New constructions of symplectically fat bundles

Artur Woike

University of Warmia and Mazury in Olsztyn

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joint work with M. Bocheński, A. Szczepkowska and A. Tralle

The origin of fat bundles

1. A fat bundle is a bundle which has a lot of curvature, as opposed to a flat bundle.
2. In 1968 Weinstein introduces unflat bundles in a preprint written at MIT.
3. Unflat bundles were introduced to study Riemannian manifolds of positive and non-negative curvature.
4. The term unflat bundle was a slightly misleading one. A bundle which is not flat might not be unflat either, in fact most bundles are somewhere in between.
5. As a term fat bundles shows up for a first time in a paper written by Weinstein in 1980.
6. The same paper contains the Sternberg-Weinstein theorem relating fatness and symplectic fibrations.

Notations

- $G \longrightarrow P \longrightarrow B$ – a principal fiber bundle
- $\mathcal{H}$ – a horizontal distribution defining connection
- θ, Ω – the connection form and the curvature form
- $\langle\,,\rangle$ – the natural pairing between $\mathfrak{g}$ and its dual $\mathfrak{g}^*$
- Ω is a 2-form with values in $\mathfrak{g}$
- $\langle X, u\rangle = u(X)$ for all $X \in \mathfrak{g}$ and $u \in \mathfrak{g}^*$

Fat vectors

Definition

A vector $u \in \mathfrak{g}^*$ is fat, if the 2-form

$$(X, Y) \longmapsto \langle\Omega(X, Y), u\rangle$$

is non-degenerate for all horizontal vector fields X, Y.

Remark

If $u \in \mathfrak{g}^*$ is a fat vector, then its whole coadjoint orbit $\mathcal{O}(u)$ is fat.

Symplectically fat bundles

Theorem 1. (Sternberg, Weinstein)

Let there be given a principal fiber bundle

$$G \longrightarrow P \longrightarrow B$$

and a symplectic G-manifold F with a Hamiltonian G-action and a moment map $\mu : F \to \mathfrak{g}^*$. If there exist a connection in the above principal bundle such that all vectors in $\mu(F) \subset \mathfrak{g}^*$ are fat, then the total space of the associated bundle

$$F \longrightarrow P \times_G F \longrightarrow B$$

admits a fiberwise symplectic structure.

Thurston's theorem on symplectic fibrations

Theorem 2. (Thurston)

Let there be given a fiber bundle $F \longrightarrow M \longrightarrow B$ over a compact symplectic base B and a symplectic fiber (F, σ).
Assume that:

1) the structure group of the bundle reduces to the group of symplectomorphisms of the fiber;
2) there exists a cohomology class $a \in H^2(M)$ which restricts to the cohomology class $[\sigma]$ on the fiber.

Under these assumptions M admits a fiberwise symplectic form.

Remark

Thurston's theorem requires a compact and symplectic base.

Symplectically fat bundles

Proposition 1.

Let u be a fat vector in a principal fiber bundle

$$G \longrightarrow P \longrightarrow B.$$

Then the associated bundle

$$\mathcal{O}(u) \longrightarrow P \times_G \mathcal{O}(u) \longrightarrow B$$

is a symplectically fat bundle.

Notations

- G – a semisimple Lie group with a Lie algebra $\mathfrak{g}$
- $H \subset G$ – a maximal rank compact Lie subgroup
- B – the Killing form for G, which is non-degenerate on $\mathfrak{h} \subset \mathfrak{g}$
- $\mathfrak{g}^c$, $\mathfrak{h}^c$ – complexifications of the Lie algebra $\mathfrak{g}$ and $\mathfrak{h}$
- $\mathfrak{t}$ – a maximal abelian subalgebra in $\mathfrak{h}$
- $\mathfrak{t}^c$ – a Cartan subalgebra in $\mathfrak{g}^c$
- $\Delta = \Delta(\mathfrak{g}^c, \mathfrak{t}^c)$ – the root system for $\mathfrak{g}^c$ with respect to $\mathfrak{t}^c$
- $\Delta(\mathfrak{h})$ – the root system for $\mathfrak{h}^c$ with respect to $\mathfrak{t}^c$
- $\Delta(\mathfrak{h})$ is a subsystem of Δ

Notations

- $\mathfrak{m}$ – the orthogonal complement to $\mathfrak{h}$ in $\mathfrak{g}$ with respect to the Killing form B
- the decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ complexifies to $\mathfrak{g}^c = \mathfrak{h}^c \oplus \mathfrak{m}^c$
- we have following root decompositions

$$\mathfrak{g}^c = \mathfrak{t}^c + \sum_{\alpha \in \Delta} \mathfrak{g}^\alpha$$

$$\mathfrak{h}^c = \mathfrak{t}^c + \sum_{\alpha \in \Delta(\mathfrak{h})} \mathfrak{g}^\alpha$$

$$\mathfrak{m}^c = \sum_{\alpha \in \Delta \setminus \Delta(\mathfrak{h})} \mathfrak{g}^\alpha$$

