Czech Technical University in Prague
Faculty of Nuclear Sciences and Physical Engineering
Břehová 7, CZ - 115 19 Prague, Czech Republic

J. Tolar

From Lie gradings to gradings of operator algebras*

37th Winter School "Geometry and Physics", Srní 2017

*Based on collaboration with V. Teska

Outline

1. Introduction

2. The Pauli grading of the Lie algebra $\mathrm{gl}(n, \mathbb{C})$

3. Quantum kinematics of single $N-$level systems

4. Complementarity

5. Gradings of operator algebras

1. Introduction

Gradings of Lie algebras have been implicitly used since the beginning of Lie theory. Among the gradings of complex semisimple Lie algebras the most important ones are the gradings by the maximal torus, also called root or Cartan decomposition. Such a grading means a decomposition into the eigen-spaces of the maximal torus.

The question about the existence of other gradings and their symmetries has been raised in 1989 in the seminal paper by J. Patera and H. Zassenhaus. The systematic study of gradings by the research group collaborating with J. Patera in Prague (M. Havlíček, E. Pelantová, M. Svobodová, J. Tolar) described besides the root grading many new gradings, especially the Pauli gradings.

In quantum computing, the basic operators are the generalized Pauli matrices. They define finite quantum kinematics and generate the Pauli grading of the operator algebra. Our general analysis of gradings of operator algebras then provides the classification of composite finite quantum kinematics.

J. Patera and H. Zassenhaus, *On Lie gradings I*, Lin. Alg. Appl. **112** (1989), 87–159.

M. Havlíček, J. Patera, E. Pelantová, *On Lie gradings II*, Lin. Alg. Appl. **277** (1998) 97-125.

M. Havlíček, J. Patera, E. Pelantová, *On Lie gradings III*, Lin. Alg. Appl. **314** (2000) 1-47.

M. Havlíček, J. Patera, E. Pelantová and J. Tolar, *Automorphisms of the fine grading of* sl$(n, \mathbb{C})$ *associated with the generalized Pauli matrices*, J. Math. Phys. **43** (2002) 1083-1094.

E. Pelantová, M. Svobodová and S. Tremblay, *Fine grading of* sl$(p^2, \mathbb{C})$ *generated by tensor product of generalized Pauli matrices and its symmetries*, J. Math. Phys. **47** (2006) 013512 (18pp)

2. The Pauli grading of the Lie algebra $\mathrm{gl}(n, \mathbb{C})$

A grading of a Lie algebra $\mathcal{L}$ is a decomposition of $\mathcal{L}$ as a vector space into direct sum of subspaces

$$\Gamma : \quad \mathcal{L} = \bigoplus_{i \in I} \mathcal{L}_i$$

such that for any pair of indices $i, j \in I$ there exists an index $k \in I$ with the property

$$[\mathcal{L}_i, \mathcal{L}_j] := \{[X, Y] \mid X \in \mathcal{L}_i, Y \in \mathcal{L}_j\} \subseteq \mathcal{L}_k \quad \text{or zero.}$$

According to Patera and Zassenhaus, a special class of gradings of Lie algebras can be produced very effectively from Aut $\mathcal{L}$, the group of all automorphisms of $\mathcal{L}$.

If $\phi \in$ Aut $\mathcal{L}$ is *diagonalizable* and X, Y are its eigenvectors with non-zero eigenvalues λ, μ, $\phi\, X = \lambda\, X$, $\phi\, Y = \mu\, Y$, then clearly

$$\phi\, [X, Y] = [\phi\, X, \phi\, Y] = \lambda\mu\, [X, Y]. \tag{1}$$

This means that the element $[X, Y]$ is either an eigenvector of ϕ with eigenvalue $\lambda\mu$, or is the zero element. The given automorphism ϕ thus leads to a decomposition of the linear space $\mathcal{L}$ into eigenspaces of ϕ, which, according to (1), satisfies the definition of a grading.

Refinements of a given grading, i.e. further decompositions of the subspaces, can be obtained by adjoining further automorphisms commuting with ϕ. A grading which cannot be further refined is called fine.

