Compactification, boundary calculus, and applications: Hypersurface and boundary calculus

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background:
G-., Waldon, arXiv:1506.02723, and Boundary calculus for conformally compact manifolds. Indiana U.M.J. **63** (2014).
Curry, **G-.** ⋯Conformal Geometry ⋯ GR⋯, LMS Series, Cambridge, arXiv:1412.7559
Čap, **G-.** Hammerl: Holonomy reductions etc, *Duke Math. J.* **163** (2014) 1035–1070.
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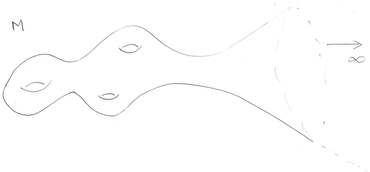
Some motivation (partly naïve) via a general question:

Taming big spaces: Suppose we have an infinite space (i.e. non-compact manifold, geodesically complete):



How do we deal with the "far region"? Can we make a notion of "infinity" that is mathematically useful? If so what geometry does it have? Are there many ways to do such things, or is any success essentialy unique?

A **Compactification** of a non-compact ("large") topological space M is an embedding of M as a dense subset of a compact ("small") space $\tilde{M}$: So $\boxed{M \hookrightarrow \overline{M} \text{ injective cts and a homeo. onto its image}}$.

Example: A compactification of Minkowski space

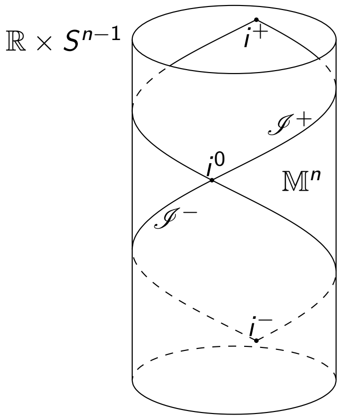


Figure: The standard embedding of n-dimensional Minkowski space $\mathbb{M}^n$ into the Einstein cylinder. This is **conformal**: $g_{\mathrm{Mink}} = \Omega^2 g_{Lorentzian\ cyln}$

Questions: Is this essentially the only way to conformally compactify $\mathbb{M}$? Is it forced that $i^\pm$ and i^0 are points? That $\mathscr{I}$ is an open subset of $\partial\mathbb{M}$?

Penrose's "generalisation" and conformal infinity

Definition

A smooth (time- and space-orientable) spacetime (M_+, g_+) is called **asymptotically simple** if there exists another smooth Lorentzian manifold (M, g) such that

1. M_+ is an open submanifold of M with smooth boundary $\partial M_+ = \mathscr{I}$;
2. there exists a smooth scalar field Ω on M, such that $g = \Omega^2 g_+$ on M_+, and so that $\Omega = 0$, $\mathrm{d}\Omega \neq 0$ on $\mathscr{I}$;
3. every null geodesic in $\overline{M}$ acquires a future and a past endpoint on $\mathscr{I}$.

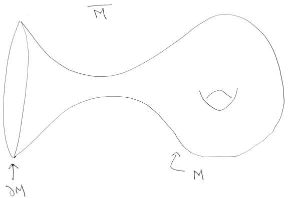
An asymptotically simple spacetime is called *asymptotically flat* if in addition $\mathrm{Ric}^{g_+} = 0$ in a neighbourhood of $\mathscr{I}$.

Questions: How would we re-discover the Einstein cylinder compactification or this useful definition? Treat other geometries similarly?

Compactification: $M \hookrightarrow \overline{M}$ smooth injective, M open dense. (In general M may be a manifold with boundary, a manifold with corners,) . . .

Question: What is a right way to do this when geometry is involved?

In many simple cases the result is a manifold with boundary $\overline{M}$ so that M is the interior and ∂M has codimension 1.



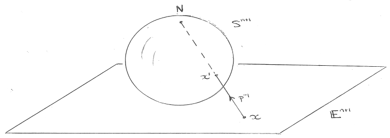
Questions: How do we find the geometry on ∂M?

Boundary calculus: Relate the goemetries/fields on ∂M and M?

Applications: 1. Discovery new links between different geometries; geometric tools PDE boundary problems; invariant theory and invariant operators; representation theory; scattering and non-local operators; AdS/CFT correspondence in physics

A flat model – compactifying Euclidean space

Stereographic projection $p : S^{n+1} \setminus \{\mathrm{N}\} \to \mathbb{E}^{\mathrm{n}+1}$ is a diffeomorphism (even C^∞)



and $p^{-1} : \mathbb{E}^{n+1} \to S^{n+1}$ is a **conformal embedding**. That is it preserves angles and circles are mapped to circles.

BUT: the **conformal infinity** of this compactification is the one point $N \in S^{n+1}$ – so this cannot encode much information about $\mathbb{E}^{n+1}$, or analysis on $\mathbb{E}^{n+1}$. For example:

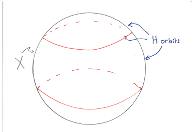
• The **Euclidean group** action on $\mathbb{E}^{n+1}$ extends to S^{n+1}, but it acts trivially on N.

• It is clearly a bad compactification for **Euclidean scattering**.

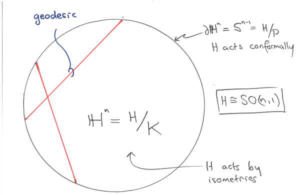
Q: Other ways to compactify?

Answering these questions has several steps. The first step is linked to another problem:

Problem: Suppose a Lie group H acts on a manifold X with a finite number of orbits. Then: (i) understand and relate the different (Klein) geometries on the orbits; and (ii) construct and treat a well defined curved version of this theory.

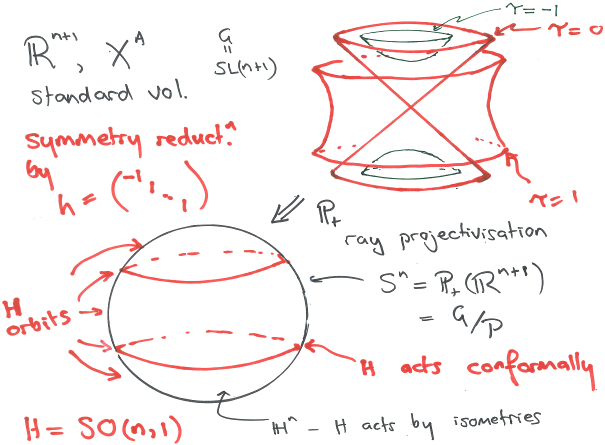


E.g. **Example** $H := SO(n,1)$ **orbits** $X =$ the **Klein Ball**:



Note: The Klein ball is a **compactification** of $\mathbb{H}^n$ linked to projective geometry.

$H = SO(n,1)$ orbits on the sphere



$S^n = \mathbb{P}_+(\mathbb{R}^{n+1} \setminus \{0\})$ is model of flat projective geometry.

Symmetry reduction by h (plus time↑): $\Rightarrow$ North polar cap is projective compactification of $\mathbb{H}^n$; $\tau = 0$ projective ∞ with conformal str.

NB: Embeddings relate the orbits – but these encoded in $H \hookrightarrow G$.

Conformal compactification of $\mathbb{H}^{n+1}$ – the Poincaré ball

Escher's circle limit

$$\overline{\mathbb{H}^2} = \mathbb{H}^2 + \partial\mathbb{H}^2$$

The embedding gives the compactification

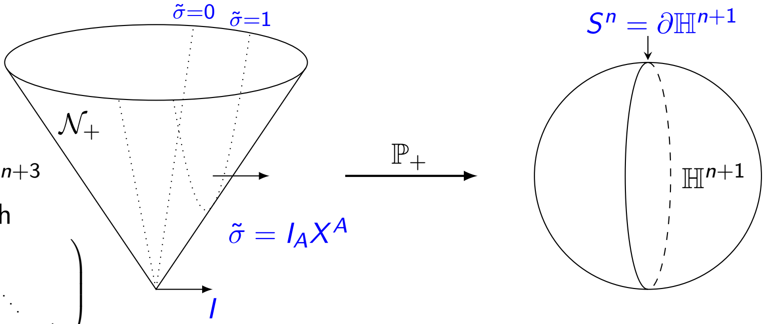
$\mathbb{H}^{n+1}$ embedded conformally in Euclidean $\mathbb{E}^{n+1}$

$S^n = \partial\mathbb{H}^{n+1}$

$$g_+ = \frac{4}{(1-|x|^2)^2}\sum^{n+1} dx_i^2$$

$$\overline{\mathbb{H}^{n+1}} = \mathbb{H}^{n+1} + \partial\mathbb{H}^{n+1}$$

Conformal compactification of $\mathbb{H}^{n+1}$ by **symmetry breaking**:



i.e. $\mathbb{R}^{n+3}$ with

$$\begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

$S^{n+1} = \mathbb{P}_+(\mathcal{N}_+ \subset \mathbb{R}^{n+3} \setminus \{0\})$ is model of flat conformal geometry.
$G := SO_o(n+2,1)$ acts transitively. $I \in \mathbb{R}^{n+3}$, spacelike $h(I,I) = 1$
Symmetry reduction by I: $\Rightarrow$ $H = SO_o(n+1,1)$ orbits. Right hemi. is conf
compactification $\overline{M}_c$ of $\mathbb{H}^{n+1}$; $\sigma = 0$ conformal ∞ with conformal str.