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Notations

- the Killing form B for G is non-degenerate (since G is semisimple)
- we can identify Lie algebra $\mathfrak{g}$ with its dual $\mathfrak{g}^*$ via the Killing form B

$$\forall_{u \in \mathfrak{g}^*} \; u \longmapsto X_u$$

- this identification preserves the identification of $\mathfrak{h}$ and $\mathfrak{h}^*$ (since B is by assumption non-degenerate on $\mathfrak{h}$)
- $C \subset \mathfrak{t}$, C_α – a closed Weyl chamber and its wall determined by the root α

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Generalized Lerman's Theorem

Theorem 3.

Let G be a semisimple Lie group, and $H \subset G$ a compact subgroup of maximal rank. Suppose that the Killing form B of G is non-degenerate on the Lie algebra $\mathfrak{h} \subset \mathfrak{g}$ of the subgroup H.
The following conditions are equivalent:

1. A vector $u \in \mathfrak{h}^*$ is fat with respect to the canonical invariant connection in the principal bundle $H \longrightarrow G \longrightarrow G/H$.
2. The vector X_u does not belong to the set

$$Ad_H(\cup_{\alpha \in \Delta \setminus \Delta(\mathfrak{h})} C_\alpha).$$

3. The isotropy subgroup $V \subset H$ of $u \in \mathfrak{h}^*$ with respect to the coadjoint action is the centralizer of a torus in G.

Generalized Lerman's Theorem

Remark

1. The original Lerman's theorem dates back to 1988.
2. Lerman's original assumptions were as follows:
   - G is a compact semisimple Lie group;
   - G/H is a coadjoint orbit (and therefore symplectic).
3. Generalized Lerman's theorem was published in 2011.

Generalized Lerman's Theorem

Definition

We will call the vector $u \in \mathfrak{h}^*$ an admissible vector, if and only if its dual vector $X_u \in \mathfrak{h}$ (according to the identification via the Killing form) does not belong to the set

$$Ad_H(\cup_{\alpha \in \Delta \setminus \Delta(\mathfrak{h})} C_\alpha).$$

Homogeneous case

Theorem 4.

Let G be a semisimple Lie group and $H \subset G$ a compact subgroup. The canonical invariant connection in the principal bundle

$$H \longrightarrow G \longrightarrow G/H$$

admits fat vectors, if $\operatorname{rank} G = \operatorname{rank} H$.
If G is compact, the converse is also true.

Coadjoint orbits as fibers

Remark.

1. Any compact simply connected homogeneous symplectic manifold is symplectomorphic to a coadjoint orbit. However Proposition 1 is applicable only to coadjoint orbits of admissible vectors.
2. Only homogeneous spaces can have coadjoint orbits as images of the moment map. There is no possibility to extend the class of symplectically fat fiber bundles in a following way: take any symplectic G-manifold and require that $\mu(F)$ is a coadjoint orbit of some fat vector.

Homogeneous bundles

1. Bundles of the form

$$H/K \longrightarrow G/K = G \times_H (H/K) \longrightarrow G/H,$$

where G is a semisimple Lie group, $H \subset G$ a compact subgroup of maximal rank, $K = Z_G(T) \subset H$ for some torus $T \subset G$, are symplectically fat.

Twistor bundles

2. Twistor bundles of the form

$$SO(2n)/U(n) \longrightarrow \mathcal{T}(M) \longrightarrow M,$$

where (M^{2n}, g) is an even-dimensional Riemannian manifold with $\frac{3}{2n+1}$-pinched sectional curvature K_g (that is K_g satisfies $1 - \frac{3}{2n+1} \leq |K_g| \leq 1$), are symplectically fat.

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Locally homogeneous complex manifolds

3. Locally homogeneous complex manifolds fibered over locally symmetric Riemannian manifolds as follows

$$K/V \longrightarrow \Gamma \setminus G/V \longrightarrow \Gamma \setminus G/K,$$

where G is a semisimple Lie group of noncompact type, Γ is a uniform lattice in G, $K \subset G$ a maximal compact subgroup and $V = Z_G(T) \subset K$ for some torus $T \subset G$, are symplectically fat.

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Motivation

Let $H \longrightarrow G \longrightarrow G/H$ be a principal bundle satisfying assumptions of the generalized Lerman's theorem, (M, ω) a symplectic manifold with a Hamiltonian action of a Lie group H and a moment map μ.