Conversely, if a grading of a simple Lie algebra $\mathcal{L}$ is given, it defines a particular abelian subgroup $\mathrm{Diag}\,\Gamma \subset \mathrm{Aut}\mathcal{L}$ consisting of those automorphisms $\phi \in \mathrm{Aut}(\mathcal{L})$ which preserve Γ, $\phi(\mathcal{L}_i) = \mathcal{L}_i$, and are diagonal, $\phi X = \lambda_i X \quad \forall X \in \mathcal{L}_i, i \in I$, where $\lambda_i \neq 0$ depends only on ϕ and $i \in I$.

Theorem 1. (Patera-Zassenhaus)

Let $\mathcal{L}$ be a finite–dimensional simple Lie algebra over an algebraically closed field of characteristic zero. Then the grading Γ is fine, if and only if the diagonal subgroup $\mathrm{Diag}\,\Gamma$ *is a Maximal Abelian group of Diagonalizable automorphisms (MAD-group).*

If the Lie algebra is $\mathrm{gl}(n,\mathbb{C})$, then the classification of all MAD-groups of *inner* automorphisms corresponding to Ad-groups in $\mathrm{GL}(n,\mathbb{C})$ is given by

Theorem 2. (Havlíček, Patera, Pelantová)

$G \subset \mathrm{GL}(n,\mathbb{C})$ is an Ad-group, if and only if G is conjugated to one of the finite groups

$$\Pi_{\pi_1} \otimes \cdots \otimes \Pi_{\pi_s} \otimes D(n/\pi_1 \ldots \pi_s),$$

where $\pi_1, \ldots, \pi_s$ are powers of primes and their product $\pi_1 \ldots \pi_s$ divides n.[†]

[†]With the exception of the case $\Pi_2 \otimes \cdots \otimes \Pi_2 \otimes D(1)$.

Here $D(n)$ denotes the subgroup of $\mathrm{GL}(n,\mathbb{C})$ containing all regular diagonal matrices. Further, special unitary matrices are $k \times k$ diagonal matrices

$$Q_k = \mathrm{diag}(1, \omega_k, \omega_k^2, \ldots, \omega_k^{k-1}),$$

where ω_k is a primitive k-th root of unity, $\omega_k = \exp(2\pi i/k)$, and

$$P_k = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & & & \ddots & & \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

The unitary matrices P_k, Q_k are called generalized Pauli matrices and satisfy $Q_k^k = P_k^k = I_k$, $P_k Q_k = \omega_k Q_k P_k$. They are important in finite-dimensional quantum mechanics.

The discrete subgroup Π_k of $\mathrm{GL}(k,\mathbb{C})$ generated by powers of P_k, Q_k is called the *Pauli group* or the *finite Weyl-Heisenberg group*; for k odd

$$\Pi_k = \{\omega_k^l Q_k^i P_k^j | i, j, l = 0, 1, \ldots, k-1\}.$$

The simplest Ad-group is $G = D(n)$. The MAD-group corresponding to this Ad-group leads to the Cartan grading or root decomposition $\mathrm{gl}(n,\mathbb{C}) = h \oplus (\bigoplus_\alpha g_\alpha)$ which is a fine grading.

At the other extreme is the Pauli Ad-group $\Pi_n \otimes D(1) = \Pi_n$. The corresponding fine grading decomposes $\mathrm{gl}(n,\mathbb{C})$ into a direct sum of n^2 one-dimensional subspaces

$$\Gamma_\Pi : \quad \mathrm{gl}(n,\mathbb{C}) = \bigoplus_{(r,s)\in\mathbb{Z}_n\times\mathbb{Z}_n} \mathcal{L}_{rs},$$

where $\mathcal{L}_{rs} = \mathbb{C}X_{rs}$, with $X_{rs} = Q^r P^s$ forming the basis of $\mathrm{gl}(n,\mathbb{C})$ representing n^2 cosets of Π_n with respect to its center $\{\omega^l | l \in \mathbb{Z}_n\}$. Their products and commutators

$$[X_{rs}, X_{r's'}] = Q^r P^s Q^{r'} P^{s'} - Q^{r'} P^{s'} Q^r P^s = (\omega^{sr'} - \omega^{rs'}) X_{r+r',s+s'}$$

clearly satisfy the grading property with the index set I being the abelian group $\mathbb{Z}_n \times \mathbb{Z}_n$.