Bigger groups: H vs G

In each example above there is implicitly a larger group $G \supset H$:

• **Poincaré ball and compactifying boundary** arise as two $H = SO_+(n+1,1)$ orbits on $S^{n+1} = G/P$ where $G = S0_+(n+2,1)$ and P maximal parabolic in .

The larger homogeneous space G/P encodes how the orbits **smoothly** fit together – i.e. the conformal compactification. Similarly:

- **Stereographic (conformal) compactification** of $\mathbb{E}^{n+1}$ arises as two $H =$ Euclidean group orbits on $S^{n+1} = G/P$ – with same G and P. I.e. Stereo. encoded by $H \hookrightarrow G$
- **Klein (projective) compactification** of $\mathbb{H}^{n+1}$ arises as two $H = SO_+(n+1,1)$ orbits on $S^{n+1} = G/P$ where now $G = SL(n+2)$ and P maximal parabolic in $SL(n+2)$.
- Projective **compactification** of $\mathbb{E}^{n+1}$ arises as two $\cdots$ (Exercise!)

Curving homogeneous spaces

For a Lie group G and closed Lie group P, homogeneous spaces G/P are **geometries** in the sense of **Klein**. There are often canonical curved generalisations:

Theorem (Cartan, Tanaka, $\cdots$)

*If P is a parabolic subgroup of a semisimple Lie group G then there is a **canonical** notion of geometry*

$$\begin{array}{ccc} \mathcal{G} & \leftarrow & P \\ \downarrow & & \\ M & & \end{array} \quad \textit{modelled on} \quad \begin{array}{ccc} G & \leftarrow & P \\ \downarrow & & \\ G/P & & \end{array}$$

where $\mathcal{G}$ is equipped with a Cartan connection ω – viz. a suitably equivariant Lie($\tilde{G}$)-valued 1-form, cf. Maurer-Cartan form on $\tilde{G}$.

E.g. Conformal geometry, projective DG, CR geometry, $\cdots$
For **conformal DG**: $G = SO_o(p+1, q+1)$, and P subgroup stabilising a ray in $\mathbb{R}^{p+q+2}$.

Tractor bundles

If we have a representation $\mathbb{V}$ of the group G then we have an associated **vector bundle** $\mathcal{G} \times_P \mathbb{V}$ with a **linear connection** ∇. This is the associated **tractor connection**. In fact for $(\mathcal{G}, \omega)$ modelled on (G, P), with G semi-simple, P parabolic:

Theorem (Čap+G.)

Cartan bundle $\mathcal{G}$ + connection ω $\Leftrightarrow$ Tractor bundle and tractor connection.

Then:

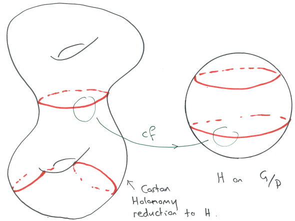
parallel tractors lead to curved analogues of orbit decompositions.

The point is that even on the model we can think of the orbit decomposition as arising from a parallel tractor of some type. Then the corresponding parallel tractor leads to a corresponding stratification of the curved manifold.

Theorem (**Curved orbit decomposition** - Čap,G., Hammerl)

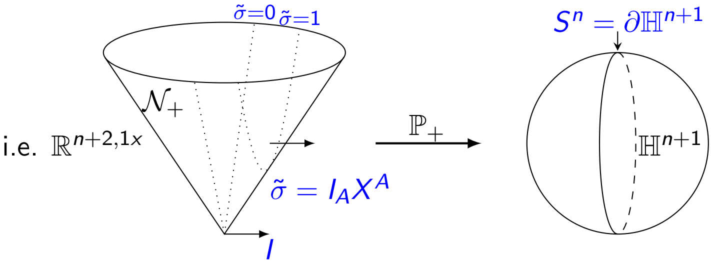
Suppose $(\mathcal{G}, \omega) \to M$ *is a Cartan geometry (modelled on* $G \to G/P$*) endowed with a parallel tractor field h giving a Cartan holonomy reduction with* **holonomy group** H*. Then:*
(1) M *is canonically stratified* $M = \bigcup_{i \in H \backslash G/P} M_i$ *in a way locally diffeomorphic to the the* H-**orbit decomposition** *of* G/P*; and*
(2) there $\exists$ *a* **Cartan geometry on** M_i *of the same type as the model.*

Thus there is a **general way to define a curved analogue of an orbit decomposition of a homogeneous space**.

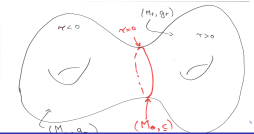


Curving the conformal compactification of $\mathbb{H}^{n+1}$

Recall the $H = SO_o(n+1,1)$ orbits on conformal sphere G/P, where $G = SO_o(n+2,1)$, H fixes $I \in \mathbb{R}^{n+3}$ spacelike:



Curved: A conformal manifold has a canonical Cartan bundle $\mathcal{G}$ modeled on (G, P). If this supports a **parallel spacelike tractor** I then the **curved orbit theorem** (**plus** some interpretation) states either M Einstein or M stratifies into disjoint union $M = M_- \cup M_0 \cup M_+$ and M_0 is a separating hypersurface. Moreover $M \setminus M_\mp$ is a **compactification** of the **Einstein** $M_\pm$.



Compactification Programme

Given some non-compact geometry of interest (e.g. pseudo-Riemannian):

Part 1 (homogeneous): Identify a homogeneous model $X_i = H/K$ of the geometry as an open $H < G$ orbit M in a compact homogeneous space $X = G/P$. (E.g. G semi-simple and P parabolic.) Then the topological closure $\overline{X}_i \subset X$ is a compactification of X_i.

Part 2 (curved I): Given a compact Cartan geometry $(\mathcal{G}, \omega) \to M$ modelled on $G \to G/P$, with a Cartan holonomy reduction with **holonomy group** H and an open curved orbit M_i (with same Cartan geometry type as X_i), then $\overline{M_i}$ is its compactification.

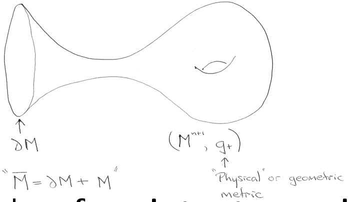
The Cartan/tractor machinery relates geometries of M_i & ∂M_i etc

Part 3 (curved II): Typically the geometry on M_i has restrictions on e.g. Einstein or symmetries, . . . (as a **normal solution of a BGG equation holds on** M). In some cases we can drop restrictions yet still exploit the Cartan/tractor machinery.

Curving Poincaré II: conformal compactification

Henceforth in these talks, **conformal compactification** of pseudo-Riemannian manifold (M^{n+1}, g_+) is a manifold $\overline{M}$ with boundary ∂M s.t.:

- $\exists\ \overline{g}$ on $\overline{M}$, with
- $g_+ = r^{-2}\overline{g}$, where r a defining function for ∂M.



$\Rightarrow$ canonical **conformal structure on boundary**: $(\partial M, [\overline{g}|_{\partial M}])$ (where dr not null).

• Called a **Poincaré-Einstein** metric if also g_+ negative Einstein. **This is the case rediscovered above by a parallel spacelike tractor** I **on** $\overline{M} = M_+ \cup M_0$.

Generalisations and applications

It is natural to seek generalising (even) less rigid notions than holonomy reductions.
Above we saw the Poincare Ball compactification of $\mathbb{H}^{n+1}$ generalises via curved orbit decomposition methods to a notion of conformal compactifcation of complete **Einstein** spaces – this recovers the usual notion of Poincaré-Einstein spaces, but in a new and very useful way.

We want to drop the "Einstein" in this and similar cases.

But we want to do this in a way that retains the **Cartan tractor** picture – as **this encodes deep information**.
Here in particular we will attempt to treat conformally compact manifolds in general (almost).

Part II: Conformal geometry and the geometry of scale

A conformal $n+1$-manifold ($n \geq 2$) is the structure $(M, \mathbf{c})$ where

- M is an $d = n+1$-manifold,
- $\mathbf{c}$ is a conformal equivalence class of signature (p,q) metrics, i.e. $g, \widehat{g} \in \mathbf{c} \stackrel{\text{def.}}{\Longleftrightarrow} \widehat{g} = \Omega^2 g$ and $C^\infty(M) \ni \Omega > 0$.

