Gauge invariant PDE, AKSZ sigma models, and Higher Spin fields

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Motivations and perspectives

- Higher spin (HS) gauge theories, underlying geometry, gauge structure, formalism

- Interactions are typically tightly related to underlying gauge symmetries (e.g. fields responsible for fundamental interactions are gauge fields) which in turn can be understood in terms of underlying geometry.

- Batalin-Vilkovisky (BV) approach to gauge systems (or its generalizations) is probably the most powerful. *Batalin, (Fradkin), Vilkovisky, 1981 . . .* . In particular, it gives tools to extract geometrical structures underlying gauge invariance of the theory.

- For various topological models their BV formulation can be cast into the form of AKSZ sigma model.

Alexandrov, Kontsevich, Schwartz, Zaboronsky, 1994.

AKSZ can be seen as BV generalization for *Sullivan* free differential algebras (FDA) with constraints. *Vasiliev's* unfolded formalism in HS theory.

- It turns out the same can be done for not necessarily topological models at the price of infinite-dimensional target space *G. Barnich, M.G. 2010, M.G. 2010*

The construction shows deep relations with Vinogradov's approach to PDE.

- For a local gauge theory AKSZ-like formulation has certain advantages over the usual jet-bundle version of the BV formalism *Henneaux; Barnich, Brandt, . . .* .

- This has to do with the manifest background independence of AKSZ. In particular, this offers remarkably powerful framework for studying relation between fields in the boundary and in the bulk e.g. in the context of AdS/CFT correspondence (cf. lectures by *R.Gover*) in the case of gauge systems. Same applies to manifest realization of symmetries.

- An underlying ideology: try to work with local gauge field theory (gauge-invariant PDE) as a basic geometrical object. Just like we work with PDE, understood in invariant terms à la Vinogradov.

PDE's and jet-bundles

Fiber-bundle $\mathcal{F} \to X$ (global aspects are not discussed):
base space (independent variables or space-time coordinates): x^a, $a = 1, \ldots, n$.
Fiber: (dependent variables or fields ϕ^i)

Jet-bundle:
A point of J^n is a pair $(x, [s])$, where $[s]$ is an equivalence class of sections $s : X \to \mathcal{F}$ such that their partial derivatives coincide up to order n at x. In coordinates:

$$\frac{\partial^l \phi^i(s(x))}{\partial x^{a_1} \ldots \partial x^{a_l}} = \frac{\partial^l \phi^i(s'(x))}{\partial x^{a_1} \ldots \partial x^{a_l}} \qquad l = 0, 1, \ldots n$$

In particular, $J^0(\mathcal{F}) = \mathcal{F}$.

One can use x^i, and values of above derivatives as coordinates:

$$J^0(\mathcal{F}): \quad x^a, \phi^i, \qquad J^1(\mathcal{F}): \quad x^a, \phi^i, \phi^i_a\,, \qquad J^2(\mathcal{F}) \quad x^a, \phi^i, \phi^i_a, \phi^i_{ab}\,, \qquad \ldots$$

Projections:

$$\ldots \to J^N(\mathcal{F}) \to J^{N-1}(\mathcal{F}) \to \ldots \to J^1(\mathcal{F}) \to J^0(\mathcal{F}) = \mathcal{F}$$

Useful to work with $\mathcal{J} := \mathcal{J}^\infty$ (projective limit).

A local function is a pull-back of a function from $J^N(\mathcal{F})$ for some N. i.e. it depends on only a finite number of the coordinates.
A local function $f = f(x, \phi, \phi_a, \phi_{ab} \ldots)$ can be evaluated on a section $s : X \to \mathcal{F}$ as

$$f(s) := f(x, \phi^i(s), \partial_a \phi^i(s), \ldots)$$

Total derivative: (imitates the action of standard partial derivative)

$$\partial_a^T := \frac{\partial}{\partial x^a} + \phi_a^i \frac{\partial}{\partial \phi^i} + \phi_{ab}^i \frac{\partial}{\partial \phi_a^i} + \dots$$

Main property:

$$\partial_a(f(s)) = (\partial_a^T f)(s)\,.$$

Similarly one defines local forms. These are forms that can be obtained by pullback from finite jets.

