Integrability conditions for Killing equations on homogeneous spaces

Petr Zima
(joint with P. Somberg)

Charles University,
Prague, Czech Republic

37th Winter School Geometry and Physics
January 18, 2017

Introduction

Killing equations are certain natural systems of partial differential equations defined on a (pseudo-)Riemannian (spin-)manifold.

- Different types: vectors, symmetric tensors, forms, spinors, spinor-valued forms
- Invariant and overdetermined systems of PDEs
- Important applications both in mathematics and physics
- Closely related to some special geometric structures: Sasakian, nearly parallel Kähler, nearly parallel G_2-manifolds

Introduction

Killing equations are certain natural systems of partial differential equations defined on a (pseudo-)Riemannian (spin-)manifold.

- Different types: vectors, symmetric tensors, forms, spinors, spinor-valued forms
- Invariant and overdetermined systems of PDEs
- Important applications both in mathematics and physics
- Closely related to some special geometric structures: Sasakian, nearly parallel Kähler, nearly parallel G_2-manifolds

Explicitly solve the system on homogeneous space G/H:

1. The so called **prolongation** procedure transforms the system into a closed system by introducing new indeterminates.
2. The **integrability conditions** of the prolongated system restrict possible values in a point by a condition involving curvature.
3. Extend the solution from a point to its neighborhood or possibly the whole G/H.

Homogeneous Cartan geometry

Let $(\mathsf{P} \to \mathsf{M} = \mathsf{G}/\mathsf{H}, \omega)$ be a homogeneous Cartan geometry of type (L, K). It corresponds to homomorphisms $i \colon \mathsf{H} \to \mathsf{K}$ and $\alpha \colon \mathfrak{g} \to \mathfrak{l}$, such that

(a) α is $\mathrm{Ad}(\mathsf{H})$-invariant,

(b) $\alpha|_{\mathfrak{h}} = i'$, and

(c) $\underline{\alpha} \colon \mathfrak{g}/\mathfrak{h} \to \mathfrak{l}/\mathfrak{k}$ is an isomorphism.

Homogeneous Cartan geometry

Let $(\mathsf{P} \to \mathsf{M} = \mathrm{G}/\mathrm{H}, \omega)$ be a homogeneous Cartan geometry of type (L, K). It corresponds to homomorphisms $i\colon \mathrm{H} \to \mathsf{K}$ and $\alpha\colon \mathfrak{g} \to \mathfrak{l}$, such that

(a) α is $\mathrm{Ad}(\mathrm{H})$-invariant,

(b) $\alpha|_{\mathfrak{h}} = i'$, and

(c) $\underline{\alpha}\colon \mathfrak{g}/\mathfrak{h} \to \mathfrak{l}/\mathfrak{k}$ is an isomorphism.

Description of the geometry

- $\mathsf{P} = \mathrm{G} \times_{\mathrm{H}} \mathsf{K}$ and i induces a bundle map $I\colon \mathrm{G} \to \mathsf{P}$.
- All the natural bundles are associated also to G.
- $I^*\omega = \alpha \circ \omega_{\mathrm{G}}$, where ω_{G} is the Maurer-Cartan form on G.
- The curvature function is given by

$$\kappa(\xi, \eta) = \{\xi, \eta\}_{\mathfrak{l}} - \alpha(\{\alpha^{-1}(\xi), \alpha^{-1}(\eta)\}_{\mathfrak{g}}), \qquad \forall \xi, \eta \in \mathcal{A}\mathsf{M}, \qquad (1)$$

  where $\{,\}_{\mathfrak{l}}$ and $\{,\}_{\mathfrak{g}}$ are the (algebraic) brackets in $\mathfrak{l}$ and $\mathfrak{g}$ respectively.
- G acts transitively on $(\mathsf{P} \to \mathsf{M}, \omega)$ by automorphisms.

Infinitesimal automorphisms

≡ Vector fields whose flow is a one-parameter group of automorphisms.