1. If M is nonhomogeneous (for example not a coadjoint orbit) then there is no general way of checking fatness condition.
2. If H is an abelian group then the Delzant's theorem states that for any Delzant polytope there exist a toric H-manifold M, whose moment map has the image which is exactly the given Delzant polytope.
3. There exist symplectically fat fiber bundles with abelian structure group and toric manifolds as fibers. We can obtain them by taking Delzant polytopes omitting ,,forbidden" walls C_α from Lerman's theorem.

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Symplectically fat bundles with arbitrary Hamiltonian fibers

Weyl chambers

- K – a compact connected Lie group
- $T \subset K$ – a maximal torus in K
- $\mathfrak{k}$, $\mathfrak{t}$ – Lie algebras of K and T
- $W_K = N(T)/T$ – the Weyl group of K
- W_K acts on $\mathfrak{t}$ and its dual $\mathfrak{t}^*$
- every adjoint orbit in $\mathfrak{k}$ intersects $\mathfrak{t}$ in a single W_K-orbit

  $\mathfrak{k}^*/K = \mathfrak{t}^*/W_K$

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Weyl chambers

- choose any connected component of $\mathfrak{t}^*_{reg} = \{\xi \in \mathfrak{t}^* | K_\xi = T\}$ and denote its closure by $\mathfrak{t}^*_+$
- K_ξ – the isotropy subgroup of ξ under the coadjoint action
- $\mathfrak{t}^*_+$ – a closed Weyl chamber of $\mathfrak{k}^*$
- $\mathfrak{t}^*_+$ is a fundamental domain of the coadjoint action of K on $\mathfrak{k}^*$

$$\mathfrak{k}^*/K = \mathfrak{t}^*/W_K = \mathfrak{t}^*_+$$

Kirwan map

- (M, ω) – a compact connected symplectic manifold with a Hamiltonian action of K and a moment map $\mu : M \to \mathfrak{k}^*$
- $\Phi : M \to \mathfrak{t}^*_+$ – the composition of μ with the natural projection $\mathfrak{k}^* \to \mathfrak{k}^*/K = \mathfrak{t}^*_+$
- Φ – the Kirwan map
- $\Phi(M)$ is a convex polyhedron
- μ is equivariant with respect to the given action of K on M and the coadjoint action of K on $\mathfrak{k}^*$
- $\mu(M)$ is a union of coadjoint orbits in $\mathfrak{k}^*$
- $K \cdot \Phi(M)$ – the union of K-orbits of elements of $\Phi(M)$
- $\mu(M) = K \cdot \Phi(M)$ and $\Phi(M) = \mu(M) \cap \mathfrak{t}^*_+$

Kirwan map

Proposition 2.

Let (M, ω) be a symplectic manifold with a Hamiltonian action of a Lie group K and a moment map $\mu : M \to \mathfrak{k}^*$. Let T be a maximal torus in K and $\mu_T : M \to \mathfrak{t}^*$ its moment map for the restricted action of T on M.
If the set $\mu_T(M)$ consist of fat vectors with respect to some connection in the principal bundle of the form $K \to P \to B$, then $\mu(M)$ consists of fat vectors with respect to the same connection as well.

Notations

- K – a semisimple connected Lie group
- $H \subset K$ – a connected and compact subgroup of maximal rank
- $\mathfrak{k}$, $\mathfrak{h}$ – Lie algebras of K and H
- assume that the Killing form of $\mathfrak{k}$ is non-degenerate on $\mathfrak{h}$

Remark.

These assumptions allows for the application of generalized Lerman's theorem.

Arbitrary Hamiltonian fibers

Theorem 5.

Let H be the centralizer of a torus in K. Assume that H is compact and acts in a Hamiltonian fashion with the moment map μ on a compact and connected symplectic manifold (M, ω). Then the associated bundle

$$M \longrightarrow K \times_H M \longrightarrow K/H$$

is symplectically fat.

Motivation

Theorem 6. (Reznikov)

The total space of the twistor bundle over an even-dimensional compact Riemannian manifold (B^{2n}, g), whose sectional curvature K_g satisfies the inequality $1 - \frac{3}{2n+1} \leq |K_g| \leq 1$, admits a fiberwise symplectic structure.

Remark.

1. Reznikov used his theorem to construct examples of closed symplectic manifolds with no Käehler structure.
2. Reznikov's theorem turned out to be a consequence of the fact that these bundles are symplectically fat.