Space-time differentials dx^a. Horizontal differential:

$$d_{\mathsf{h}} \equiv dx^a \partial_a^T \,, \qquad d_{\mathsf{h}}^2 = 0\,.$$

Differential forms:

$$\alpha = \alpha(x, dx, \phi, \phi_a, \ldots)_{I_1 \ldots I_k} d_{\mathsf{v}}\phi^{I_1} \ldots d_{\mathsf{v}}\phi^{I_k}\,, \qquad \phi^I = \{\phi^i_{a_1 \ldots a_m}\}$$

Vertical differential:

$$d_{\mathsf{v}} \equiv d - d_{\mathsf{h}} = d_{\mathsf{v}}\phi^I \frac{\partial}{\partial \phi^I}$$

Variational bicomplex:

$$d_{\mathsf{v}}^2 = 0, \qquad d_{\mathsf{v}} d_{\mathsf{h}} + d_{\mathsf{h}} d_{\mathsf{v}} = 0\,, \qquad d_{\mathsf{h}}^2 = 0$$

Bidegree (l, p). On the jet space $H^{>0}(d_{\mathsf{v}}) = 0 = H^{<n}(d_{\mathsf{h}})$ (unless global geometry!), where $n = \dim(X)$. $H^n(d_{\mathsf{h}}) =$ local functionals

A system of partially differential equations (PDE) is a collection of local functions on $\mathcal{J}$

$$E_\mu[\phi, x]\,.$$

It's usually assumed that E_μ define regular submanifold and x^a are not constrained, so that $\mathcal{E}$ is a bundle over X (space-time).

The equation manifold (stationary surface): $\mathcal{E} \subset \mathcal{J}$ singled out by: (prolonged equation)

$$\partial^T_{a_1} \ldots \partial^T_{a_l} E_\mu = 0\,, \qquad l = 0, 1, 2, \ldots$$

understood as the algebraic equations in $\mathcal{J}$.

∂^T_a are tangent to $\mathcal{E}$ and hence restricts to $\mathcal{E}$. So do the differentials d_{h} and d_{v}. $\partial^T_a|_{\mathcal{E}}$ determine a dim-n integrable distribution (Cartan distribution).

Definition: *[Vinogradov]* *A PDE is a manifold $\mathcal{E}$ equipped with an integrable distribution.*

In addition one typically assumes regularity, constant rank, and that $\mathcal{E}$ is a bundle over the spacetime. Use notation $(\mathcal{E}, d_{\mathsf{h}})$.

PDEs are isomorphic when respective distributions are.

Differential forms on $\mathcal{E}$ form the variational bicomplex of $\mathcal{E}$. Note that in general $H^k(d_{\mathsf{h}}) \neq 0$ for $k < n$.

Example: mechanics

ODE: $x_{tt} = f(x, x_t, t)$, as cooridnates on $\mathcal{E}$ one can take t, x, x_t (*cf. talk by A.Waldron*) i.e. $\mathcal{E}$ is a phase space extended by time variable. If the equation arise from $L(x, x_t, t)$ then $\mathcal{E}$ acquires presymplectic (and contact) structure.

Example: scalar field

Start with:

$$L = \frac{1}{2}\eta^{ab}\phi_a\phi_b - V(\phi)\,, \qquad \partial_a\partial^a\phi + \frac{\partial V}{\partial \phi} = 0\,.$$

$\mathcal{E}$ is coordinatized by $x^a, \phi, \phi_a, \phi_{ab}, \ldots$. Already ϕ_{ab} are not independent. One can e.g. take $\phi_{abc\ldots}$ traceless. The d_{h}-differential on $\mathcal{E}$ reads as

$$d_{\mathsf{h}} x^a = dx^a\,, \quad d_{\mathsf{h}}\phi = dx^a\phi_a,\,, \quad d_{\mathsf{h}}\phi_a = dx^b(\phi_{ab} - \frac{1}{n}\eta_{ab}\frac{\partial V}{\partial \phi})\,, \quad \ldots$$

So if the system is nonlinear, i.e. $\frac{\partial V}{\partial \phi}$ nonlinear in ϕ, d_{h} is also nonlinear.

Linear PDE:

$$E_\mu = D_{\mu i}(x, \partial_a^T)\phi^i$$

Equation manifold is a vector bundle over X.
Formal solutions at $x \in X$:

$$\phi^i(x_0, y) = \phi^i(x) + \partial_a \phi^i y^a + \frac{1}{2}\partial_a \partial_b \phi^i y^a y^b + \dots$$

$$W_{x_0} = \{\phi^i(x_0, y) \ : \ D_{\alpha i}(x_0 + y, \frac{\partial}{\partial y})\phi^i(x_0, y) = 0\}$$

give a fiber at x_0.