- By the prolongation they naturally correspond to adjoint tractor fields $\xi \in \mathfrak{a} \subset \Gamma(\mathcal{A}M)$, such that

$$D_\eta \, \xi = \kappa(\eta, \xi) - \{\eta, \xi\}_\mathrm{I}, \qquad \forall \eta \in \mathcal{A}M, \tag{2}$$

  where D is the fundamental derivative.

- **1st integrability condition** of (2)

$$D_\xi \, \kappa = 0. \tag{3}$$

Infinitesimal automorphisms

≡ Vector fields whose flow is a one-parameter group of automorphisms.

- By the prolongation they naturally correspond to adjoint tractor fields $\xi \in \mathfrak{a} \subset \Gamma(\mathcal{A}M)$, such that

$$D_\eta \, \xi = \kappa(\eta, \xi) - \{\eta, \xi\}_{\mathfrak{l}}, \qquad \forall \eta \in \mathcal{A}M, \tag{2}$$

  where D is the fundamental derivative.

- **1st integrability condition** of (2)

$$D_\xi \, \kappa = 0. \tag{3}$$

- Lie bracket of infinitesimal automorphisms is algebraic

$$[\xi_1, \xi_2] = \kappa(\xi_1, \xi_2) - \{\xi_1, \xi_2\}_{\mathfrak{l}} =: \{\xi_1, \xi_2\}_{\mathfrak{a}}, \qquad \forall \xi_1, \xi_2 \in \mathfrak{a}. \tag{4}$$

- Bianchi identity implies (3) ⇒ Jacobi identity for $\{,\}_{\mathfrak{a}}$

$$\sum_{\text{cycl.}} (D_{\xi_1} \kappa)(\xi_2, \xi_3) = \sum_{\text{cycl.}} \{\xi_1, \{\xi_2, \xi_3\}_{\mathfrak{a}}\}_{\mathfrak{a}}, \qquad \forall \xi_1, \xi_2, \xi_3 \in \mathcal{A}M. \tag{5}$$

Infinitesimal automorphisms

At $x \in M$ we define $\mathfrak{a}_x^1 = \{\xi \in \mathcal{A}M_x \mid \mathrm{D}_\xi\, \kappa = 0\}$ and consider $\{,\}_{\mathfrak{a}}$.

- Problem: In general $\mathfrak{a}_x^1$ is not closed under $\{,\}_{\mathfrak{a}}$!
- We need **higher integrability conditions** of (2)

$$\mathrm{D}_\xi(\mathrm{D}^i\, \kappa) = 0, \qquad i = 0, 1, 2, \ldots. \tag{6}$$

Infinitesimal automorphisms

At $x \in M$ we define $\mathfrak{a}_x^1 = \{\xi \in \mathcal{A}M_x \mid D_\xi \kappa = 0\}$ and consider $\{,\}_\mathfrak{a}$.

- Problem: In general $\mathfrak{a}_x^1$ is not closed under $\{,\}_\mathfrak{a}$!
- We need **higher integrability conditions** of (2)

$$D_\xi(D^i \kappa) = 0, \qquad i = 0, 1, 2, \dots. \tag{6}$$

We define $\mathfrak{a}_x^k = \{\xi \in \mathcal{A}M_x \mid D_\xi(D^i \kappa) = 0,\ i = 0, \dots, k-1\}$.

- Ricci identity implies $\xi_1, \xi_2 \in \mathfrak{a}_x^{k+1} \Rightarrow \{\xi_1, \xi_2\}_\mathfrak{a} \in \mathfrak{a}_x^k$

$$D^2_{\xi_1,\xi_2} - D^2_{\xi_2,\xi_1} = -D_{\{\xi_1,\xi_2\}_\mathfrak{a}}, \qquad \forall \xi_1, \xi_2 \in \mathcal{A}M. \tag{7}$$

Infinitesimal automorphisms

At $x \in M$ we define $\mathfrak{a}_x^1 = \{\xi \in \mathcal{A}M_x \mid D_\xi \kappa = 0\}$ and consider $\{,\}_\mathfrak{a}$.

