Homology Smale-Barden manifolds with K-contact and Sasakian structures

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Symplectic and Kaehler manifolds

(M^{2n}, ω), $\omega \in \Omega^2(M)$ such that $d\omega = 0$ and $\omega^n = \omega \wedge \cdots \wedge \omega \neq 0$.
$J : TM \to TM, J^2 = -\mathrm{Id}$ is called *compatible* with ω:

$$\omega(JX, JY) = \omega(X, Y), \text{ and } g(X, Y) = \omega(JX, Y)$$

is a Riemannian metric on M.
Kaehler manifolds:

(g, ω, J), J is integrable, if

$$[JX, JY] - J[JX, Y] - J[X, JY] - [X, Y] = 0$$

If J is integrable, we say that (M, ω, J, g) is **Kaehler.**

Contact manifolds

(Closed) M^{2n+1}.
A contact form: $\eta \in \Omega^1(M)$ such that

$$\eta \wedge (d\eta)^n \neq 0$$

. Associates:

- $\mathcal{F} \subset TM$ a distribution $\mathcal{F} = \mathrm{Ker}\eta$ of codimension 1;
- *Reeb vector field* $\xi : M \to TM$ on M with the properties

$$\eta(\xi) = 1, i_\xi d\eta = 0.$$

K-contact manifolds

(M, η) is K***-contact*** if there is an endomorphism Φ of TM such that:

1. $\Phi^2 = -Id + \xi \otimes \eta$,
2. η is compatible with Φ:
$$d\eta(\Phi X, \Phi Y) = d\eta(X, Y)$$
for all X, Y and $d\eta(\Phi X, X) > 0$ for all nonzero $X \in \operatorname{Ker} \eta$;
3. $$g(X, Y) = d\eta(\Phi(X), Y) + \eta(X)\eta(Y)$$
is a Riemannian metric on M;
4. ξ is a Killing vector field with respect to the Riemannian metric g, $(\mathcal{L}_\xi g = 0)$.

Sasakian manifolds

For $(M, \eta, \Phi, \xi, g))$ define the ***metric cone*** as

$$\mathcal{C}(M) = (M \times \mathbb{R}^{>0}, t^2 g + dt^2).$$

Define $I : T\mathcal{C}(M) \to T\mathcal{C}M$:

1. $I(X) = \Phi(X)$ on $\operatorname{Ker}\eta$;
2. $I(\xi) = t\frac{\partial}{\partial t}, I(t\frac{\partial}{\partial t}) = -\xi$.

Definition

(M, η, ξ, Φ, g) is **Sasakian** if I is integrable.

Examples of K-contact and Sasakian: Boothby-Wang

(B, ω_B) symplectic, $[\omega] \in H^2(B, \mathbb{Z})$, $S^1 \to M \to B$, $[\omega_B] \in H^2(B, \mathbb{Z})$.
Kobayashi connection:

$$\theta \in \Omega^1(M, L(S^1)) = \Omega(M), \ d\theta = \pi^* \omega_B, \ \pi : M \to B.$$

$\implies$ a connection metric g on M, θ is contact.

Prototype

Any Boothby-Wang S^1-bundle carries a K-contact structure. If B is Kaehler, then M is Sasakian.

the Hopf bundle

$$S^1 \to S^{2n+1} \to \mathbb{C}P^n$$

General case

(M, η, ξ, Φ, g), ξ is Killing $\implies$ $S^1 \times M \to M$, ξ $\implies$ a Riemannian foliation.

Definition

(M, η, ξ, Φ, g) is quasi-regular, if $\exists$ a positive $q \in \mathbb{Z}$ such that $\forall m \in M \exists U_m$ each leaf passes U_m at most q times.

Theorem

(M, η, ξ, Φ, g) is quasi-regular, then M/S^1 has a natural structure of a symplectic cyclic orbifold, and the projection $M \to M/S^1$ is a Seifert bundle. If (M, η, ξ, Φ, g) is Sasakian, then M/S^1 is a cyclic Kaehler orbifold.