Strictly speaking, in general it's not a vector bundle. Often in applications: there is a transitive space-time symmetry which forces all fibers to be isomorphic.
Moreover, fiber is a module over the space-time symmetry algebra. Sometimes called Weyl module in HS context *(cf. Vasiliev unfolded formalism).*

Example: KG field on Minkowski. Equation for W takes the form (in this case nothing depend on x_0)

$$\eta^{ab}\frac{\partial}{\partial y^a}\frac{\partial}{\partial y^b}\phi(y) = 0$$

This is clearly a Poincare module and equation manifold is a product of Minkowski space with the space of totally symmetric tracelss tensors.

Example: KG field on AdS_{d+1}. In terms of $\mathbb{R}^{d,2}$ with coordinates X^A equations read

$$\eta^{AB}\frac{\partial}{\partial Y^A}\frac{\partial}{\partial Y^B}\phi(X_0, Y) = 0\,, \qquad (X_0^A + Y^A)\frac{\partial}{\partial Y^A}\phi(X_0, Y) = 0$$

One can assume $\eta_{AB} = diag(-1, -1, 1, \dots, 1)$ and for $X_0 \in AdS_{d+1}$ take $X_0 = (1, 0, \dots, 0)$ using $O(d, 2)$-symmetry. Note that $o(d, 2)$-acts as

$$\rho(J^A{}_B) = (X_0^A + Y^A)\frac{\partial}{\partial Y^B} - (X_0^B + Y^B)\frac{\partial}{\partial Y^A}$$

Anticipating a bit, note that KG equation on AdS_{d+1} can be equivalently written as *cf. Vasiliev's unfolded form*

$$\nabla\Psi = 0$$

where Ψ is defined on AdS_{d+1} and takes values in W_0 and ∇ is determined by a natural flat $o(d,2)$-connection and representation ρ:

$$\nabla = dx^\mu(\frac{\partial}{\partial x^\mu} + \omega_{\mu A}^{B}(X_0^A + Y^A)\frac{\partial}{\partial Y^B})$$

Note that although we started with describing KG field on AdS_{d+1} in terms of $\mathbb{R}^{d,2}$ we managed to write the equation in terms of $AdS_{d+1} \subset \mathbb{R}^{d,2}$. This feature is useful for studying boundary values.

Gauge PDE (local gauge field theories)

Toy model: $\dim X = 0$ case. I.e. usual geometry, $\mathcal{J} = \mathcal{F}$. PDE is just a collection of regular functions $E_\mu(\phi)$ determining equation submanifold $\mathcal{E}$ of $\mathcal{J}$.

For $n = 0$: gauge PDE is a manifold equipped with an integrable distribution.

For future use, let us pick a (local) basis of vector fields R_α. Integrability:

$$[R_\alpha, R_\beta] = U^\gamma_{\alpha\beta}(\phi) R_\gamma$$

Ghost variables:

$$c^\alpha, \quad |c^\alpha| = 1\,, \qquad \mathrm{gh}(c^\alpha) = 1$$

Geometrically, c^α are coordinates on the fibers of the vector bundle of gauge parameters (with reversed parity).

If $V(\mathcal{E}) \subset T\mathcal{E}$ is the distribution, ghost extended equation: $\widehat{\mathcal{E}} = V[1](\mathcal{E})$. It is equipped with the BRST differential (homological vector field):

$$\gamma = c^\alpha R_\alpha - \frac{1}{2} c^\alpha c^\beta U^\gamma_{\alpha\beta} R_\gamma \,, \qquad \gamma^2 = 0 \,, \quad \mathrm{gh}(\gamma) = 0$$

Cohomology: $H(\gamma, \text{functions on } \mathcal{E})$. In particular, $H^0(\gamma) =$ invariant functions on $\mathcal{E}$.

Generalization: reducible gauge theories, $Z^\alpha_A(\phi) R_\alpha = 0$ (linear dependence of generators). New ghosts ξ^A:

$$\mathrm{gh}(\xi^A) = 2 \,, \qquad \gamma = c^\alpha R_\alpha + \xi^A Z^\alpha_A \frac{\partial}{\partial \xi^\alpha} + \ldots \,, \quad \gamma^2 = 0$$

Similarly, next level reducibility etc. In this setting it is somewhat superfluous.