Because there is no distinguished metric on $(M, \mathbf{c})$ an important role is played by the **density bundles**. Note $(\Lambda^{n+1}TM)^2$ is an oriented real line bundle $\mathcal{K}$. We write $\mathcal{E}[w]$ for the roots

$$\mathcal{E}[w] = \mathcal{K}^{\frac{w}{2n+2}}, \qquad \text{so} \qquad \mathcal{K} = \mathcal{E}[2n+2],$$

$\mathcal{E}[0] := \mathcal{E}$ (the trivial bundle with fibre $\mathbb{R}$), and $\mathcal{E}_+[w]$ for the positive elements. With this notation there is tautologically a **conformal metric**

$$\boldsymbol{g} \in S^2T^*M[2], \qquad \text{so that} \qquad g^\sigma := \sigma^{-2}\boldsymbol{g} \in \mathbf{c}, \quad \sigma \in \Gamma(\mathcal{E}_+[1]),$$

and

$$\otimes^{n+1}\boldsymbol{g} : (\Lambda^{n+1}TM)^2 \xrightarrow{\simeq} \mathcal{E}[2n+2].$$

Scales

In view of the 1-1 relation between sections σ of $\mathcal{E}_+[1]$ and metrics g^σ in $\mathbf{c}$ (via $g^\sigma := \sigma^{-2}\boldsymbol{g} \in \mathbf{c}$) be call

$$\sigma \in \Gamma(\mathcal{E}_+[1])$$

a **strict scale**. Such sections provide the "symmetry breaking" which reduces us from conformal geometry to pseudo-Riemannian. Since any Levi-Civita connection ∇^g, for $g \in \mathbf{c}$, acts on $\mathcal{E}[2n+2]$ via the isomorphism $\otimes^{n+1}\boldsymbol{g} : \left(\Lambda^{n+1}TM\right)^2 \xrightarrow{\simeq} \mathcal{E}[2n+2]$ it follows easily that

$$\nabla^{g^\sigma}\sigma = 0.$$

So σ is parallel for the Levi-Civita connection it determines.

The tractor connection

On a conformal manifold $(\overline{M}, \mathbf{c})$ there is no distinguished connection on TM. But we have the conformally invariant **tractor bundle** $\mathcal{T}$ and **connection** $\nabla^{\mathcal{T}}$. Given $\overline{g} \in \mathbf{c}$ this is given by

$$\mathcal{T} \stackrel{\overline{g}}{=} \mathcal{E}[1] \oplus T^*M[1] \oplus \mathcal{E}[-1], \qquad \mathcal{E}[1] := (\Lambda^{n+1} TM)^{\frac{2}{2(n+1)}}$$

$$\nabla^{\mathcal{T}}_a(\sigma, \mu_b, \rho) = (\nabla_a \sigma - \mu_a,\ \nabla \mu_b + P_{ab}\sigma + \boldsymbol{g}_{ab}\rho,\ \nabla_a \rho - P_{ab}\mu^b),$$

and $\nabla^{\mathcal{T}}$ preserves a conformally invariant **tractor metric** h

$$\mathcal{T} \ni V = (\sigma, \mu_b, \rho) \mapsto 2\sigma\rho + \mu_b \mu^b = h(V, V).$$

There is also a second order **Thomas operator**:

$$\Gamma(\mathcal{E}[w]) \in f \mapsto D_A f \stackrel{\overline{g}}{=} \begin{pmatrix} (n + 2w - 2)wf \\ (n + 2w - 2)\nabla_a f \\ -(\Delta f + wJf) \end{pmatrix}$$

where J is $\mathrm{trace}^{\overline{g}}(P_{ab})$, so a number times $\mathrm{Sc}(\overline{g})$.

Parallel standard tractors

Note that from the formula

$$\nabla^{\mathcal{T}}_a(\sigma,\mu_b,\rho) = (\nabla_a\sigma - \mu_a,\ \nabla\mu_b + P_{ab}\sigma + \boldsymbol{g}_{ab}\rho,\ \nabla_a\rho - P_{ab}\mu^b),$$

if $I_A \stackrel{g}{=} (\sigma,\mu_a,\rho)$ is a parallel tractor then $\mu_a = \nabla_a\sigma$, and $\rho = -(\Delta\sigma + w\mathrm{J}\sigma)$. This gives the first statement of:

Proposition

I parallel implies $I_A = \frac{1}{d}D_A\sigma$. So $I \neq 0 \Rightarrow \sigma$ is nonvanishing on an open dense set $M_{\sigma\neq 0}$. On $M_{\sigma\neq 0}$, $g^o = \sigma^{-2}g$ is Einstein. Conversely if $g^o = \sigma^{-2}g$ is Einstein then $I := \frac{1}{d}D\sigma$ is parallel.

Proof.

On $M_{\sigma\neq 0}$ we have locally $\pm\sigma \in \Gamma(\mathcal{E}_+[1])$ so $\mu_a = \nabla_a\sigma = 0$ for $\nabla = \nabla^{g^\sigma}$. Thus

$$P_{ab} + \rho\boldsymbol{g}_{ab} = 0.$$

The converse is easy. □

So we say $(M, \mathbf{c})$ with parallel $I \neq 0$ is **almost Einstein**.

The Curved orbits.

Concerning $M_0 = \mathcal{Z}(\sigma)$. (Here and throughout $I^2 = I^A I_A$.)

Theorem

The curved orbit decomposition of an almost Einstein manifold $(M, \mathbf{c}, I)$ is according to the strict sign of $\sigma = I_A X^A$. The zero locus satisfies:

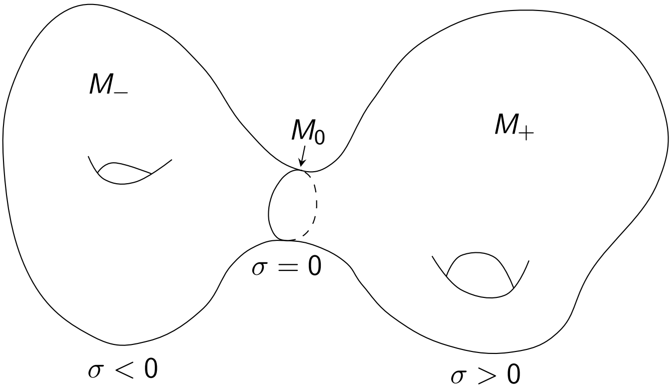
- *If $I^2 \neq 0$ (i.e. g^o Einstein and not Ricci flat) then $\mathcal{Z}(\sigma)$ is either empty or is a smoothly embedded separating hypersurface.*
- *If $I^2 = 0$ (i.e. g^o Ricci flat) then $\mathcal{Z}(\sigma)$ is either empty or, after excluding isolated points from $\mathcal{Z}(\sigma)$, is a smooth embedded hypersurface.*

Proof.

The local aspects follow from the general curved orbit theorem. Using the above formulae they are also easily recovered directly and one sees the separating statement. □

The picture so far

Thus if $I^2 \neq 0$ we have the picture:



$M \setminus M_\pm$ is evidently conformally compact and hence **Poincaré-Einstein**. Conversely all Poincaré-Einstein manifolds arise this way.

Almost pseudo-Riemannian geometry

We want now to drop the Einstein condition and understand e.g. general conformally compact manifolds.

For convenience we say that a structure

$$(M^d, \mathbf{c}, \sigma) \quad \text{where} \quad \sigma \in \Gamma(\mathcal{E}[1])$$

is **almost pseudo-Riemannian** if the scale tractor

$$I_A := \frac{1}{d} D_A \sigma \qquad \text{is nowhere zero.}$$

Note then that σ is non-zero on an open dense set, since $D_A\sigma$ encodes part of the 2-jet of σ. So on an almost pseudo-Riemannian manifold there is the pseudo-Riemannian metric $g^o = \sigma^{-2}\boldsymbol{g}$ on the same open dense set. In the following the notation I will always refer to a scale tractor, so $I = \frac{1}{d} D\sigma$, for some $\sigma \in \Gamma(\mathcal{E}[1])$. Then we often mention I instead of σ and refer to $(M, \mathbf{c}, I)$ as an almost pseudo-Riemannian manifold. Evidently:

Lemma

A conf. compact mfld is an almost Riemannian manifold $(\overline{M}, \mathbf{c}, \sigma)$ with boundary ($\overline{M} = M_+ \cup \partial M_+$) such that σ **defines** ∂M_+

Now recall from the formula for I and the metric we have

$$I^A I_A =: I^2 \stackrel{g}{=} \boldsymbol{g}^{ab}(\nabla_a \sigma)(\nabla_b \sigma) - \frac{2}{d}\sigma(\mathrm{J} + \Delta)\sigma \qquad (1)$$

where g is any metric from **c** and ∇ its Levi-Civita connection. This is well-defined everywhere on an almost pseudo-Riemannian manifold, while where σ is non-zero, it computes

$$I^2 = -\frac{2}{d}\mathrm{J}^{g^o} = -\frac{\mathrm{Sc}^{g^o}}{d(d-1)} \quad \text{where} \quad g^o = \sigma^{-2}\boldsymbol{g}.$$

Thus I^2 gives a **generalisation of the scalar curvature** (up to a constant factor $-1/d(d-1)$); it is canonical and smoothly extends the scalar curvature to include the zero set of σ. We shall use the term *ASC manifold* (where ASC means almost scalar constant) to mean an almost pseudo-Riemannian manifold with $I^2 = constant$. Since the tractor connection preserves h, then I parallel implies $I^2 = constant$. So an almost Einstein manifold is ASC, just as Einstein manifolds have constant scalar curvature.