3. Quantum kinematics of single $N-$level systems

Already H. Weyl and J. Schwinger recognized the special role of generalized Pauli matrices in quantum physics. They are now used as basic operators in finite-dimensional quantum mechanics in general and in quantum computing in particular. The Pauli group Π_N (identical with the finite Weyl-Heisenberg group) is a subgroup of the unitary group U(N) acting in an N-dimensional complex Hilbert space.

Schwinger J., Unitary operator bases *Proc. Nat. Acad. Sci. U.S.A.* **46** (1960), 570–579, 1401–1415

Šťovíček P. and Tolar J., Quantum mechanics in a discrete space-time, *Rep. Math. Phys.* **20** (1984), 157–170

In the N-dimensional Hilbert space $\mathcal{H}_N = \mathbb{C}^N$ of a single N-level system an orthonormal basis $\mathcal{B} = \{|0\rangle, |1\rangle, \ldots |N-1\rangle\}$ and special unitary operators Q_N, P_N are defined by the relations

$$\begin{aligned} Q_N|j\rangle &= \omega_N^j|j\rangle, \\ P_N|j\rangle &= |j-1 \pmod{N}\rangle, \end{aligned}$$

where $j = 0, 1, \ldots, N-1$, $\omega_N = \exp(2\pi i/N)$. In the canonical or computational basis $\mathcal{B}$ the operators P_N and Q_N are just the generalized Pauli matrices.

Elements of $\mathbb{Z}_N = \{0, 1, \dots N-1\}$ label the vectors of the basis $\mathcal{B}$ with the physical interpretation that $|j\rangle$ is the (normalized) eigenvector of position at $j \in \mathbb{Z}_N$. In this sense the cyclic group $\mathbb{Z}_N$ is the configuration space for an N-level quantum system.

The action of $\mathbb{Z}_N$ on $\mathbb{Z}_N$ via addition modulo N is represented by unitary operators $U(k) = P_N^k$ with cyclic action on vectors $|j\rangle$

$$U(k)|j\rangle = P_N^k|j\rangle = |j - k \pmod{N}\rangle$$

Quantum kinematics of an N-level system is then expressed by the unitary operators $U(j) = P_N^j$, $V(\rho) = Q_N^\rho$, $j, \rho = 0, 1, \dots, N-1$ satisfying

$$U(j)V(\rho) = \omega_N^{j\rho} V(\rho)U(j).$$

The center $Z(\Pi_N)$ of the finite Weyl-Heisenberg group is the set of all those elements of Π_N which commute with all elements in Π_N, i.e. (for odd N) the complex numbers ω_N^k. Since the center is a normal subgroup, one can go over to the quotient group $\Pi_N/Z(\Pi_N)$. Its elements are the cosets labeled by pairs (i,j), $i,j = 0,1,\ldots,N-1$.

The quotient group is identified with the finite phase space $\mathbb{Z}_N \times \mathbb{Z}_N$. Denoting the cosets corresponding to elements (i,j) of the phase space by $Q^i P^j = \left\{\omega_N^l Q_N^i P_N^j | \quad l = 0,1,\ldots,N-1\right\}$, the correspondence

$$\Phi : \Pi_N/Z(\Pi_N) \to \mathbb{Z}_N \times \mathbb{Z}_N : Q^i P^j \mapsto (i,j),$$

is an *isomorphism of Abelian groups*, since

$$\Phi\left(\left(Q^i P^j\right)\left(Q^{i'} P^{j'}\right)\right) = \Phi\left(\left(Q^i P^j\right)\right)\Phi\left(\left(Q^{i'} P^{j'}\right)\right) =$$

$$= (i,j) + (i',j') = (i+i', j+j').$$

4. Complementarity

The quantal notion of complementarity concerns very specific relation among quantum observables. Consider two almost identical definitions.