- Problem: In general $\mathfrak{a}_x^1$ is not closed under $\{,\}_\mathfrak{a}$!
- We need **higher integrability conditions** of (2)

$$D_\xi(D^i \kappa) = 0, \qquad i = 0, 1, 2, \dots . \tag{6}$$

We define $\mathfrak{a}_x^k = \{\xi \in \mathcal{A}M_x \mid D_\xi(D^i \kappa) = 0,\ i = 0, \dots, k-1\}$.

- Ricci identity implies $\xi_1, \xi_2 \in \mathfrak{a}_x^{k+1} \Rightarrow \{\xi_1, \xi_2\}_\mathfrak{a} \in \mathfrak{a}_x^k$

$$D^2_{\xi_1,\xi_2} - D^2_{\xi_2,\xi_1} = -D_{\{\xi_1,\xi_2\}_\mathfrak{a}}, \qquad \forall \xi_1, \xi_2 \in \mathcal{A}M. \tag{7}$$

Hence $\mathfrak{a}_x^\infty = \bigcap \mathfrak{a}_x^k$ with $\{,\}_\mathfrak{a}$ is a well-defined Lie algebra.

- The chain $\mathfrak{a}_x^1 \supseteq \mathfrak{a}_x^2 \supseteq \cdots \supseteq \mathfrak{a}_x^\infty$ stabilizes after finitely many steps.
  Question: How many steps are needed?
- Evaluation at x is a Lie algebra homomorphism $\mathfrak{a} \to \mathfrak{a}_x^\infty$.

Infinitesimal automorphisms

Now let $(\mathsf{P} \to \mathsf{M} = \mathsf{G}/\mathsf{H}, \omega)$ be homogeneous.

- The infinitesimal left action $l' \colon \mathfrak{g} \to \mathfrak{a}$ yields infinitesimal automorphisms (in every tangent direction).
- Evaluating at x and composing with the adjoint representation of $\mathfrak{a}_x^\infty$ we get a representation $\lambda_x \colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{a}_x^\infty)$

$$\lambda_x(X)\,\xi = \{l'(X)_x, \xi\}_{\mathfrak{a}}, \qquad \forall X \in \mathfrak{g}, \xi \in \mathfrak{a}_x^\infty. \tag{8}$$

Infinitesimal automorphisms

Now let $(\mathrm{P} \to \mathrm{M} = \mathrm{G}/\mathrm{H}, \omega)$ be homogeneous.

- The infinitesimal left action $l' \colon \mathfrak{g} \to \mathfrak{a}$ yields infinitesimal automorphisms (in every tangent direction).
- Evaluating at x and composing with the adjoint representation of $\mathfrak{a}_x^\infty$ we get a representation $\lambda_x \colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{a}_x^\infty)$

$$\lambda_x(X)\,\xi = \{l'(X)_x, \xi\}_{\mathfrak{a}}, \qquad \forall X \in \mathfrak{g}, \xi \in \mathfrak{a}_x^\infty. \tag{8}$$

Proposition

On a homogeneous space, for each $\xi_0 \in \mathfrak{a}_x^\infty$ there exists a unique locally defined infinitesimal automorphism $\xi \in \mathfrak{a}$ given by

$$\xi(\exp(X)\,x) = \exp(\lambda_x(X))\,\xi_0, \qquad \forall X \in U(0) \subset \mathfrak{g}. \tag{9}$$

General closed invariant system

Let E be a natural vector bundle over $(\mathsf{P} \to \mathsf{M} = \mathsf{G}/\mathsf{H}, \omega)$.

- We consider an equation for $\Phi \in \Gamma(\mathsf{E})$ in form

$$D_\xi \Phi = B_\xi \Phi, \qquad \text{where } B \in \Gamma(\mathcal{A}^* \mathsf{M} \otimes \mathsf{E}). \tag{10}$$

- We suppose B is invariant in the following sense

$$D_\xi (D^i B) = 0, \qquad \forall \xi \in \mathfrak{a},\ i = 0, 1, 2, \dots. \tag{11}$$

- This is satisfied, e.g. if B comprises of K-invariant tensors and the curvature κ.