Non-zero generalised scalar curvature.

Much of the almost Einstein curved orbit picture remains in the almost pseudo-Riemannian setting when I^2 is non-vanishing:

Theorem

Let $(M, \mathbf{c}, I)$ be an almost pseudo-Riemannian manifold with I^2 **nowhere zero**. *Then $\mathcal{Z}(\sigma)$, if not empty, is a smooth embedded separating hypersurface. This has a spacelike (resp. timelike) normal if g^o has negative scalar (resp. positive) scalar curvature. If $\mathbf{c}$ has Riemannian signature and $I^2 < 0$ then $\mathcal{Z}(\sigma)$ is empty.*

Key aspect of Proof.

From $I^2 \stackrel{g}{=} \boldsymbol{g}^{ab}(\nabla_a\sigma)(\nabla_b\sigma) - \frac{2}{d}\sigma(\mathrm{J} + \Delta)\sigma$: Along $\mathcal{Z}(\sigma)$ we have

$$I^2 = \boldsymbol{g}^{ab}(\nabla_a\sigma)(\nabla_b\sigma).$$

in particular $\nabla\sigma$ is nowhere zero on $\mathcal{Z}(\sigma)$, and so σ is a **defining density**. Thus $\mathcal{Z}(\sigma)$ is a smoothly embedded hypersurface by the implicit function theorem.

Conformally compact manifolds

Summary: A conformal manifold equipped with a scale tractor $I = \frac{1}{d} D\sigma$, with I^2 nowhere zero has I nowhere zero and so is almost pseudo-Riemanian. Where $\sigma = X^A I_A$ is nonzero (almost everywhere) there is the pseudo-Riemannian metric $g^o = \sigma^{-2}\boldsymbol{g}$, and σ is a defining density for the separating hypersurface $M_0 = Z(\sigma)$.

Thus we again have a stratification

$$M = M_- \cup M_0 \cup M_+.$$

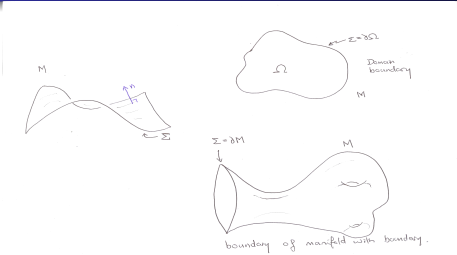
Moreover $(M, \mathbf{c}, I) \setminus M_\mp$ is conformally compact, as any scale $\tau \in \Gamma(\mathcal{E}_+[1])$ gives $\overline{g} = \tau^{-2}\boldsymbol{g} \in \mathbf{c}$, and $r := \tau^{-1}\sigma$ is a defining function for M_0 in $\overline{M}_\pm = M_\pm \cup M_0$. It is clear all conformally compact manifolds with scalar curvature bounded away from zero arise arise this way.

We want to develop a boundary calculus for these conformally compact manifolds. First we digress to understand hypersurfaces.

Part III: Hypersufaces in conformal manifolds

Part IV: Geometry of conformal infinity

Part III: Hypersurfaces in conformal geometry - a digression



To treat boundary calculus we need to understand the mathematics of hypersurfaces.

Defn: *hypersurface* Σ in a manifold M means a smoothly embedded codimension 1 submanifold of $(M, \mathbf{c})$.

• we restrict to Σ with the property that the any conormal field along Σ is nowhere null (i.e. to nondegenerate hypersurfaces).

Then:

• restriction of any $g \in \mathbf{c}$ gives metric $\bar{g}$ on $\Sigma \rightsquigarrow \mathbf{c}$ induces $\bar{\mathbf{c}}$ on Σ.

• It is natural to work with a weight 1 co-normal n_a along Σ satisfying $\boldsymbol{g}^{ab} n_a n_b = \pm 1$.

Basic hypersurface invariants

For $g \in \mathbf{c}$, the **second fundamental form** L_{ab} is the restriction of $\nabla_a n_b$ to $T\Sigma \times T\Sigma \subset (TM \times TM)|_\Sigma$, where $\nabla = \nabla^g$; i.e.

$$L_{ab} := \nabla_a n_b \mp n_a n^c \nabla_c n_b \quad \text{along} \quad \Sigma.$$

This is not conformally invariant. But under a conformal rescaling, $g \mapsto \widehat{g} = e^{2\omega} g$, L_{ab} transforms according to

$$L^{\widehat{g}}_{ab} = L^{g}_{ab} + \overline{\boldsymbol{g}}_{ab} \Upsilon_c n^c, \qquad \text{where} \qquad \Upsilon = d\omega$$

Thus:

Proposition

The trace-free part of the second fundamental form

$$\mathring{L}_{ab} = L_{ab} - H\overline{\boldsymbol{g}}_{ab}, \quad \textit{where}, \quad H := \frac{1}{d-1}\overline{\boldsymbol{g}}^{cd} L_{cd}$$

is conformally invariant.

Here $d = n + 1$ is the dimension of the ambient manifold M.

The normal tractor

Evidently, under a conformal rescaling $g \mapsto \widehat{g} = e^{2\omega} g$, the **mean curvature** H^g transforms to $H^{\widehat{g}} = H^g + n^a \Upsilon_a$. Thus we obtain a conformally invariant section N of $\mathcal{T}|_\Sigma$

$$N_A \stackrel{g}{=} \begin{pmatrix} 0 \\ n_a \\ -H^g \end{pmatrix},$$

and $h(N, N) = \pm 1$ along Σ. This is the **normal tractor** of Bailey-Eastwood-G. Differentiating N tangentially along Σ using $\nabla^{\mathcal{T}}$, we obtain the following result.

Proposition (Conformal Shape operator)

$$\mathbb{L}_{aB} := \underline{\nabla}_a N_B \stackrel{g_{cb}}{=} \begin{pmatrix} 0 \\ \mathring{L}_{ab} \\ -\frac{1}{d-2} \nabla^b \mathring{L}_{ab} \end{pmatrix}$$

where $\underline{\nabla}$ is the pullback to Σ of the ambient tractor connection. Thus Σ is **totatally umbilic** *iff N is parallel along Σ.*

Conformal hypersurface calculus

The classical **Gauss formula**

$$\underline{\nabla}_a v^b = \overline{\nabla}_a v^b \mp n^b L_{ac} v^c \qquad v \in T\Sigma \subset TM,$$

is the basis of Riemannian hypersurface calculus.

We want the conformal analogue. First we need this:

Proposition (Branson-G., Grant)

There is a natural conformally invariant (isometric) isomorphism

$$\mathcal{T}|_\Sigma \supset N^\perp \xrightarrow{\simeq} \overline{\mathcal{T}} = \textit{std tractor bdle of } (\Sigma, \bar{\boldsymbol{c}}).$$

Proof.

Calculating in a scale g on M the tractor bundle $\mathcal{T}$, and hence also $N^\perp$, decomposes into a triple. Then the mapping of the isomorphism is

$$[N^\perp]_g \ni \begin{pmatrix} \sigma \\ \mu_b \\ \rho \end{pmatrix} \mapsto \begin{pmatrix} \sigma \\ \mu_b \mp H n_b \sigma \\ \rho \pm \frac{1}{2} H^2 \sigma \end{pmatrix} \in [\overline{\mathcal{T}}]_{\overline{g}}.$$

The tractor Gauss equation

The above reveals two connections on $\overline{\mathcal{T}} \cong N^{\perp}$ that we can compare. Namely the **intrinsic tractor connection** $\overline{\nabla}^{\overline{\mathcal{T}}}$ determined by $(\Sigma, \overline{\boldsymbol{c}})$, and the **projected ambient tractor connection** $\tilde{\nabla}$. The latter is defined by

$$\tilde{\nabla}_a U^B := \Pi^B_C(\Pi^c_a \nabla_c U^C) \qquad U \in \Gamma(N^{\perp}) \text{ extended arb. off } \Sigma$$

where Π^B_C and Π^c_a are the orthog. projections due to N and n. Including the tractor derivative of Π^B_C gives:

Proposition (Tractor Gauss formula – Stafford,Vyatkin)

$$\underline{\nabla}_a V^B = \overline{\nabla}_a V^B \mp S_a{}^B{}_C V^C \mp \mathrm{N}^B \mathbb{L}_{aC} V^C,$$

where $\mathrm{S}_{aBC} = \overline{\mathbb{X}}_{BC}{}^c \mathcal{F}_{ac}$, *($\overline{\mathbb{X}}_{BC}{}^c$ an invariant bundle injector), and*

$$\mathcal{F}_{ab} = \frac{1}{n-2}\Big(W_{acbd} n^c n^d + \mathring{L}^2_{ab} - \frac{|\mathring{L}|^2}{2(n-1)} \overline{\boldsymbol{g}}_{ab} \Big).$$

Recall $\mathbb{L}_{aC} = \underline{\nabla}_a N_C$. This shows that $\mathcal{F}_{ab}$ is a conformal invariant of hypersurfaces. It is the so-called **Fialkow tensor**.