Definition 1.

Two observables $\mathbf{A}$ *and* $\mathbf{B}$ *of a quantum system with Hilbert space of finite dimension* N *are called complementary, if their eigenvalues are non-degenerate and any two normalized eigenvectors* $|u_i\rangle$ *of* $\mathbf{A}$ *and* $|v_j\rangle$ *of* $\mathbf{B}$ *satisfy*

$$|\langle u_i|v_j\rangle| = \frac{1}{\sqrt{N}}.$$

In other words, if the system is prepared in any eigenstate of A, then the transition probabilities to all eigenstates of the complementary observable B are the same (equal to $1/N$). This means that exact knowledge of the measured value of $\mathbf{A}$ implies maximal uncertainty to any measured value of $\mathbf{B}$.

The basic unitary operators $A = Q_N$ and $B = P_N$ exactly satisfy this criterion of complementarity, since

$$|u_i\rangle = |i\rangle$$

and

$$|v_j\rangle = \frac{1}{\sqrt{N}} \sum_k \omega_N^{jk} |k\rangle.$$

For the next definition note that the (non-degenerate) eigenvalues a_i of $\mathbf{A}$ and b_j of $\mathbf{B}$ are in fact irrelevant, since only the corresponding orthonormal bases $|u_i\rangle$ and $|v_j\rangle$ are involved.

Definition 2.

Two orthonormal bases in an N-dimensional complex Hilbert space $\{|u_i\rangle | i = 1, 2, \ldots, N\}$ and $\left\{|v_j\rangle | j = 1, 2, \ldots, N\right\}$ are called mutually unbiased, if inner products between all possible pairs of vectors taken from distinct bases have the same magnitude $1/\sqrt{N}$,

$$|\langle u_i | v_j \rangle| = \frac{1}{\sqrt{N}} \quad \text{for all} \quad i, j \in \{1, 2, \ldots, N\}.$$

In the sense of these definitions one may call two measurements to be mutually unbiased, if the bases composed of the eigenstates of the corresponding observables (with non-degenerate spectra) are mutually unbiased.

5. Gradings of operator algebras

The operators of a finite quantum system with the complex Hilbert space of finite dimension N are elements of the full matrix algebra $M_N(\mathbb{C})$. In quantum information, $M_N(\mathbb{C})$ is equipped with the Hilbert-Schmidt inner product defined by the standard trace (defining also the Hilbert-Schmidt norm).

Since the automorphisms of $M_N(\mathbb{C})$ are only inner, we studied a class of fine gradings of $M_N(\mathbb{C})$ determined by MAD-groups of commuting inner automorphisms.

As a result, among the gradings of this class also the Pauli gradings emerge and they define inequivalent complementarity structures in $M_N(\mathbb{C})$.

Definition 3.

A grading of an associative algebra A is a direct sum decomposition of A as a vector space

$$\Gamma \;:\quad A = \bigoplus_{\alpha} A_{\alpha}$$

satisfying the property

for all $\quad x \in A_{\alpha}, \quad y \in A_{\beta} \quad$ the product $\quad xy \in A_{\gamma} \quad$ or zero.

As for Lie algebras, we define the grading subspaces A_α as eigen-spaces of automorphisms of A with non-zero eigenvalues.

For an automorphism ϕ of A, $\phi(xy) = \phi(x)\phi(y)$ holds for all $x, y \in A$. Now if $\phi(x) = \lambda_\alpha x$ defines the subspace A_α and $\phi(y) = \lambda_\beta y$ defines the subspace A_β, then

$$\phi(xy) = \phi(x)\phi(y) = \lambda_\alpha \lambda_\beta xy$$

defines the subspace A_γ with $\lambda_\gamma = \lambda_\alpha \lambda_\beta$.

In this way ϕ defines the grading decomposition

$$A = \bigoplus_\alpha \mathrm{Ker}(\phi - \lambda_\alpha).$$

Further commuting automorphisms may refine the grading.

Definition 4.