Amounts to Q-manifolds ((generalized) Lie algebroids)

Batalin-Vilkovisky formalism:

If one works in terms of $\mathcal{J}$:

Equations E_μ, gauge symmetries R^i_α, reducibility relations,....

the BRST differential:

$$s = \delta + \gamma + \ldots\,, \qquad s^2 = 0\,, \quad \mathrm{gh}(s) = 1$$

$$\delta = E_\mu \frac{\partial}{\partial \mathcal{P}_\mu} + F^\mu_A \mathcal{P}_\mu \frac{\partial}{\partial \pi^A} \ldots\,, \qquad \gamma = c^\alpha R^i_\alpha \frac{\partial}{\partial \phi_i} + \ldots\,.$$

δ – (Koszule-Tate) restriction to the stationary surface

γ – implements gauge invariance condition

ϕ^i – fields, c^α – ghosts,

$\mathcal{P}_\mu$ – ghost momenta, π^A – reducibility ghost momenta

$$\mathrm{gh}(\phi^i) = 0\,, \quad \mathrm{gh}(c^\alpha) = 1\,, \quad \mathrm{gh}(\mathcal{P}_\mu) = -1\,, \quad \ldots$$

BRST differential completely defines the system.

Equations of motion and gauge symmetries can be read off from s:

$$s\mathcal{P}_\mu|_{\mathcal{P}=0,\, c=0,\dots} = 0\,, \qquad \delta_\epsilon \phi^i = (s\phi^i)|_{c^\alpha=\epsilon^\alpha,\, \mathcal{P}=0,\, \dots}$$

Two Q-manifold $(\widehat{\mathcal{J}}, s)$ and $(\widehat{\mathcal{E}}, \gamma)$ are equivalent provided δ is acyclic in nonnegative degree in $\mathcal{P}_\mu, \pi_A$.

In BRST formalism different realizations of the same gauge system arise as homotopy equivalent Q-manifolds.

If the theory is Lagrangian then:

$T_i = \frac{\delta S_0}{\delta \phi^i}$, reducibility relations $R^i_\alpha T_i = 0$ so that $Z^i_\alpha = R^i_\alpha$

Natural bracket structure (antibracket)

$$\left(\phi^i, \mathcal{P}_j\right) = \delta^i_j \qquad \left(c^\alpha, \mathcal{P}_\beta\right) = \delta^\alpha_\beta$$

BV master action

$$s = \left(\cdot, S_{BV}\right), \qquad S_{BV} = S_0 + \mathcal{P}_i R^i_\alpha c^\alpha + \dots$$

Master equation:

$$\left(S_{BV}, S_{BV}\right) = 0 \quad \Longleftrightarrow \quad s^2 = 0$$

Example: YM theory

Fields: A_μ, C (with values in the Lie algebra)

Antifields: $A^{*\mu}$, C^*

Gauge part BRST differential: $\gamma A_\mu = \partial_\mu C + [A_\mu, C]$

Master action:

$$S_{BV} = S_0 + \int d^n x \mathrm{Tr}[A^{*\mu}(\partial_\mu C + [A_\mu, C]) + \frac{1}{2} C^*[C, C]]$$

Extension to gauge theories

If $(\mathcal{E}, d_{\mathsf{h}})$ has gauge symmeries there are parameters ϵ^{α} which are arbitary space time functions. Promote them to ghost variables c^{α} and consider the extension $\bar{\mathcal{E}}$ of $\mathcal{E}$ by the jet-space for c^{α}:

$$C^I = \{c^{\alpha}, \quad c^{\alpha}_{a}, \quad c^{\alpha}_{ab}, \quad \ldots\}$$

The gauge symmetry is encoded in vector field γ satisfying

$$[d_{\mathsf{h}}, \gamma] = 0\,, \qquad \gamma^2 = 0\,, \qquad \mathrm{gh}(\gamma) = 1$$

It can be written as

$$\gamma = C^I R_I^A(\psi)\frac{\partial}{\partial \psi^A} - \frac{1}{2} C^I C^J U_{IJ}^K(\psi)\frac{\partial}{\partial C^K}$$

Vector fields R_I determine an integrable distribution on $\mathcal{E}$ (gauge-distribution), compatible (in the sense of $[d_{\mathsf{h}}, \gamma] = 0$) with Cartan distribution.

The above motivates the following (somewhat provisional) definition:

Gauge PDE is a graded manifold $\widehat{E}$ equipped with Cartan distribution and BRST differential s such that $[d_h, \gamma] = 0$.

It turns out that any local gauge system gives rise to $(\widehat{\mathcal{E}}, d_{\mathsf{h}}, s)$ (e.g. just by constructing BV formulation). Other way around, given $(\widehat{\mathcal{E}}, d_{\mathsf{h}}, s)$ one can systematically reconstruct certain explicit realization of this system.