Arbitrary K-contact and Sasakian

Theorem (Rukimbira)

Any K-contact (Sasakian) admits a quasi-regular K-contact (Sasakian) structure.

Understanding the construction of K-contact manifolds

- Orbifolds are oriented 4-dimensional with atlas $(\{\tilde{U}_\alpha, \phi_\alpha, \Gamma_\alpha)\}, \tilde{U}_\alpha \subset \mathbb{R}^4, \Gamma_\alpha < SO(4), \Gamma_\alpha = \mathbb{Z}_{m_\alpha}$.
- Locally: $x \in X$, a chart $\phi : \tilde{U} \to U$, $\phi(0) = x$, isotropy group $\Gamma = \mathbb{Z}_m \implies$ any $\gamma \in \Gamma$ is conjugate to $\mathrm{diag}(\exp(2\pi i j_1/m), \exp(2\pi i j_2/m))$,
- $\tilde{U} = \mathbb{C}^2, \Gamma = \langle \xi \rangle, \xi = e^{\frac{2\pi i}{m}}$,

$$\xi \cdot (z_1, z_2) = (\xi^{j^1} z_1, \xi^{j_2} z_2).$$

- m is an order of a point $x \in X$.
- set $m_1 = gcd(j_1, m), m_2 = gcd(j_2, m_2)$ - multiplicities of the isotropy surfaces.

Understanding, local classification of points

1. Regular points: $m(x) = m = 1$,
2. isotropy points: $m(x) > 1$
3. smooth points: $m(x) > 1$, but $U \cong$ to a ball in $\mathbb{C}^2$:
   $m(x) > 1, m_1 > 1, m_2 > 1, m_1 m_2 = m$
4. singular points $=$ not smooth.

Extra assumption of semi-regularity

We consider orbifolds X, whose all points are smooth.

Example: smooth points

$$m = m_1 m_2, \mathbb{Z}_m = \langle \xi \rangle, m_1 > 1, m_2 > 1, gcd(j_1, j_2, m) = 1$$

Locally

$$D_1 = \{(z_1, 0)\}, D_2 = \{(0, z_2)\}$$

D_1 and D_2 are the isotropy surfaces intersecting transversely in one point, e.g. the multiplicity of D_1 is m_2, the multiplicity of D_2 is m_1, and

$\mathbb{Z}_m$ acts on $\mathbb{C}^2$ as $\mathbb{Z}_{m_2} \times \mathbb{Z}_{m_1} \implies \mathbb{C}^2/\mathbb{Z}_m \cong \mathbb{C}^2$.

Construction and determining orbit invariants

Proposition

Let X be a symplectic smooth oriented 4-manifold with symplectic surfaces D_i intersecting transversely and positively, and coefficients $m_i > 1$ such that $gcd(m_i, m_j) = 1$ if D_i and D_j intersect. Then, there exists a smooth symplectic orbifold X with isotropy sufaces D_i of multiplicities m_i.

Seifert bundles

A Seifert bundle over X is an oriented 5-manifold equipped with a smooth S^1-action and a continuous map $\pi : M \to X$ such that for each orbifold chart $(\tilde{U}, \phi, \mathbb{Z}_m)$, there is a commutative diagram

$$\begin{array}{ccc} (S^1 \times \tilde{U})/\mathbb{Z}_m & \xrightarrow{\cong} & \pi^{-1}(U) \\ \pi \downarrow & & \pi \downarrow \\ \tilde{U}/\mathbb{Z}_m & \xrightarrow{\cong} & U \end{array}$$

where the action of $\mathbb{Z}_m$ on S^1 is by multiplication by ξ, and the top diffeomorphism is S^1-equivariant.