Applications

The above results and tools provide the first steps in an **invariant calculus for conformal hypersurfaces** that is somewhat analogous to the local invariant calculus for Riemannian hypersurfaces. In particular combining these with usual tractor calculus it is easy to proliferate **hypersurface conformal invariants** and **conformally invariant operators**. E.g.:

Families of boundary operators along a conformal hypersurface.

Background: In **Riemannian geometry** the Neumann operator is $n^a\nabla_a$. Higher transverse order transverse boundary operators similarly given: $n^a n^b \nabla_a \nabla_b$ etc.

Conformal Robin op: The tools above allow an immediate analogue. Recall $\delta_1 :\stackrel{g}{=} n^a\nabla_a - wH^g$, is the conformal Cherrier-Robin operator – it gives a conformal boundary Robin operator for the conformal Laplacian. This is recovered by

$$(n+2w-2)\delta_1 = N^A D_A \qquad \text{along } \Sigma^{n-1} \text{ in } M^n.$$

Higher order analogues are important for PDE boundary problems, and the construction of higher order conformal Dirichlet-Neumann operators. **Juhl**, **Kobyashi et al** have sought continuous families of such, which in the flat case they describe as **symmetry breaking operators** and interpret as **intertwinors of the spherical principal series** representations of the conformal group.

Here is an **immediate construction** of such:

Lemma (G.-Peterson)

Given a conformal hypersurface embedding $\Sigma \hookrightarrow (M, \mathbf{c})$*,*

$$\delta_{j+1} := N^{A_1} N^{A_2} \cdots N^{A_j} \delta_1 D_{A_1} D_{A_2} \cdots D_{A_j}$$

constructs a family of natural conformally invariant hypersurface operators $\delta_K : \mathcal{T}^\Phi[w] \to \mathcal{T}^\Phi[w-K]|_\Sigma$ *along* Σ*.*

This observation can be refined **significantly** (G.-Peterson). The key is identify and treat the **special weights** w.

Hidden problems, hidden treasures

It would **appear** from the formula

$$\delta_K := N^{A_1} N^{A_2} \cdots N^{A_{K-1}} \delta_1 D_{A_1} D_{A_2} \cdots D_{A_{K-1}} \quad \text{along} \quad \Sigma$$

that the operator has "high" transverse order and is always at least of transverse order 1. But e.g.: (where $\bar{n} = \dim(\Sigma)$ etc)

$$\delta_2 f = -(\bar{\Delta} - \frac{\bar{n}-2}{4(\bar{n}-1)} \bar{\mathrm{Sc}}) f + \frac{\bar{n}-2}{4(\bar{n}-1)} \mathring{L}^{ab} \mathring{L}_{ab} f\ , \quad \text{for } f \in \mathcal{E}\left[1 - \frac{\bar{n}}{2}\right] .$$

This is the **intrinsic to Σ Yamabe operator** of $(\Sigma, \mathbf{c}_\Sigma)$ (plus the conformal invariant $\mathring{L}^{ab} \mathring{L}_{ab}$). So:

at this weight δ_2 has transverse order 0.

At the **interior Yamabe weight** $1 - \frac{n}{2}$ we have instead

$$\delta_2 = -(\Delta - \frac{n-2}{4(n-1)} \mathrm{Sc}) \quad \text{along} \quad \Sigma.$$

– i.e. the **interior Yamabe operator**.

Part IV: Geometry of conformal infinity

We return now to **conformally compact geometries** $(M, \mathbf{c}, I)$. Recall the **scale tractor** I is given $I = (\sigma, \nabla\sigma, -\frac{1}{d}(\Delta\sigma + J\sigma))$. We will consider in particular $(M, \mathbf{c}, I)$ which near the conformal infinity are **asymptotically of constant nonzero scalar curvature**. By imposing a constant dilation we may assume that I^2 approaches ± 1.

The σ, equivalently scale tractor I, strongly links the geometry of $\Sigma = \mathcal{Z}(\sigma)$ to the ambient by a beautiful agreement of I and the normal tractor:

Proposition

Let $(M^d, \mathbf{c}, I)$ be an almost pseudo-Riemannian structure with scale singularity set $\Sigma \neq \emptyset$ and $I^2 = \pm 1 + \sigma^2 f$ for some smooth (weight -2) density f. Then Σ is a smoothly embedded hypersurface and, with N denoting the normal tractor for Σ, we have $N = I|_\Sigma$.

Proof.

For simplicity assume the case $I^2 = \pm 1$ (so $f = 0$ and the structure is ASC). As usual let us write $\sigma := h(X, I)$. Along $\mathcal{Z}(\sigma)$

$$I_A = \frac{1}{d} D_A \sigma \stackrel{g}{=} \begin{pmatrix} 0 \\ \nabla_a \sigma \\ -\frac{1}{d}\Delta\sigma \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{g}^{ab}(\nabla_a \sigma)\nabla_b \sigma = \pm 1$$

so $n_a := \nabla_a \sigma$ is the unit conormal and a computation gives $\frac{1}{d}\Delta\sigma = -\frac{1}{d-1}\boldsymbol{g}^{ab} L^g_{ab} = -H^g$. □

Corollary

Let $(M^d, \mathbf{c}, I)$ be an almost pseudo-Riemannian structure with scale singularity set $\Sigma \neq \emptyset$, and that is asymptotically Einstein in the sense that $I^2|_\Sigma = \pm 1$, and $\nabla_a I_B = \sigma f_{aB}$ for some smooth (weight -1) tractor valued 1-form f_{aB}. Then Σ is a **totally umbilic hypersurface**.

Agreement of tractor connections

If we assume the stronger asymptotics: $I^2|_\Sigma = \pm 1$, and $\nabla_a I_B = \sigma^2 f_{aB}$ Then along Σ, I_B is parallel to the given order, and so the tractor curvature satisfies

$$\kappa_{ab}{}^C{}_D I^D = \kappa_{ab}{}^C{}_D N^D = 0 \quad \text{along} \quad \Sigma.$$

This implies

$$\boxed{W_{ab}{}^c{}_d n^d = 0}, \quad \text{along} \quad \Sigma = \mathcal{Z}(\sigma)$$

$\therefore$ Fialkow $\mathcal{F}_{ab} = \frac{1}{n-2}(W_{acbd} n^c n^d + \mathring{L}^2_{ab} - \frac{|\mathring{L}|^2}{2(n-1)} \overline{\boldsymbol{g}}_{ab})$ vanishes, &

Theorem

Let $(M^{d\geq 4}, \mathbf{c}, I)$ be an almost pseudo-Riemannian structure with scale singularity set $\Sigma \neq \emptyset$, and that is asymptotically Einstein in the sense that $I^2|_\Sigma = \pm 1$, and $\nabla_a I_B = \sigma^2 f_{aB}$. Then the tractor connection of $(M, \mathbf{c})$ preserves the intrinsic tractor bundle of Σ, where the latter is viewed as a subbundle of the ambient tractors: $\mathcal{T}_\Sigma \subset \mathcal{T}$. Furthermore the restriction of the parallel transport of $\nabla^{\mathcal{T}}$ coincides with the intrinsic tractor parallel transport of $\nabla^{\mathcal{T}_\Sigma = \overline{\mathcal{T}}}$.

Summary to this point

Any almost pseudo-Riemannian manifold with **non-zero generalised scalar curvature** (i.e. I^2 nowhere zero) has zero locus $\Sigma = \mathcal{Z}(\sigma)$ a smoothly embedded hypersurface.

If $g_\pm = \sigma^{-2}\boldsymbol{g}$ is (asymptotically) Einstein in that ∇I vanishes (sufficiently quickly) along Σ then for Σ the conformal invariants $\mathring{L}_{ab}$ and $\mathcal{F}_{ab}$ both vanish everywhere along Σ. This:
• gives agreement of the ambient and intrinsic tractor connections;
• **excludes interesting embeddings of** Σ – i.e. such Σ are not useful for studying general hypersurface geometry.

This is the classical Poincaré-Einstein setting. From the work of Fefferman-Graham and others we know that if $g_\pm = \sigma^{-2}\boldsymbol{g}$ is (asymptotically) Einstein (to sufficiently high order) then the conformal geometry of $(\Sigma, \bar{\boldsymbol{c}})$ actually formally determines the full geometry of $(M, g_\pm)$, at least up to order approximately $d-1$. Thus a **powerful link** intrinsic geometry of $(\Sigma, \bar{\boldsymbol{c}})$ to $(M, g_\pm)$ – **little freedom at all in the embedding** of Σ.