Subtleties: - generic $(\widehat{\mathcal{E}}, d_{\mathsf{h}}, s)$ may give untractable theory with inifinite ammount of fields

- in contrast to usual PDE one and the same gauge PDE can be described by many equivalent $(\widehat{\mathcal{E}}, d_{\mathsf{h}}, s)$. Possible way out is to ask for “minimal” $(\widehat{\mathcal{E}}, d_{\mathsf{h}}, s)$

Linear Gauge PDE

Work in terms of $\widehat{\mathcal{J}}$ – jet-bundle extended by ghosts and antifields (needed for Koszule-Tate). $(\widehat{\mathcal{J}}, d_{\mathsf{h}}, s)$
$(s = \delta + \gamma + \ldots)$

$\psi^A = \{\phi^i, c^\alpha, \mathcal{P}_\mu, \ldots\}$. Linear s

$$s\psi^A = \Omega^A_B(x, \partial^T_a)\psi^B\,, \qquad s^2 = 0 \quad \rightarrow \quad \Omega^A_B \Omega^B_C = 0$$

Introduce a graded vector bundle $\mathcal{H}(X)$ over X underlying the space of fields, ghosts, etc. Sections:

$$\Phi = \phi^A(x) e_A\,, \quad \mathrm{deg}(e_A) = -\mathrm{gh}(\psi^A)$$

In contrast to ψ^A coefficients ϕ^A are always commuting!

“First quantized BRST complex”

Decompose $\mathcal{H} = \bigoplus \mathcal{H}_l$ and $\Phi = \sum \Phi^{(l)}$ accordingly.
Taking into account $\deg \Omega = 1$, $\Omega^2 = 0$ we have a complex of vector spaces:

$$\dots \xrightarrow{\Omega^{(-2)}} \Gamma(\mathcal{H}_{-1}) \xrightarrow{\Omega^{(-1)}} \Gamma(\mathcal{H}_0) \xrightarrow{\Omega^{(0)}} \Gamma(\mathcal{H}_1) \xrightarrow{\Omega^{(1)}} \dots$$

Equations of motion and gauge symmetries can be explicitly written as:

$$\Omega^{(0)}\Phi^{(0)} = 0\,, \qquad \delta\Phi^{(0)} = \Omega^{(1)}\Phi^{(1)}\,, \qquad \dots$$

Has a clear interpretation as a (formal) quantum mechanics of a constrained system.

AKSZ sigma models

M - supermanifold (target space) with coordinates Ψ^A:
Ghost degree – gh()
(odd)symplectic structure σ, gh(σ) $= n-1$ and hence
(odd)Poisson bracket $\{\cdot\,,\,\cdot\}$, gh($\{\cdot\,,\,\cdot\}$) $= -n+1$
"BRST potential" $S_M(\Psi)$, gh(S_M) $= n$, master equation
$\{S_M, S_M\} = 0$
(QP structure: $Q = \{\cdot\,, S_M\}$ and $P = \{\cdot\,,\,\cdot\}$)

$\mathcal{X}$ - supermanifold (source space)
Ghost degree gh( )
$\boldsymbol{d}$ – odd vector field, $\boldsymbol{d}^2 = 0$, gh($\boldsymbol{d}$) $= 1$
Tipically, $\mathcal{X} = T[1]X$, coordinates $x^\mu, \theta^\mu \equiv dx^\mu$, $\boldsymbol{d} = \theta^\mu \frac{\partial}{\partial x^\mu}$,
$\mu = 0, \ldots n-1$

$\Phi : \mathcal{X} \to M$. Fields $\Psi^A(x,\theta) \equiv \Phi^*(\Psi^A)$.

BV master action

$$S_{BV} = \int \left[(\Phi^*(\chi))(\boldsymbol{d}) + \Phi^*(S_M) \right] , \qquad \text{gh}(S_{BV}) = 0$$

χ is potential for $\sigma = d\chi$. In components:

$$S_{BV} = \int d^n x d^n \theta \left[\chi_A(\Psi(x,\theta)) \boldsymbol{d}\Psi^A(x,\theta) + S_M(\Psi(x,\theta)) \right]$$

BV antibracket

$$\big(F, G\big) = \int d^n x d^n \theta \left(\frac{\delta^R F}{\delta \Psi^A(x,\theta)} \sigma^{AB} \frac{\delta G}{\delta \Psi^B(x,\theta)} \right) , \qquad \text{gh}\big(, \big) = 1$$

$\sigma^{AB}(\Psi)$ – components of the Poisson bivector.