Construction of Seifert bundles from orbit invariants

Proposition

Let X be an orbifold with orbit invariants D_i, m_i. Let $0 < j_i < m_i$ with $gcd(j_i, m_i) = 1$. Let $0 < b_i < j_i$ such that $j_i b_j = 1 (\bmod m_i)$. Finally, let B be a complex line bundle over X. Then there exists a Seifert bundle $f : M \to X$ with orbit invariants $\{(D_i, m_i, j_i)\}$ and the first Chern class

$$c_1(M/X) = c_1(B) + \sum_i \frac{b_i}{m_i}[D_i].$$

First Chern class

If $f : M \to X$ is a Seifet bundle, and $\mathbb{Z}_m \subset S^1$, then $S^1/\mathbb{Z}_m = S^1$ $\implies M/\mathbb{Z}_m$ is again a Seifert bundle. If $\mu = lcm_x(m(x))$ then $M/\mu = M/\mathbb{Z}_\mu$ is a manifold.

$$c_1(M/X) = \frac{1}{\mu} c_1(M/\mu) \in H^2(X, \mathbb{Q}).$$

Conclusion

Now we view K-contact and Sasakian manifolds as Seifert bundles over smooth cyclic orbifolds with symplectic (Kaehler) structures. Our data are:

- a smooth symplectic (Kaehler) manifold X,
- orbit invariants $\{D_i, m_i, j_i\}$
- $c_1(M/X)$

Boyer and Galicki program, 2009

Topological obstructions to the existence of K-contact/Sasakian structures on a compact manifold M of dimension $2n+1$:

1. the evenness of the p-th Betti number for p odd with $1 \leq p \leq n$, of a Sasakian manifold,
2. some torsion obstructions in dimension 5 discovered by Kollár,
3. the fundamental groups of Sasakian manifolds are special,
4. the cohomology algebra of a Sasakian manifold satisfies (a version of) the hard Lefschetz property,
5. formality properties of the rational homotopy type.

Program

Study topological properties of K-contact and Sasakian manifolds, in particular, understand the question of “K-contact vs. Sasakian”.

K-contact vs Sasaki

Do there exist simply connected closed K-contact manifolds which do not carry Sasakian structures?

- Biswas, Fernández, Muñoz, A.T. : yes, dim $M \geq 17$, 2016
- Hajduk, A.T. , independently Cappelletti-Montano et. al., yes, dim $M \geq 9$
- Muñoz, A.T., dim $M = 7$

Challenging: dim $M = 5$

Do there exist Smale-Barden manifolds which carry K-contact but do not carry Sasakian structures?

Dimension 5?

Converse

Let (X, ω) be a symplectic orbifold and let $M \to X$ be a Seifert bundle with Chern class $c_1(M/X) = [\omega] \in H_2(X, \mathbb{Z})$. Then M admits a quasi-regular K-contact structure. If (X, ω) is Kaehler, then this structure is Sasakian.

Kodaira-Baily theorem

The Hodge orbifold is a projective algebraic variety.

Possible strategy?

1. It is easy (now) to find symplectic manifold/orbifold which is not Kaehler, and to build a Seifert bundle

$$S^1 \to M \to X$$

determined by $c_1(M/X) = [\omega] \in H^2(X, \mathbb{Z})$.

2. How to prove that there is NO other Seifert fibering

$$S^1 \to M \to X'$$

with X' as a projective algebraic variety?

Result: Muñoz, Rojo, A.T.

An orbifold X is **semi-regular** if all points of X are smooth. We say that a 5-manifold M is ***homology Smale-Barden*** if $H_1(M) = 0$.

Theorem

There exists a homology Smale-Barden manifold which admits K-contact structures and does not carry any semi-regular Sasakian structure.

