^2}}$

- Ooguri-Vafa limit (1996): As $\rho \to \infty$, $V \to -\ln(\rho^2)$.

- Solution to $SU(\infty)$-Toda eq. $u_{xx} + u_{yy} + (e^u)_{tt} = 0$

  $u(x, t) = \ln(\rho^2)$ : $x = z\ln(\rho^2)$, $t = z^2 - \frac{1}{2}\rho^2(\ln(\rho^2) - 1)$

$$u^2(u-1)\,e^u + 2(t - x^2) = 0,$$

$u$ constant on the cylinder $x^2 - t = k$.

  - Solutions constant on central ellipsoids and planes (Tod 1995, Dunajski-PP 2011)

# More explicit expression $u_{xx} + (e^u)_{tt} = 0$

- Ward (1990): axially symmetric solution to Laplace's eq.

$$\rho V_{zz} + (\rho V_\rho)_\rho = 0$$

- Taub-NUT: $$u(x,t) = \ln\left(\frac{1/4 - t^2}{x^2}\right)$$

- multi-Taub-NUT: roots of polynomials

e.g. $N = 2$, $$u(x,t) = \ln\left(\frac{1/4}{(x+2t-zx)^2 - z^2}\right)$$

$$(x+2t-zx)^2(zx-2t)^2(2z-1) = 2x(zx-2t) - x^2$$

# Conclusion

- Construct Skyrmions from axially symmetric multi-Taub-NUT instantons
  - Demonstrate gauge dependence
- Relation with $SU(\infty)$-Toda equation
  - Induced metric on the space where Skyrmions live is governed by solution of $SU(\infty)$-Toda eq.
  - Obtain implicit expressions.