Part V: Boundary calculus

Part VI: The Loewner-Nirenberg problem and higher Willmore

Part V: Boundary calculus

Recall: By embedding a conformal manifold $(\Sigma, \bar{\mathbf{c}})$ as the boundary at infinity of a Poincaré-Einstein manifold (M, g_+), the Fefferman-Graham programme led to powerful tools: New approaches to the construction of conformal invariants (of $(\Sigma, \bar{\mathbf{c}})$), The GJMS operators and Q-curvature, scattering theory, $\cdots$, Applications to AdS/CFT conjecture, $\cdots$

From the early discussion we will see that there is an **analogous program** in for conformally embdedded hypersurfaces. In part the **generalises** the FG program but in other ways it is a **different program**.

We work on $M^{d=n+1}$.

Differential operators by prolonged coupling

On an almost pseudo-Riemannian manifold $(M, \mathbf{c}, I)$ there is a canonical differential operator by **coupling** I^A **to** D_A , namely

$$I{\cdot}D := I^A D_A .$$

This acts on any weighted tractor bundle, preserving its tensor type but lowering the weight:

$$I{\cdot}D : \mathcal{E}^\Phi[w] \to \mathcal{E}^\Phi[w-1].$$

It will be useful to define define the *weight operator* $\mathbf{w}$: if $\beta \in \Gamma(\mathcal{B}[w_0])$ we have

$$\mathbf{w}\,\beta = w_0\beta.$$

Then on $\mathcal{E}^\Phi[w]$ we have

$$I{\cdot}D \stackrel{g}{=} \begin{pmatrix} -\frac{1}{d}(\Delta\sigma + \mathrm{J}\sigma) & \nabla^a\sigma & \sigma \end{pmatrix} \begin{pmatrix} \mathbf{w}(d+2\mathbf{w}-2) \\ \nabla_a(d+2\mathbf{w}-2) \\ -(\Delta + \mathrm{J}\mathbf{w}) \end{pmatrix}.$$

$$= -\sigma\Delta + (d+2w-2)[(\nabla^a\sigma)\nabla_a - \frac{w}{d}(\Delta\sigma)] - \frac{2w}{d}(d+w-1)\sigma\mathrm{J}$$

The canonical degenerate Laplacian

Now on $M \setminus \mathcal{Z}(\sigma)$ in the metric $g_\pm = \sigma^{-2}\boldsymbol{g}$, with densities trivialised accordingly, we have

$$I \cdot D \stackrel{g_\pm}{=} \mp\Big(\Delta^{g_\pm} + \frac{2w(d+w-1)}{d} \mathrm{J}^{g_\pm}\Big).$$

In particular if $g_\pm$ satisfies $\mathrm{J}^{g_\pm} = \mp\frac{d}{2}$ (*i.e.* $\mathrm{Sc}^{g_\pm} = \mp d(d-1)$ or equivalently $I^2 = \pm 1$) then, relabeling $d+w-1 =: s$ and $d-1 =: n$, we have

$$\boxed{I \cdot D \stackrel{g_\pm}{=} \mp\big(\Delta^{g_\pm} \pm s(n-s)\big)}.$$

so solutions are **eigenvectors of the Laplacian** (and s is called the **spectral parameter**) as in **scattering theory**.

But on $\Sigma = \mathcal{Z}(\sigma) \neq \emptyset$, the conformal infinity, $I \cdot D$ degenerates and there the operator is first order. In particular if the structure is asymptotically ASC in the sense that $I^2 = \pm 1 + \sigma f$, for some smooth f, then along Σ

$$\boxed{I \cdot D = (d+2w-2)\delta_n}, \qquad \delta_1 \stackrel{g}{=} n^a \nabla_a^g - wH^g = \text{ conformal Robin}$$

Thus $I \cdot D$ is a **degenerate Laplacian**, natural to $(M, \mathbf{c}, I)$.

The $\mathfrak{sl}(2)$-algebra

$(M, \mathbf{c})$ be a conformal structure of dimension $d \geq 3$, $\sigma \in \Gamma(\mathcal{E}[1])$ and $I_A = \frac{1}{d} D_A \sigma$ (as usual). Then a direct computation gives

Lemma

Acting on any section of a weighted tractor bundle we have

$$[I \cdot D, \sigma] = I^2(d + 2\mathbf{w}),$$

where $\mathbf{w}$ *is the weight operator.*

Thus with **only the restriction that generalised scalar curvature is non-vanishing** we have:

Proposition (G.-Waldron)

Suppose that (M, c, σ) *is such that* I^2 *is nowhere vanishing. Setting* $x := \sigma$, $y := -\frac{1}{I^2} I \cdot D$, *and* $h := d + 2\mathbf{w}$ *we obtain the commutation relations*

$$[h, x] = 2x, \quad [h, y] = -2y, \quad [x, y] = h,$$

of standard $\mathfrak{sl}(2)$*-algebra generators.*

Application: Conformal Laplacian powers

Theorem

Let $\mathcal{E}^\Phi$ be any tractor bundle and $k \in \mathbb{Z}_{\geq 1}$. Then, for each $k \in \mathbb{Z}_{\geq 1}$, along $\Sigma = \mathcal{Z}(\sigma)$

$$P_k : \mathcal{E}^\Phi[\frac{k-n}{2}] \to \mathcal{E}^\Phi[\frac{-k-n}{2}] \quad \textit{given by} \quad P_k := \left(-\frac{1}{I^2} I\!\!D\right)^k \quad (2)$$

is a tangential differential operator, and so determines a canonical differential operator $P_k : \mathcal{E}^\Phi[\frac{k-n}{2}]|_\Sigma \to \mathcal{E}^\Phi[\frac{-k-n}{2}]|_\Sigma$. For k even this takes the form

$$P_k = \overline{\Delta}^k + \textit{lower order terms}. \quad (3)$$

Proof.

From the $\mathfrak{sl}(2)$-identities we have $[x, y^k] = y^{k-1}k(h-k+1)$.
Thus on $\mathcal{E}^\Phi[\frac{k-n}{2}]$

$$P_k(f + \sigma h) = y^k(f + xh) = P_k f + \sigma \widetilde{P}_k h.$$

So P_k is **tangential**. Expanding the $I{\cdot}D$s yields (3). □

Natural boundary problems

Suppose on a conformally compact manifold M_+ (with $M_+ \cup \partial M_+ = \overline{M}$) we wish to study solutions to

$$Pf := \left(\Delta^{g_+} + \frac{2w(d+w-1)}{d} \mathsf{J}^{g_+}\right) f = 0.$$

E.g. this is what is studied in the usual Poincaré-Einstein scattering program.

Then one needs to fix suitable boundary conditions. E.g. in the case of Riemannian signature one wants some elliptic boundary problem. Since the boundary ∂M_+ is at infinity, with g_+ singular along ∂M_+, this is non-trivial.

But if we view f as the trivialisation of a density of weight w then $Pf \stackrel{g_+}{=} I{\cdot}Df$ and $I{\cdot}D$ is well defined on all of $\overline{M}$ (and its smooth extension to M beyond ∂M_+). Thus it is natural to study the $I{\cdot}D$ problem. We do this **formally**.

First we treat an obvious Dirichlet-like problem where we view $f|_\Sigma$ as the initial data.

Asymptotic solutions of the first kind

Problem

Given $f|_\Sigma$, and an arbitrary extension f_0 of this to $\mathcal{E}^\Phi[w_0]$ over M, find $f_i \in \mathcal{E}^\Phi[w_0 - i]$ (over M), $i = 1, 2, \cdots$, so that

$$f^{(\ell)} := f_0 + \sigma f_1 + \sigma^2 f_2 + \cdots + O(\sigma^{\ell+1})$$

solves $I{\cdot}Df = O(\sigma^\ell)$, off Σ, for $\ell \in \mathbb{N} \cup \infty$ as high as possible.

$I{\cdot}Df = 0 \Leftrightarrow -\frac{1}{I^2} I{\cdot}Df = 0$ so we recast this via $\mathfrak{sl}(2) = \langle x, y, h\rangle$.
Set $h_0 = d + 2w_0$. By the identity $[x^k, y] = x^{k-1}k(h + k - 1)$:

$$yf^{(\ell+1)} = yf^{(\ell)} - x^\ell(\ell + 1)(h + \ell)f_{\ell+1} + O(x^{\ell+1}).$$

Now $hf_{\ell+1} = \big(h_0 - 2(\ell + 1)\big) f_{\ell+1}$, thus

$$yf^{(\ell+1)} = yf^{(\ell)} - x^\ell(\ell + 1)(h_0 - \ell - 2)f_{\ell+1} + O(x^{\ell+1}). \qquad (4)$$

By assumption $yf^{(\ell)} = O(x^\ell)$, thus if $\boxed{\ell \neq h_0 - 2}$ we can solve $yf^{(\ell+1)} = O(x^{\ell+1})$ and this **uniquely determines** $f_{\ell+1}|_\Sigma$.