If $\Gamma\ :\quad A = \bigoplus_{\alpha\in I} A_\alpha$ and $\tilde{\Gamma}\ :\quad \tilde{A} = \bigoplus_{\beta\in \tilde{I}} \tilde{A}_\beta$ are two gradings of A, then $\tilde{\Gamma}$ is a refinement of Γ, if for any $\beta \in \tilde{I}$ there is $\alpha \in I$ such that $\tilde{A}_\beta \subseteq A_\alpha$.

Proper refinement: if for some β dim $\tilde{A}_\beta <$ dim A_α.

Γ is fine if it admits no proper refinement.

Γ is a coarsening of $\tilde{\Gamma}$.

Any grading is a coarsening of a fine grading.

For $M \in \mathsf{GL}(N, \mathbb{C})$, let $\mathsf{Ad}_M \in \mathsf{Int}(M_N(\mathbb{C}))$ denote the inner automorphism of $M_N(\mathbb{C})$ induced by the operator $M \in \mathsf{GL}(N, \mathbb{C})$, i.e.

$$\mathsf{Ad}_M(X) = MXM^{-1} \quad \text{for} \quad X \in M_N(\mathbb{C}).$$

Since the commuting inner automorphisms form an abelian subgroup of $\mathsf{Int}(M_N(\mathbb{C}))$, fine gradings of $M_N(\mathbb{C})$ are obtained using the *Maximal Abelian groups of Diagonalizable automorphisms* – the MAD-groups.

Definition 5.

We define $\mathcal{P}_N$ as the group

$$\mathcal{P}_N = \{\mathsf{Ad}_{Q_N^i P_N^j} | (i,j) \in \mathbb{Z}_N \times \mathbb{Z}_N\}.$$

It is an abelian subgroup of Int$(M_N(\mathbb{C}))$ because it is generated by two commuting automorphisms Ad_{Q_N}, Ad_{P_N}, each of order N.

$\mathcal{P}_N$ is isomorphic to the *quantum phase space* $\Pi_N / Z(\Pi_N)$ of a single N-level system identified with the abelian group $\mathbb{Z}_N \times \mathbb{Z}_N$.

All MAD-groups for $M_N(\mathbb{C})$ are given in the following

Theorem 3.

Any MAD-group contained in $\mathrm{Int}(M_N(\mathbb{C}))$ *is conjugated to one and only one of the groups of the form*

$$\mathcal{P}_{N_1} \otimes \mathcal{P}_{N_2} \otimes \ldots \otimes \mathcal{P}_{N_f} \otimes D(m),$$

where $N_i = p_i^{r_i}$ *are powers of primes,* $N = mN_1N_2\ldots N_f$ *and* $D(m)$ *is the image in* $\mathrm{Int}(M_N(\mathbb{C}))$ *of the group of* $m \times m$ *complex diagonal matrices under the adjoint action.*

If the MAD-group is $\mathcal{P}_N \otimes D(1)$, the corresponding fine grading is the *Pauli grading* and it decomposes $M_N(\mathbb{C})$ into N^2 one-dimensional subspaces spanned by the Weyl-Schwinger operators

$$M_N(\mathbb{C}) = \bigoplus_{\rho,j=0}^{N} \mathbb{C}W(\rho,j).$$

Note that the system of N^2 unitary Weyl-Schwinger operators

$$W(\rho,j) = \frac{1}{\sqrt{N}}\omega_N^{j\rho/2} Q_N^{\rho} P_N^{j}$$

forms a complete orthonormal basis in the linear space $M_N(\mathbb{C})$ of $N \times N$ complex matrices with respect to the Hilbert-Schmidt inner product.