General closed invariant system

Let E be a natural vector bundle over $(\mathsf{P} \to \mathsf{M} = \mathsf{G}/\mathsf{H}, \omega)$.

- We consider an equation for $\Phi \in \Gamma(\mathsf{E})$ in form

$$\mathrm{D}_\xi \Phi = \mathrm{B}_\xi \Phi, \qquad \text{where } \mathrm{B} \in \Gamma(\mathcal{A}^*\mathsf{M} \otimes \mathsf{E}). \tag{10}$$

- We suppose B is invariant in the following sense

$$\mathrm{D}_\xi(\mathrm{D}^i \mathrm{B}) = 0, \qquad \forall \xi \in \mathfrak{a},\ i = 0, 1, 2, \ldots. \tag{11}$$

- This is satisfied, e.g. if B comprises of K-invariant tensors and the curvature κ.
- **Integrability conditions** using the Ricci identity

$$\begin{aligned} \mathrm{B}_{\{\eta,\xi\}_\mathfrak{a}} \mathrm{B}_{\zeta_1} \cdots \mathrm{B}_{\zeta_i} \Phi = (\mathrm{B}_\eta \mathrm{B}_\xi - \mathrm{B}_\xi \mathrm{B}_\eta) \mathrm{B}_{\zeta_1} \cdots \mathrm{B}_{\zeta_i} \Phi + \\ + \text{terms with } \mathrm{D}^j \mathrm{B},\ j \geq 1. \end{aligned} \tag{12}$$

Hence B defines a representation of $\mathfrak{a}_x$ on
$S^\infty_x = \{\Phi \in \mathsf{E}\mathsf{M}_x \mid \text{such that (12) holds for each } i = 0, 1, 2, \ldots\}$.

General closed invariant system

Now let again $(\mathsf{P} \to \mathsf{M} = \mathsf{G}/\mathsf{H}, \omega)$ be homogeneous.

- Evaluating the infinitesimal left action at x and composing with B we get a representation $\mu_x \colon \mathfrak{g} \to \mathfrak{gl}(S^\infty_x)$

$$\mu_x(X)\,\Phi = \mathrm{B}_{l'(X)_x}\,\Phi, \qquad \forall X \in \mathfrak{g}, \Phi \in S^\infty_x. \tag{13}$$

General closed invariant system

Now let again $(\mathsf{P} \longrightarrow \mathsf{M} = \mathsf{G}/\mathsf{H}, \omega)$ be homogeneous.

- Evaluating the infinitesimal left action at x and composing with B we get a representation $\mu_x \colon \mathfrak{g} \longrightarrow \mathfrak{gl}(S_x^\infty)$

$$\mu_x(X)\,\Phi = \mathrm{B}_{l'(X)_x}\,\Phi, \qquad \forall X \in \mathfrak{g}, \Phi \in S_x^\infty. \tag{13}$$

Proposition

On a homogeneous space, for each $\Phi_0 \in S_x^\infty$ there exists a unique locally defined solution $\Phi \in \Gamma(\mathsf{E})$ of the equation (10) *given by*

$$\Phi(\exp(X)\,x) = \exp(\mu_x(X))\,\Phi_0, \qquad \forall X \in U(0) \subset \mathfrak{g}. \tag{14}$$

Killing forms

Let (M, g) be a (pseudo-)Riemannian manifold, ∇ the Levi-Civita connection and $\mathcal{R}$ its curvature.

- A **Killing *p*-form** is a differential p-form α such that

$$\nabla_X \alpha = \frac{1}{p+1} X \lrcorner \mathrm{d}\alpha, \qquad \forall X \in \mathcal{T}M. \tag{15}$$

- Generalization of Killing vectors which correspond to 1-forms.

Killing forms

Let (M, g) be a (pseudo-)Riemannian manifold, ∇ the Levi-Civita connection and $\mathcal{R}$ its curvature.