Master equation:

$$\big(S_{BV}, S_{BV}\big) = 0 ,$$

BRST differential:

$$s^{AKSZ}\Psi^A(x,\theta) = \boldsymbol{d}\Psi^A(x,\theta)+Q^A(\Psi(x,\theta))\,, \qquad Q^A = \left\{\Psi^A, S_M\right\}$$

Natural lift of Q and $\boldsymbol{d}$ to the space of maps.

Dynamical fields: those of vanishing ghost degree

$$\Psi^A(x,\theta) = \overset{0}{\Psi}{}^A(x) + \overset{1}{\Psi}{}^A_\mu(x)\theta^\mu + \ldots \qquad \text{gh}(\overset{k}{\Psi}{}^A_{\mu_1\ldots\mu_k}) = \text{gh}(\Psi^A) - k$$

If $\text{gh}(\Psi^A) = k$ with $k \geqslant 0$ then $\overset{k}{\Psi}{}^A_{\mu_1\ldots\mu_k}(x)$ is dynamical.

AKSZ equations of motion

$$\sigma_{AB}(\boldsymbol{d}\Psi^A + Q^A) = 0\,, \quad \Rightarrow \quad \boldsymbol{d}\Psi^A(x,\theta) + Q^A(\Psi(x,\theta)) = 0$$

(recall: σ_{AB} is invertible)

AKSZ at the level of equations of motion (nonlagrangian)

$$\{,\}\,,\, S_M \quad \Rightarrow \quad Q = Q^A \frac{\partial}{\partial \Psi^A} \quad Q^2 = 0\,.$$

I.e. target is a generic Q manifold.
target doesn't know dim X! (Recall gh$(S_M) = n =$ dim X)

If gh$(\Psi^A) \geqslant 0 \ \forall\ \Psi^A$ then BV-BRST extended FDA.
Otherwise BV-BRST extended FDA with constraints.

Examples:

Chern-Simons: *AKSZ, 1994*

Target space M:

$M = \mathfrak{g}[1]$, $\mathfrak{g}$ – Lie algebra with invariant inner product.

e_i –basis in $\mathfrak{g}$, C^i – coordinates on $\mathfrak{g}$, $\mathrm{gh}(C^i) = 1$, $C = C^i e_i$

$$S_M = \frac{1}{6}\langle C, [C, C]\rangle\,, \qquad \left\{C^i, C^j\right\} = \langle e_i, e_j\rangle^{-1}$$

Source space:

$\mathcal{X} = T[1]X$, X – 3-dim manifold. Fied content

$$C^i(x,\theta) = c^i(x) + \theta^\mu A^i_\mu(x) + \theta^\mu\theta^\nu A^{*i}_{\mu\nu} + (\theta)^3 c^{*i}$$

BV action

$$S_{BV} = \int (\frac{1}{2}\langle C, \boldsymbol{d}C\rangle + \frac{1}{6}\langle C, [C, C]\rangle) = \int \frac{1}{2}\langle A, \boldsymbol{d}A\rangle + \frac{1}{6}\langle A, [A, A]\rangle) + \ldots$$

1d AKSZ systems

Target space M – Extended phase space: $\{,\}$ – Poisson bracket, $S_M = \Omega - \theta H$, Ω – BRST charge, H - BRST invariant Hamiltonian

Source space $\mathcal{X} = T[1](\mathbb{R}^1)$, coordinates t, θ

BV action *M.G., Damgaard, 1999*

$$S_{BV} = \int dt d\theta (\chi_A \boldsymbol{d}\psi^A + \Omega - \theta H)$$

Integration over θ gives BV for the Hamiltoninan action *Fisch, Henneaux, 1989, Batalin, Fradkin 1988*.

Example: coordinates on M: $\widetilde{c}, \widetilde{\mathcal{P}}, \widetilde{x}^\mu, \widetilde{p}_\mu$, BRST charge $\Omega = \widetilde{c}(\widetilde{p}^2 - m^2)$,

$$S_{BV} = \int dt d\theta (\widetilde{p}_\mu \boldsymbol{d}\widetilde{x}^\mu + \widetilde{\mathcal{P}}\boldsymbol{d}\widetilde{c} + \widetilde{c}(p^2 - m^2)) = \int dt (p_\mu \dot{x}^\mu + \lambda(p^2 + m^2))$$

$$\widetilde{x}^\mu(t,\theta) = x^\mu(t) + \theta p^\mu_*(t)\,, \qquad \widetilde{p}_\mu(t,\theta) = p_\mu(t) + \theta x^*_\mu(t)\,,$$

$$\widetilde{c}(t,\theta) = c(t) + \theta\lambda(t)\,, \qquad \ldots$$

– Background-independent

– AKSZ is both Lagrangian and Hamiltonian

AKSZ model: $(M, S_M, \{,\})$ and $(\mathcal{X}, \boldsymbol{d})$.