Scheme

Theorem 1

There exists a simply connected symplectic 4-manifold (X, ω) with $b_2(X) = 36$ and with 36 disjoint symplectic surfaces $S_1, ..., S_{36}$ such that:

- $[S_i]$ generate $H_2(X, \mathbb{Z})$,
- $g(S_1) = ... = g(S_9) = g(S_{11}) = ... = g(S_{19}) = g(S_{21}) = ... = g(S_{29}) = 1, g(S_{10}) = g(S_{20}) = g(S_{30}) = 3, g(S_{31)} = g(S_{32}) = g(S_{33}) = 2, g(S_{34}) = g(S_{35}) = g(S_{36}) = 1,$
- $S_i \cdot S_i = -1, i = 1, ..., 9.11, ..., 19, 21, ..., 29, S_j \cdot S_j = 1, j = 10, 20, 30, S_{31} \cdot S_{31} = -1, S_{32} \cdot S_{32} = -1, S_{33} \cdot S_{33} = 1, S_{34} \cdot S_{34} = -1, S_{35} \cdot S_{35} = -1, S_{36} \cdot S_{36} = 1.$

Scheme, continiuation

Proposition 1

Take a prime p and choose $g_i = g(S_i)$ as in Theorem 1. There exists a 5-dimensional K-contact manifold with $H_1(M,\mathbb{Z}) = 0$ and

$$H_2(M,\mathbb{Z}) = \mathbb{Z}^{35} \oplus (\oplus_{i=1}^{36}(\mathbb{Z}/p^i)^{2g_i}. \qquad (*)$$

Scheme, continuation

Proposition 2

Suppose M is a 5-manifold with

$$H_1(M,\mathbb{Z}) = 0, H_2(M,\mathbb{Z}) = \mathbb{Z}^k \oplus (\oplus_{i=1}^{k+1}(\mathbb{Z}/p^i)^{2g_i}, \ k \geq 0,$$

with p prime and $g_i \geq 1$. If $M \to X$ is a semi-regular Seifert bundle, then, necessarily

$$H_1(X) = 0, H_2(X,\mathbb{Z}) = \mathbb{Z}^{k+1}$$

and the ramification locus of X has $k+1$ disjoint surfaces D_i linearly independent in rational homology and of genus $g(D_i) = g_i$.

Proposition 3

Under the assumptions of Proposition 2, if X is a smooth Kaehler orbifold, then X is a smooth complex manifold and D_i are complex curves intersecting transversally.

Scheme, completion

Theorem 2

Let S be a smooth Kaehler surface with $H_1(S, \mathbb{Z}) = 0$ and containing $b = b_2(S)$ smooth disjoint complex curves with $g(D_i) = g_i > 0$ and spanning $H_2(S, \mathbb{Z})$. Assume that

- at least two $g_i > 1$,
- $g = max\{g_i\} \leq 3$

Then $b \leq 2g + 3$.

Main steps of understanding

- Kollár's work on homology of Smale-Barden manifolds with circle actions (Topology, 2006). There are no geometric restrictions on formulas, relating $H_*(M, \mathbb{Z})$ and $H_*(X, \mathbb{Z})$.
- An algebraic geometry argument. From the data, using geometric genus, irregularity, Noether formula, Riemann-Hodge relations, data on line bundles $\mathcal{O}(D_1)$... one derives $b_2 \leq 2g + 3$ (under the assumptions on genera)
- Constructing a simply connected symplectic 4-manifold X with a pattern of symplectic surfaces S_i generating the second homology, intersecting transversely, with the same genera but violating $b_2 \leq 2g + 3$.
- Defining an orbifold structure on X (with smooth points) by declairing S_i to constitue a ramification locus (isotropy sufaces), miltiplicities $m_i = p^i$, and taking a Seifert bundle $M \to X$ with $c_1(M/X) = [\omega]$.

Construction of X

Needed: (X, ω) 4-dimensional, simply connected, $S_1, ..., S_{36}$ symplectic surfaces with $g(S_i) \leq 3$, at least 2 have $g(S_i) > 1$ and such that S_i generate $H_2(X, \mathbb{Z})$.