For illustration, we give the list of MAD-groups in low dimensions:

- $n = 2$: $\mathcal{P}_2 \otimes D(1)$, $D(2)$
- $n = 3$: $\mathcal{P}_3 \otimes D(1)$, $D(3)$
- $n = 4$: $\mathcal{P}_4 \otimes D(1)$, $\mathcal{P}_2 \otimes \mathcal{P}_2 \otimes D(1)$, $\mathcal{P}_2 \otimes D(2)$, $D(4)$
- $n = 5$: $\mathcal{P}_5 \otimes D(1)$, $D(5)$
- $n = 6$: $\mathcal{P}_3 \otimes \mathcal{P}_2 \otimes D(1)$, $\mathcal{P}_3 \otimes D(2)$, $\mathcal{P}_2 \otimes D(3)$, $D(6)$

- $n = 7$: $\mathcal{P}_7 \otimes D(1)$, $D(7)$

- $n = 8$: $\mathcal{P}_8 \otimes D(1)$, $\mathcal{P}_4 \otimes \mathcal{P}_2 \otimes D(1)$, $\mathcal{P}_2 \otimes \mathcal{P}_2 \otimes \mathcal{P}_2 \otimes D(1)$, $\mathcal{P}_4 \otimes D(2)$, $\mathcal{P}_2 \otimes \mathcal{P}_2 \otimes D(2)$, $\mathcal{P}_2 \otimes D(4)$, $D(4)$

In each dimension N one MAD-group $\mathcal{P}_N \otimes D(1)$ with the trivial diagonal subgroup $D(1)$ is obtained and it induces exactly the Weyl-Schwinger decomposition. However, there are still fine gradings induced by $D(m)$, $m = 2, 3 \ldots$. They lead to decompositions which have the form of the Cartan root decompositions of Lie algebras sl$(m, \mathbb{C})$ extended by the unit matrix. They contain the Abelian Cartan subalgebra of dimension $m - 1$, the unit matrix and one-dimensional root subspaces spanned by nilpotent matrices.

For physical interpretation of these Cartan decompositions one may speculate that the "root subspaces" are connected with transition operators governing quantum time evolutions. Leaving these decompositions aside, we are left with the *Pauli gradings* which decompose $M_N(\mathbb{C})$ in direct sums of N^2 one-dimensional subspaces.

According to Theorem 3, for given N and $D(1)$ there is in general a multitude of inequivalent gradings with $N = N_1 N_2 \dots N_f$. On the other hand one should realize that $M_N(\mathbb{C})$ encompasses all operators of any quantum system with N-dimensional Hilbert space.

In this way we are forced to accept a new view on the relation between the general mathematical formalism and physical realizations of finite quantum systems.

Tolar J., A classification of finite quantum kinematics, *J. Phys.: Conf. Series* **538** (2014), 012020

It seems that there is a contradiction that $M_N(\mathbb{C})$ is the operator algebra of any N-dimensional quantum system and at the same time there may exist inequivalent quantum kinematics (complementarity structures) for given N.

However, there is no conflict, since $M_N(\mathbb{C})$ is the operator algebra not only for a single N-level system, but also for all other members of the set of inequivalent quantum kinematics for this N which just correspond to possible physical realizations of composite quantum systems. Note that $M_{N_1}(\mathbb{C}) \otimes \cdots \otimes M_{N_f}(\mathbb{C})$ is isomorphic with $M_N(\mathbb{C})$ for $N = N_1 N_2 \ldots N_f$.

Of course, each composite system has its *preferred quantum operators*. Let the Hilbert space of a composite system be the tensor product

$$\mathcal{H}_{N_1} \otimes \cdots \otimes \mathcal{H}_{N_f}$$

of dimension $N = N_1 \dots N_f$. The corresponding quantum phase space is an abelian subgroup of $\mathrm{Int}(M_N(\mathbb{C}))$ defined by

$$\mathcal{P}_{(N_1,\dots,N_f)} = \{\mathsf{Ad}_{M_1\otimes\cdots\otimes M_f} |\ M_i \in \Pi_{N_i}\}.$$

Its generating elements are the inner automorphisms

$$e_j := \mathsf{Ad}_{A_j} \quad \text{for} \quad j = 1, \dots, 2f,$$

where (for $i = 1, \dots, f$)

$$A_{2i-1} := I_{N_1 \cdots N_{i-1}} \otimes P_{N_i} \otimes I_{N_{i+1} \cdots N_f},$$

$$A_{2i} := I_{N_1 \cdots N_{i-1}} \otimes Q_{N_i} \otimes I_{N_{i+1} \cdots N_f}.$$

Thank you for your attention