Invariant connections in G-structures

- K/H – a reductive homogeneous space ($\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{m}$, $Ad_H(\mathfrak{m}) \subset \mathfrak{m}$)
- $G \to P \to K/H$ – a K-invariant G-structure
- $\lambda : H \to G$ – the linear isotropy representation (assume that λ is faithful)
- following Kobayashi one can identify λ with the restriction of the adjoint representation of H on $\mathfrak{m}$
- θ – a K-invariant canonical connection in $G \to P \to K/H$

Theorem 7.

The curvature form of the canonical connection in P is given by the formula

$$\Theta(X, Y) = -\lambda([X, Y]_{\mathfrak{h}}), \ X, Y \in \mathfrak{m}.$$

Notations

- $\lambda : \mathfrak{a} \to \mathfrak{b}$ – a monomorphism of Lie algebras
- $\lambda^* : \mathfrak{b}^* \to \mathfrak{a}^*$ – the dual map ($\lambda^*(f)(X) = f(\lambda(X))$, $X \in \mathfrak{a}$)
- $B_{\mathfrak{a}}$, $B_{\mathfrak{b}}$ – non-degenerate bilinear invariant forms on $\mathfrak{a}$ and $\mathfrak{b}$
- $B_{\mathfrak{a}}(Y_g, Y) =< g, Y >$, $g \in \mathfrak{a}^*$, $Y \in \mathfrak{a}$
  $B_{\mathfrak{b}}(X_f, X) =< f, X >$, $f \in \mathfrak{b}^*$, $X \in \mathfrak{b}$
- X_f^λ – the $B_{\mathfrak{a}}$-dual vector of $\lambda^*(f) \in \mathfrak{a}^*$
  $B_{\mathfrak{a}}(X_f^\lambda, Y) :=< \lambda^*(f), Y >$

Fatness condition

Proposition 3.

Let K be a semisimple Lie group, and $H \subset K$ a compact subgroup of maximal rank. Suppose that the Killing form $\mathcal{K}$ is non-degenerate on the Lie algebra $\mathfrak{h} \subset \mathfrak{g}$ of the subgroup H. Assume that the homogeneous space K/H is equipped with a K-invariant G-structure and that the isotropy representation λ is faithful. Then, $v \in \mathfrak{g}^*$ is fat with respect to the canonical connection, if

$$X_v^\lambda \notin Ad_H(\bigcup_{\alpha \in \Delta \setminus \Delta(\mathfrak{h})} C_\alpha).$$

Twistor bundle

Definition.

The twistor bundle over an even-dimensional Riemannian manifold (B^{2n}, g) is the bundle associated with the orthonormal frame bundle of B with the fiber $SO(2n)/U(n)$.

Remark.

1. In our case $B = K/H$ and the twistor bundle has the form

$$SO(2n)/U(n) \to SO(K/H) \times_{SO(2n)} (SO(2n)/U(n)) \to K/H,$$

where $\dim K/H = 2n$ and $SO(K/H)$ is the total space of the principal $SO(2n)$-bundle of oriented frames.

2. We will denote the total space of the twistor bundle by $\mathcal{T}(K/H)$.

Fiber of the twistor bundle

- $J \in \mathfrak{so}(2n)$ – the matrix consisting of n blocks of the form

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

- $J^* \in \mathfrak{so}(2n)^*$ – the vector dual to J with respect to the Killing form $B_{\mathfrak{so}(2n)}$
- $SO(2n)/U(n)$ – the coadjoint orbit of J^* with respect to the standard transitive action of $SO(2n)$ on $SO(2n)/U(n)$

Fatness condition

Theorem 8.

Consider the twistor bundle over a reductive homogeneous space K/H satisfying the following conditions:

1. K is semisimple, H is compact, and the Killing form of $\mathfrak{k}$ restricted to $\mathfrak{h}$ is non-degenerate;
2. K/H is a reductive homogeneous space of maximal rank (that is, $\operatorname{rank} K = \operatorname{rank} H$);
3. there exists $T \in \mathfrak{t} \subset \mathfrak{h}$ in the Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{h}$ and $\mathfrak{k}$ such that

$$(ad\ T|_{\mathfrak{m}})^2 = -id, \;\; T \notin \bigcup_{\alpha \in \Delta \setminus \Delta(\mathfrak{h})} C_\alpha .$$

Then, the corresponding twistor bundle is symplectically fat.

Twistor bundles over even-dimensional Grassmanians

Theorem 9.

The twistor bundles over even-dimensional Grassmanians of maximal rank

$$SO(2n+2m)/SO(2n)\times SO(2m),\ \ m,n\neq 1$$
$$SO(2(n+m)+1)/SO(2n)\times So(2m+1),\ \ n\neq 1$$
$$Sp(n+m)/Sp(n)\times Sp(m)$$
$$U(m+n)/U(m)\times U(n)$$

are symplectically fat.

References

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