So we can solve to all orders provided we do not hit $\ell = h_0 - 2$ i.e. provided $w_0 \notin \{\frac{k-n}{2} \ : \ k \in \mathbb{Z}_{\geq 1}\}$. Otherwise (4) shows that

$$\ell = h_0 - 2 \quad \Rightarrow \quad yf^{(\ell)} = y\big(f^{(\ell)} + x^{\ell+1} f_{\ell+1}\big), \qquad \text{modulo } O(x^{\ell+1}),$$

regardless of $f_{\ell+1}$. It follows that the map $f_0 \mapsto x^{-\ell} y f^{(\ell)}$ is tangential and $x^{-\ell} y f^{(\ell)}|_\Sigma$ is the obstruction to solving $yf^{(\ell+1)} = O(x^{\ell+1})$. Then by a simple induction this is seen to be a non-zero multiple of $y^{\ell+1} f_0|_\Sigma$:

Proposition

If $\ell = h_0 - 2$ then the smooth extension is (in general) obstructed by $P_{\ell+1} f_0|_\Sigma$, where $P_{\ell+1} = (-\frac{1}{I^2} I\cdot Df)^{\ell+1}$ is the tangential operator on densities of weight w_0 given by Theorem 27.

If $\ell = h_0 - 2$ then the extension can be continued with **log terms**.
If $\overline{M}$ is almost Einstein to sufficiently high order then:

- the **odd order** $P_{\ell+1}$ **vanish identically**; and
- the **even order** $P_{\ell+1}$ are the **GJMS operators** on $(\partial M_+, \bar{\mathbf{c}})$.

(Formal) solutions of the second kind

Now we consider the more general type of solution:

Problem

Given $\overline{f}_0|_\Sigma \in \Gamma\mathcal{E}^\Phi[w_0 - \alpha]|_\Sigma$ *and an arbitrary extension* $\overline{f}_0$ *of this to* $\Gamma\mathcal{E}^\Phi[w_0 - \alpha]$ *over* $\overline{M}$, *find* $\overline{f}_i \in \mathcal{E}^\Phi[w_0 - \alpha - i]$ *(over* $\overline{M}$*),* $i = 1, 2, \cdots$, *so that*

$$\overline{f} := \sigma^\alpha\big(\overline{f}_0 + \sigma\,\overline{f}_1 + \sigma^2\,\overline{f}_2 + \cdots + O(\sigma^{\ell+1})\big) \qquad (5)$$

solves $I \cdot D\overline{f} = O(\sigma^{\ell+\alpha})$, *off* ∂M_+, *for* $\ell \in \mathbb{N} \cup \infty$ *as high as possible.*

Now α, if not integral, this Problem takes us outside the realm of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ and its modules. But it is straightforward to show that for any $\alpha \in \mathbb{R}$:

$$[x^\alpha, y] = x^{\alpha-1}\alpha(h + \alpha - 1). \qquad (6)$$

It follows immediately from (6) that $I \cdot D\overline{f} = 0$ has:

- no solution if $\alpha \notin \{0, h_0 - 1\}$, where $h\overline{f} = h_0\overline{f}$; and
- if $\alpha = h_0 - 1$ and $\overline{f} = \sigma^{\alpha} f$ then

$$I \cdot D\overline{f} = \sigma^{\alpha} I \cdot Df \qquad \boxed{\text{So } \overline{f} \text{ is a solution iff } f \text{ is!}}$$

So in this way **second solutions** arise from **first** and vv.

For $w_0 \notin \{\frac{k-n}{2} \ : \ k \in \mathbb{Z}_{\geq 1}\}$, and writing $F = f$, $G = \sigma^{-\alpha}\overline{f}$ we can combine these to a general solution

$$F + \sigma^{h_0 - 1} G = F + \sigma^{n+2w_0} G$$

or, trivialising the densities on M_+ using the generalised scale σ:

$$f = \sigma^{n-s} F + \sigma^{s} G = \sigma^{-w_0}(F + \sigma^{h_0-1} G)$$

where $s := w_0 + n$. Which is the form of solution used in the scattering theory (of Graham-Zworski, Mazzeo-Melrose, $\cdots$). The $\mathfrak{sl}(2)$ approach above solves the asymptotics of F and G.

Part VI: The Loewner-Nirenberg problem and higher Willmore

The Poincaré-Einstein construction is a tool for studying a conformal manifold $(\Sigma, \bar{\mathbf{c}})$ **holographically**. That is for obtaining the invariants and invariant operators of $(\Sigma, \bar{\mathbf{c}})$ in terms (pseudo-)Riemannian objects on the manifold M_+ of 1 greater dimension that has $\Sigma = \partial M_+$.

Conversely the scattering theory of (M, g_+) can be understood in terms of non-local conformal operators on the boundary $(\Sigma, \bar{\mathbf{c}})$.

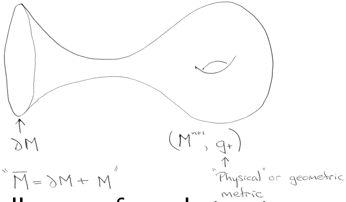
But requiring g_+ to be Einstein (even asymp. near ∂M_+) is **highly restrictive**. It means that the conformal manifold with boundary $(\overline{M}, \mathbf{c})$ has $\Sigma = \partial M_+$ totally umbilic, Fialkow vanishes, etcetera.

Here we seek to set up the analogous program for $(\overline{M}, \mathbf{c})$ **a general manifold with boundary**.

Thus, given $(\overline{M} = M_+ \cup \partial M_+, \mathbf{c})$ we **need a way to determine a distinguished metric** $g_+ \in \mathbf{c}|_{M_+}$ on M_+ so that (M_+, g_+) is conformally compact.

Recall a **conformal compactification** of a complete Riemannian manifold (M^{n+1}, g_+) is a manifold $\overline{M}$ with boundary ∂M s.t.:

- $\exists\ \overline{g}$ on $\overline{M}$, with $g_+ = r^{-2}\overline{g}$, where
- r a **defining function** for ∂M: $\partial M = \mathcal{Z}(r)$ & $dr_p \neq 0\ \forall p \in \partial M$.



$\Rightarrow$ canonically a conformal structure on boundary: $(\partial M, [\overline{g}|_{\partial M}])$.

Question/variant: Given $\overline{g}$ (or really $\mathbf{c} = [\overline{g}]$) can we find a defining function $r \in C^\infty(\overline{M})$ for $\Sigma = \partial M$ s.t.

$\mathrm{Sc}(r^{-2}\overline{g}) = -n(n+1)$? NB: This satisfied for Poincaré-Einstein

cf. Loewner-Nirenberg, Aviles and McOwen – related interior problems.

The obstruction density of ACF

Can we solve $\mathrm{Sc}(r^{-2}\overline{g}) = -n(n+1)$? formally (i.e. power series) along the boundary? **Answer: No** - in general can get:

Theorem (Andersson, Chruściel, & Friedrich)

$$\mathrm{Sc}(r^{-2}\overline{g}) = -n(n+1) + r^{n+1}\mathcal{B}_n.$$

Furthermore (they show)

$$\mathcal{B}_2 = \delta \cdot \delta \cdot \mathring{L} + \textit{lower order}$$

is a conformal invariant of $\Sigma^2 = \partial M$.

Theorem.[G. + Waldron] For $n \geq 2$ $\mathcal{B}_n$ is a conformal invariant of $\Sigma = \partial M$, and $\mathcal{B}_2$ = **Willmore Invariant** $= \overline{\Delta} +$ lower order!

•For n even the invariant $\mathcal{B}_n$ is **higher order analogue** of $\mathcal{B}_2 = \mathcal{B}$.

NB. The existence of such a higher analogue was not previously obvious as the weight and leading order of $\mathcal{B}_n$ means standard tractor/ambient metric approaches fail.

Recasting the problem and holography

Recall the constant scalar curvature condition in terms of scale. A conformal manifold has a canonical conformal metric $\boldsymbol{g} \in S^2 T^* M[2]$. A metric $g_+ \in \mathbf{c}$ is equivalent to a **scale**:

$$\boxed{g_+ = \sigma^{-2}\boldsymbol{g}} \quad \Leftrightarrow \quad \boxed{\sigma \in \Gamma(\mathcal{E}_+[1]).}$$

Via the Thomas-D operator $\bar{D} = \frac{1}{n+1} D$ the scale is equivalent to the

$$\boxed{\textbf{scale tractor} \quad I_A := \bar{D}_A \sigma,} \qquad \text{and}$$

Lemma

$\mathrm{Sc}(g_+) = -n(n+1) \Leftrightarrow I^2 := h(I, I) = 1$

So we come to a "conformal Eikonal equation" $(\bar{D}_A\sigma)(\bar{D}^A\sigma) = 1$, where σ a **defining density** for Σ. **NB:**

- If $\boxed{\text{we could solve uniquely then } \Sigma \hookrightarrow (M, \mathbf{c}) \textbf{ determines } g \in \mathbf{c}}$.

Then invariants of conf. compact (M, g_+) would be invariants of Σ.

The conformal Eikonal equation

Thus to solve the singular Yamabe problem formally we come to the following **non-linear** problem:

Problem: For a conformal manifold $(M, \mathbf{c})$ and an embedding $\iota : \Sigma \to M$ solve

$$I_A I^A = (\bar{D}_A \sigma)(\bar{D}^A \sigma) = 1 + O(\sigma^\ell)$$

for ℓ as high as possible, and σ a Σ defining density.