What to do: try to do symplectic surgery of building blocks with "visible" symplectic surfaces realizing homology classes and controlling the resulting genera and simply connectedness. Tori, elliptic fibrations?

What constructions to use? Symplectic resolution of transverse intersections, symplectic blow up, Gompf symplectic sum.

Construction technique: Gompf symplectic sum

(M_1^4, ω_1), (M_2^4, ω_2), $j_1 : N_1 \to M_1, j_2 : N_2 \to M_2$ - symplectic embeddings, $\nu_1(N_1), \nu_2(N_2)$ - normal bundles, $e(\nu(N_1)) = -e(\nu(N_2))$, $g(N_1) = g(N_2)$.

Fix a symplectomorphism $N_1 \cong N_2$ and the orientation-reversing and orinetation preseving diffeomorphisms

$$\psi : \nu_1 \to \nu_2,\ \varphi : \nu_1(N_1) \setminus N_1 \to \nu(N_2) \setminus N_2$$

where $\varphi : \theta \circ \psi$, $\theta(x) = \frac{x}{||x||^2}$.

Definition

$$M_1 \#_{N_1 = N_2} M_2 = (M_1 \setminus N_1) \coprod (M_2 \setminus N_2) / \simeq$$

where $\simeq$ denotes the gluing by φ.

Gompf

$M = M_1 \#_{N_1 = N_2} M_2$ is a symplectic manifold.

Symplectic blow-up

Given: (X, ω), $q \in X$, Darboux ball D around q, J - standard complex structure in D.

$$\tilde{D} = \{(z_1, z_2), [w_1 : w_2]) \in D \times \mathbb{C}P^1 \mid z_1 w_2 = z_2 w_1\}$$

$$E = \{(0,0)\} \times \mathbb{C}P^1 \subset \tilde{D}$$

$$\tilde{X} = X \setminus D \cup_{\partial D \cong \partial \tilde{D}} \tilde{D}.$$

Known

Topologically $\tilde{X} = X \# \overline{\mathbb{C}P}^2$, $E = \overline{\mathbb{C}P}^1 \subset \overline{\mathbb{C}P}^2$

$$[E] \cdot [E] = -1.$$

Elliptic fibration $E(1)$

Two generic cubics in $\mathbb{C}P^2$: $p_0([x:y:z]) = 0$, $p_1([x:y:z]) = 0$ intersect in 9 points $p_1, ..., p_9$. For any point $q \in \mathbb{C}P^2 \setminus \{p_1, ..., p_9\}$ $\exists$ only one cubic $t_0 p_0 + t_1 p_1$ going through $q \implies$

$$f : \mathbb{C}P^2 \setminus \{p_1, ..., p_9\} \to \mathbb{C}P^1,\ f(q) = [t_0 : t_1]$$

Blow up $\mathbb{C}P^2$ in $p_1, ..., p_9$ and extend $f : \mathbb{C}P^2 \# 9\overline{\mathbb{C}P}^2 \to \mathbb{C}P^1$.

The generic fiber is a torus.

Two important properties

Proposition

Suppose $S_1 \subset M_1, S_2 \subset M_2$ are symplectic surfaces intersecting transversely and positively with symplectic surfaces N_1, N_2. Assume $S_1 \cdot N_1 = S_2 \cdot N_2 = d$. Then one can do the Gompf symplectic sum

$$M_1 \#_{N_1 = N_2} M_2$$

in a way that $S = S_1 \# S_2 \subset M_1 \#_{N_1 = N_2} M_2$ will be a symplectic surface with selfintersection $S_1^2 + S_2^2$ and the genus $g(S) = g(S_1) + g(S_2) - 1 + d$.

Known fact

Every exceptional sphere E_i in the blow up $\mathbb{C}P^2 \# 9 \overline{\mathbb{C}P}^2$ at p_i is a section of the elliptic fibration $f : E(1) \to \mathbb{C}P^1$.