Let $X = X_S \times \mathbb{R}^1$ *Barnich, M.G, 2003*

$$\Omega_{BFV} = \int_{X_S} \left[(\Phi^*(\chi))(\boldsymbol{d}) + \Phi^*(S_M) \right] , \qquad \mathrm{gh}(\Omega_{BFV}) = 1$$

$$\{\,\cdot\,,\,\cdot\,\}_{BFV} = \int d^{n-1}x d^{n-1}\theta \, \{\,\cdot\,,\,\cdot\,\} \qquad \{\Omega_{BFV}, \Omega_{BFV}\}_{BFV} = 0\,.$$

– Higher BRST charges *Cattaneo et. all. (2012)*

Similarly: $X_k \subset X$ – dimension-k submanifold

$$\Omega_{X_k} = \int_{X_k} (\Phi^*(\chi))(\boldsymbol{d}) + \Phi^*(S_M))$$

In particular, $\Omega_{BFV} = \Omega_{X_S}$, $S_{BV} = \Omega_X$

– At the level of equations of motion AKSZ is a generalization of so-called unfolded formalism independently developed in the context of HS theories *Vasiliev 1988,. . .* .

– At the level of equations of motion the same target space gives an AKSZ model for any $X_k \subset X$ or even different X. Useful for "replacing space-time". E.g. *Vasiliev 2002*

(asymptotic) boundary values, e.g. in the context of AdS/CFT

For higher-spin fields *Vasiliev, 2012; Bekaert M.G. 2012*

– Locally in X and M: *Barnich, M.G. 2009*

$$H^g(s^{AKSZ}, \text{local functionals}) \cong H^{g+n}(Q, C^\infty(M))$$

Isomorphism sends $f \in C^\infty(M)$ to functional $F = \int f$. Compatible with the bracket.

– If M finite dimensional and $n > 1$ – the model is topological. What about non-topological?

AKSZ form of PDE

Intrinsic (unfolded) realization

Given PDE $(\mathcal{E}, d_{\mathsf{h}})$ defined intrinsically one can always find a jet space $\mathcal{J}$ such that $(\mathcal{E}, Q)$ can be realized as a stationary surface of some $E_\alpha[u, x]$.

There is an intrinsic way to realize $(\mathcal{E}, d_{\mathsf{h}})$ explicitly. If x^a, ψ^A coordinates on $\mathcal{E}$ (e.g. $\psi^A = \{\phi, \phi_a, \phi_{ab}, \dots\}$) promote ψ^A to fields $\psi^A(x) =$ of a new theory and subject them to EOM's

$$\boldsymbol{d}(\psi^A(x)) = (d_{\mathsf{h}}\psi^A)(x) \quad \text{components: } \frac{\partial}{\partial x^a}\psi^A(x) = (\partial^T_a \psi^A)(x)$$

Proposition: *The original PDE* $(\mathcal{E}, d_{\mathsf{h}})$ *is equivalent to* $\boldsymbol{d}\psi^A = d_{\mathsf{h}}\psi^A$

Comments:

- Version of the unfolded formulation (though only zero forms). Unfolded form of gauge systems involves gauge form-fields. *Vasiliev, 1987,. . .*

- Generalized version of the Proposition involving gauge forms and BRST extension was formulated and proved using BRST technique and Koszule-Tate differential. *Barnich, M.G.,Semikhatov, Tipunin 2004, Barnich, M.G 2010*

New jet-space

Because $\mathcal{E}$ is a bundle over spacetime, take $\mathcal{J}^{new} \equiv J^{\infty}(\mathcal{E})$. More precisely, if x^a, dx^a, ψ^A are coordinates on $\mathcal{E}$ then

$$x^a, \quad dx^a, \quad \psi^A, \quad \psi^A_b, \quad \psi^A_{bc}, \quad \psi^A_{bcd}, \quad \ldots$$

are coordinates on $\mathcal{J}^{new}$.