A key observation is that the **linearisation** of $I^A I_A = 1$ is $I^A D_A \dot{\sigma} = 0$ – the $I{\cdot}D$ problem on $\mathcal{E}[1]$. Thus $\exists$ hope that the $\mathfrak{sl}(2)$ generated by $x := \sigma$, $y := -\frac{1}{I^2} I^A D_A$ will again be useful.

Recall from the standard $\mathfrak{sl}(2)$ identities we have

$$[I \cdot D, \sigma^{k+1}] = I^2 \sigma^k (k+1)(n+k+1+2\boldsymbol{w}),$$

and this allows an inductive solution (using also **other tractor identities**) that mimics the linear case!

Lemma

Suppose that $\underline{\sigma \in \Gamma(\mathcal{E}[1]) \text{ defines } \Sigma = \partial M_+ \text{ in } (\overline{M}, \mathbf{c})}$ *and*

$$I_\sigma^2 = 1 + \sigma^k A_k \quad \text{where} \quad A_k \in \Gamma(\mathcal{E}[-k])$$

is smooth on M*, and* $k \geq 1$*, then*

- *if* $k \neq (n+1)$ *then* $\exists\ f_k \in \Gamma(\mathcal{E}[-k])$ *s.t.* $\sigma' := \sigma + \sigma^{k+1} f_k$ *satisfies* $I_{\sigma'}^2 = 1 + \sigma^{k+1} A_{k+1}$*, where* A_{k+1} *smooth;*
- *if* $k = (n+1)$ *then:* $I_{\sigma'}^2 = I_\sigma^2 + O(\sigma^{n+2})$.

Proof.

Squaring with the tractor metric, using the $\mathfrak{sl}(2)$, etc

$$\begin{aligned}(\bar{D}\sigma')^2 &= (\bar{D}\sigma + \bar{D}(\sigma^{k+1} f_k))^2 \\ &= I_\sigma^2 + \frac{2}{n+1} I_\sigma \cdot D(\sigma^{k+1} f_k) + (\bar{D}(\sigma^{k+1} f_k))^2 \\ &= 1 + \sigma^k A_k + \frac{2\sigma^k}{n+1}(k+1)(n+1-k) f_k + O(\sigma^{k+1}).\end{aligned}$$

□

The distinguished defining density

This applies formally off any hypersurface in a Riemannian conformal manifold $(M, \mathbf{c})$ (and even more generally) so we have:

Theorem (G.-, Waldron arXiv:1506.02723)

For Σ^n embedded in $(M^{n+1}, \mathbf{c})$ there is a distinguished defining density $\bar{\sigma}$, **unique** *modulo $+O(\sigma^{n+2})$, s.t.*

$$I_{\bar{\sigma}}^2 = 1 + \bar{\sigma}^{n+1} \mathcal{B}_{\bar{\sigma}}.$$

Moreover:

$$\mathcal{B} := \mathcal{B}_{\bar{\sigma}}|_{\Sigma} \in \Gamma(\mathcal{E}_{\Sigma}[-n-1])$$

is determined by $(M, \mathbf{c}, \Sigma)$ and is a **natural conformal invariant**.

For n even $\mathcal{B} = 0$ generalises the Willmore equation in that:

$$\mathcal{B} = \bar{\Delta}^{\frac{n}{2}} H + \text{lower order terms};$$

while for n odd $\mathcal{B}$ has no linear leading term.

All submanifold invariants via holography?

The construction can be used to obtain other submanifold invariants: Our Theorem above shows that:

$$(M, \mathbf{c}, \Sigma) \quad \text{determines} \quad \bar{\sigma} \quad \text{modulo} + O(\sigma^{n+2}).$$

Suppose that $\mathcal{I}$ is any coupled conformal invariant of $(M, \mathbf{c}, \bar{\sigma})$ involving only the jet $j^{n+1}\bar{\sigma}$. Then along Σ

$$\boxed{\mathcal{I}\big|_{\Sigma} \quad \text{is a conformal invariant of} \quad (M, \mathbf{c}, \Sigma).}$$

This **holographic** approach fails at order $n+2$ precisely because of the existence of the **obstruction invariant** $\mathcal{B}$. This is precisely an analogue of the use Fefferman-Graham's Poincaré and ambient metric constructions to find conformal invariants – that fails at order $n+1$ because of **Bach** B_{ab} in dimension 4 and the **Fefferman-Graham obstruction tensor** in higher even dimensions.

Extrinsically coupled GJMS operators

Recall on any almost Riemannian manifold (M, c, I) we had:

Theorem

Let $\mathcal{E}^\Phi$ be any tractor bundle and $k \in \mathbb{Z}_{\geq 1}$. Then, for each $k \in \mathbb{Z}_{\geq 1}$, along $\Sigma = \mathcal{Z}(\sigma)$

$$P_k^\sigma : \mathcal{E}^\Phi[\frac{k-n}{2}] \to \mathcal{E}^\Phi[\frac{-k-n}{2}] \quad \textit{given by} \quad P_k^\sigma := \Big(-\frac{1}{I^2} I \cdot D \Big)^k$$

is a tangential differential operator, and so determines a canonical differential operator $P_k^\sigma : \mathcal{E}^\Phi[\frac{k-n}{2}]|_\Sigma \to \mathcal{E}^\Phi[\frac{-k-n}{2}]|_\Sigma$. For k even this takes the form

$$P_k = \overline{\Delta}^k + \textit{lower order terms.}$$

Because $(M, \mathbf{c}, \Sigma)$ determines $\bar{\sigma}$ modulo $+ O(\sigma^{n+2})$, we have:

Theorem

For $k \leq n = d - 1$ the operators P_k are determined canonically by the data $(M, \mathbf{c}, \Sigma)$.

Higher Willmore energies

For suitable regularisations $\overline{M}_\epsilon$ of conformally compact manifolds $\overline{M}$:

$$\mathrm{Vol}_\epsilon = \int_{\overline{M}_\epsilon} \sqrt{g_+} = \frac{v_n}{\epsilon^n} + \cdots + \frac{v_1}{\epsilon} + \mathcal{A} \log \epsilon + V_{ren} + O(\epsilon).$$

Theorem (Graham 2016: arXiv:1606.00069)

If $g_+ = \bar{\sigma}^{-2}\mathbf{g}$, i.e. it is the approximate solution of the sing. Yamabe problem then $\mathcal{A}$ a conformal invariant of $\Sigma \hookrightarrow M$ and

$$\frac{\delta \mathcal{A}}{\delta \Sigma} = \frac{d(d-2)}{2} \mathcal{B}_n$$

So the anomaly term in the renormalised volume expansion provides an **energy** with **functional gradient the obstruction density**, in other words an energy generalising the Willmore energy.

Extrinsic Q-curvature and the anomaly

In fact – also in analogy with the treatment of Poincaré-Einstein manifolds – there is nice local quantity giving the anomaly:

Theorem (G.- Waldron arXiv:1603.07367)

$$\mathcal{A} = \frac{1}{(d-1)!(d-2)!} \int_\Sigma Q$$

where, with $\tau \in \Gamma\mathcal{E}_+[1]$ a scale giving the boundary metric, $Q := (-I \cdot D)^n \log \tau$.

- Q here is an **extrinically coupled** Q**-curvature** meaning e.g.

$$Q^{\widehat{g}_\Sigma} = e^{-nf}(Q^g + P_n f) \quad \text{where} \quad \widehat{g}_\Sigma = e^{2f} g_\Sigma$$

and for n even

$$P_n = \Delta_\Sigma^{\frac{n}{2}} + \text{lower order terms; } P_n \text{ FSA, and } P_n 1 = 0,$$

is an **extrinically coupled** GJMS type operator. Q and P_n are from G.-, Waldron arXiv:1104.2991 = Indiana U.M.J. 2014.

Idea of proof

Use a Heaviside function θ to "cut off" an integral over all $\overline{M}$

$$\mathrm{Vol}_\epsilon = \int_{\overline{M}} \frac{dV^{g_\tau}}{\sigma^d} \theta\left(\frac{\sigma}{\tau} - \epsilon\right).$$

Then the divergent terms and anomaly are given by

$$v_k \sim \frac{d^{d-1-k}}{d\epsilon^{d-1-k}} \left(\epsilon^d \frac{d}{d\epsilon} \mathrm{Vol}_\epsilon\right)\Big|_{\epsilon=0},$$

So

$$v_k \sim \int_{\overline{M}} \frac{\delta^{d-1-k}(\sigma)}{\tau^k} \quad \text{and} \quad \mathcal{A} \sim \int_{\overline{M}} \delta^{d-2}(\sigma) I \cdot D \log \tau$$

Then via identities, and the $sl(2)$ again

$$v_k \sim \int_\Sigma \frac{1}{\tau^k} \quad \text{and} \quad \mathcal{A} \sim \int_\Sigma (I \cdot D)^{d-1} \log \tau$$