Symplectic manifold X

A configuration of six tori in $\mathbb{T}^4$:

$$T_{12} = \{(x_1, x_2, \alpha_3, \alpha_4\}, T_{34} = \{(\alpha_1, \alpha_2, x_3, x_4)\}$$

$$T_{23} = \{(\beta_1, x_2, x_3, \beta_4\}, T_{14} = \{(x_1, \beta_2, \beta_3, x_4\}$$

$$T_{13} = \{((x_1, \gamma_2, x_3, \gamma_4)\}, T_{24} = \{(\gamma_1, x_2, \gamma_3, x_4)\}$$

Symplectic form

$$\omega = dx_1 \wedge dx_2 + dx_3 \wedge dx_4 + dx_2 \wedge dx_3 + \delta dx_1 \wedge dx_4 - \delta dx_1 \wedge dx_3, \ \delta > 0.$$

$$Z = \mathbb{T}^4 \#_{T_{12}=F_2} E(1)_2 \#_{T_{13}=F_3} E(1) \#_{T_{14}=F_4} E(1)_4$$

$F_2 \subset E(1)_2, F_3 \subset E(1)_3, F_4 \subset E(1)_4$ are generic fibers.

$$X = Z \# 2\overline{\mathbb{C}P}^2$$

Surfaces which are easily visible

$E(1)_2 = \mathbb{C}P^2 \# 9\overline{\mathbb{C}P}^2$ has 9 sections $E_1, ..., E_9$, $E_i \cdot E_i = -1$. They intersect a generic fiber F in one point. We have:

E_1 and F_2 intersect transversely, T_{12} intersects T_{34} transversely $\Longrightarrow$

$$S_1 = E_1 \# T_{34} \subset X$$

It is still a torus, with selfintersection -1. We get

$S_1, ..., S_9$ for the first copy F_2

$S_{11}, ..., S_{19}$ fo the second copy F_3,

$S_{21}, ..., S_{29}$ for the third copy F_4.

Total: 27 sufaces.

3 more surfaces

$L \subset E(1)$ a generic line.

Known

L intersects a generic fiber F in exactly 3 points.

In $\mathbb{T}^4 \#_{T_{12}=F_2} E(1)$ we see that ANY "parallel" copy of T_{34} intersects T_{12} transversely $\implies S_{10} = T'_{34} \# T''_{34} \# T'''_{34} \# L$ is a surface of genus 3. We get

$$S_{10}, S_{20}, S_{30}.$$

Problem with the fundamental group

X is one connected by Van Kampen. We don't know how to calculate $\pi_1(M)$ in the case of a Seifert bundle

$$S^1 \to M \to X.$$

Some references

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Quotations

J.-P. Bourgignon

“... several dominant figures of the mathematical scene of the XX century have, step after step along a 50 year period, transformed the subject [Kaehler geometry] into a major area of mathematics that has influenced the evolution of the discipline much further than could have conceivably been anticipated by anyone. “

Krzysztof Galicki

“Sasaki seemed to have had both the necessary intuition and a broad vision in understanding what is and what is not of true importance...”

Shigeo Sasaki (1912-1987)

- born in 1912 in Yamagate prefecture to a farmer's family, brought up by his uncle who was a superior of a Buddist Temple
- studied in Tohoku Imperial University (1932-1935)
- since 1935 worked in differential geometry under the guidance of Professor Kubota, Ph. D. in 1943
- in 1946 appointed to the vacant chair after Kubota's retirement
- 1962 - introduced the notion of "normal metric contact structure' which is equivalent to "Sasakian structure";
- Major works in 1962-1967, notes in Japanese, creation of a subfield in Riemannian geometry
- professorship in Tohoku University until 1976, visited to Princeton in 1952-1954, worked with Veblen, Morse and Chern
- His biography is not well known, even his obituary appeared only in a local newspaper