- A **Killing p-form** is a differential p-form α such that

$$\nabla_X \alpha = \tfrac{1}{p+1} X \lrcorner \mathrm{d}\alpha, \qquad \forall X \in \mathcal{T}M. \tag{15}$$

- Generalization of Killing vectors which correspond to 1-forms.
- The invariant **prolongation** of (15) is

$$\nabla_X \alpha = X \lrcorner \beta, \qquad \nabla_X \beta = \tfrac{1}{p} \mathcal{R}_X \wedge \alpha, \qquad \forall X \in \mathcal{T}M, \tag{16}$$

  where β is an additional $(p+1)$-form and $\mathcal{R}_X \wedge \alpha$ is partially skew-symmetrized action of the curvature on α.

Killing forms

Let (M, g) be a (pseudo-)Riemannian manifold, ∇ the Levi-Civita connection and $\mathcal{R}$ its curvature.

- A **Killing p-form** is a differential p-form α such that

$$\nabla_X \alpha = \tfrac{1}{p+1} X \lrcorner \mathrm{d}\alpha, \qquad \forall X \in \mathcal{T}M. \tag{15}$$

- Generalization of Killing vectors which correspond to 1-forms.
- The invariant **prolongation** of (15) is

$$\nabla_X \alpha = X \lrcorner \beta, \qquad \nabla_X \beta = \tfrac{1}{p} \mathcal{R}_X \wedge \alpha, \qquad \forall X \in \mathcal{T}M, \tag{16}$$

  where β is an additional $(p+1)$-form and $\mathcal{R}_X \wedge \alpha$ is partially skew-symmetrized action of the curvature on α.
- The **1st integrability condition** is

$$\mathcal{R} \diamond \alpha = 0, \qquad \mathcal{R} \diamond \beta = (\nabla \mathcal{R}) \triangle \alpha, \tag{17}$$

  where '$\diamond$' and '$\triangle$' are another invariant algebraic actions.

Berger sphere

Odd dimensional sphere $\mathbb{S}^{2m+1} = \mathsf{U}(m+1)/\mathsf{U}(m)$.

- Hopf fibration $\mathbb{S}^1 \longrightarrow \mathbb{S}^{2m+1} \longrightarrow \mathbb{CP}^m$.
- The fiber direction corresponds to the center of $\mathfrak{u}(m+1)$.
- Berger metrics g_s, $s \in \mathbb{R}^+$: Take the round metric g_1 and rescale the fiber direction by s.
- **Almost contact metric structure**: unit vector field ξ in the fiber direction, almost complex structure φ on $\xi^\perp$, in particular $\varphi(\xi) = 0$.

Berger sphere

Odd dimensional sphere $\mathbb{S}^{2m+1} = \mathsf{U}(m+1)/\mathsf{U}(m)$.

- Hopf fibration $\mathbb{S}^1 \longrightarrow \mathbb{S}^{2m+1} \longrightarrow \mathbb{CP}^m$.
- The fiber direction corresponds to the center of $\mathfrak{u}(m+1)$.
- Berger metrics g_s, $s \in \mathbb{R}^+$: Take the round metric g_1 and rescale the fiber direction by s.
- **Almost contact metric structure**: unit vector field ξ in the fiber direction, almost complex structure φ on $\xi^\perp$, in particular $\varphi(\xi) = 0$.
- For $s \neq 1$ the 1st integrability condition for Killing forms implies

$$\begin{aligned} \varphi \cdot \alpha &= 0, \qquad \text{if } p > 1, \text{ and} \\ \varphi \cdot \beta &= 0, \qquad \text{if } \alpha = 0. \end{aligned} \tag{18}$$

- For 1-forms ($\cong$ vectors) the condition (18) is sufficient.
- The Lie algebra of Killing vectors is exactly $\mathfrak{u}(m+1)$.
- Work in progress: higher ordinary forms, spinor-valued forms.

THANK YOU FOR YOUR ATTENTION!