New jet space is equipped with its own horizontal differential:

$$D_{\mathsf{h}} = dx^a\left(\frac{\partial}{\partial x^a} + \psi^A_a \frac{\partial}{\partial \psi^A} + \psi^A_{ab} \frac{\partial}{\partial \psi^A_b} + \ldots\right)$$

"Old" differential d_{h} on $\mathcal{E}$ extends to $\mathcal{J}^{new}$ by $[D_H, Q] = 0$. In the new jet space $\mathcal{J}^{new}$ consider the following PDE

$$D_{\mathsf{h}}\psi^A = d_{\mathsf{h}}\psi^A$$

In this form the new PDE is manifestly isomorphic to $(\mathcal{E}, Q)$ (because manifolds are isomorphic and horizontal differentials are equal by construction)

AKSZ form and reparametrization invariance

Consider dx^a as ghosts ξ^a, change notation $x^a \to z^a$ and extend $\mathcal{E}$ into a supermanifold with coordinates $\Psi^A = \{z^a, \xi^a, \phi^i, \phi^i_a, \phi^i_{ab}, \ldots\}$. It is a Q-manifold:

$$Q = -d_{\mathsf{h}} = -\xi^A \partial^T_a$$

Take $\mathcal{X} = T[1]X$ with coordinates x^μ, θ^μ and consider AKSZ model with source $(\mathcal{X}, \boldsymbol{d})$ and target $(\mathcal{E}, Q)$.

Note that now z^a is promoted to a field $z^a(x)$ and ξ^a to $e^a_\mu(x)dx^\mu$.

In fact: we are dealing with parametrized version.

$z^a(x)$ – space-time coordinates understood as fields

$e^a_\mu(x)$ – frame field components.

Gauge transf. for z^a: $\delta z^a = \xi^a$. Q is the BRST differential implementing repatametrization invariance.

Gauge condition $z^a = \delta^a_\mu x^\mu$ give un-parametrized version:

$$\boldsymbol{d}\Psi^A + Q^A(\Psi) = 0 \quad \Rightarrow \quad \boldsymbol{d}\Psi^A(x) - \theta^a(\partial^T_a \Psi^A)(x,\theta) = 0$$

Recall: ∂^T_a – total derivative (vector field in the target).

AKSZ form

Consider AKSZ model with source $(\mathcal{X}, \boldsymbol{d})$ and the target $(\bar{\mathcal{E}}, Q)$, where

$$Q = -d_{\mathsf{h}} + \gamma$$

Total differential familiar in the local BRST cohomology *Stora, 1983, Batnich, Brandt, Henneaux 1993,. . .* .

Equivalent to the parametrized version of gauge system. In addition to $e^a_\mu(x)dx^\mu$ new 1-form fields $A^I_\mu(x)dx^\mu$ associ-ated to C^I.

The equivalence was proved using *Barnich, M.G. 2010*

$$\widetilde{s} = -d_{\mathsf{h}} + \delta + \gamma + \dots$$

where δ is the Koszule–Tate differential of the stationary surface.

New feature: contractibe pairs for Q: if by local invertible change of coordinates:

$$Qw^a = v^a, \qquad Q\psi^\alpha = Q^\alpha(\psi)$$

then w^a, v^a are contractibe pairs. Their elimination results in the reduced Q-manifold $(Q, \widetilde{\mathcal{E}})$. Eliminating all such trivial pairs one arrives at "minimal" Q-manifold associated the gauge system *Brandt, 1996*

The manifold of generalized connections and tensor fields.

For the AKSZ model trivial pairs give rise to generalized auxiliary fields. Lagrangian: *Dresse, Grégoire, Henneaux, 1990*
EOM: *Barnich, M.G., Semikhatov, Tipunin, 2004*
Their elimination is an equivalence of the respective AKSZ models.

Example of Einstein gravity

For diffeomorphism-invariant theory parameterization brings nothing. It follows x^a, ξ^a can be eliminated together with d_{h}, giving $Q = \gamma$.

After elimination the contractible pairs of Q manifold $\widetilde{\mathcal{E}}$:

$$e^a, \quad \omega^{ab}, \quad W^{cd}_{ab}, \quad W^{cd}_{ab|e}, \quad W^{cd}_{ab|e\ldots}$$

– ghosts associated to frame field and spin connection and Weyl tensor and its independent covariant derivatives.

$$Qe^a = \omega^a{}_c\, e^c\,, \qquad Q\omega^{ab} = \omega^a{}_c\, \omega^{cb} + \frac{1}{2} e^c e^d W^{ab}_{cd}\,,$$

$$QW = eW + \omega W + \ldots$$

Minimal BRST complex (Q-manifold) for gravity.
Gives minimal AKSZ formulation (unfolded formulation).