Knot theory : From Chern-Simons to Goodwillie-Weiss

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Srni, Jan. 13-20, 2018

Lecture 1 :

Knots and their Chern-Simons invariants

Knot table:

Ordered by crossing #

0 : unknot

1 : ?

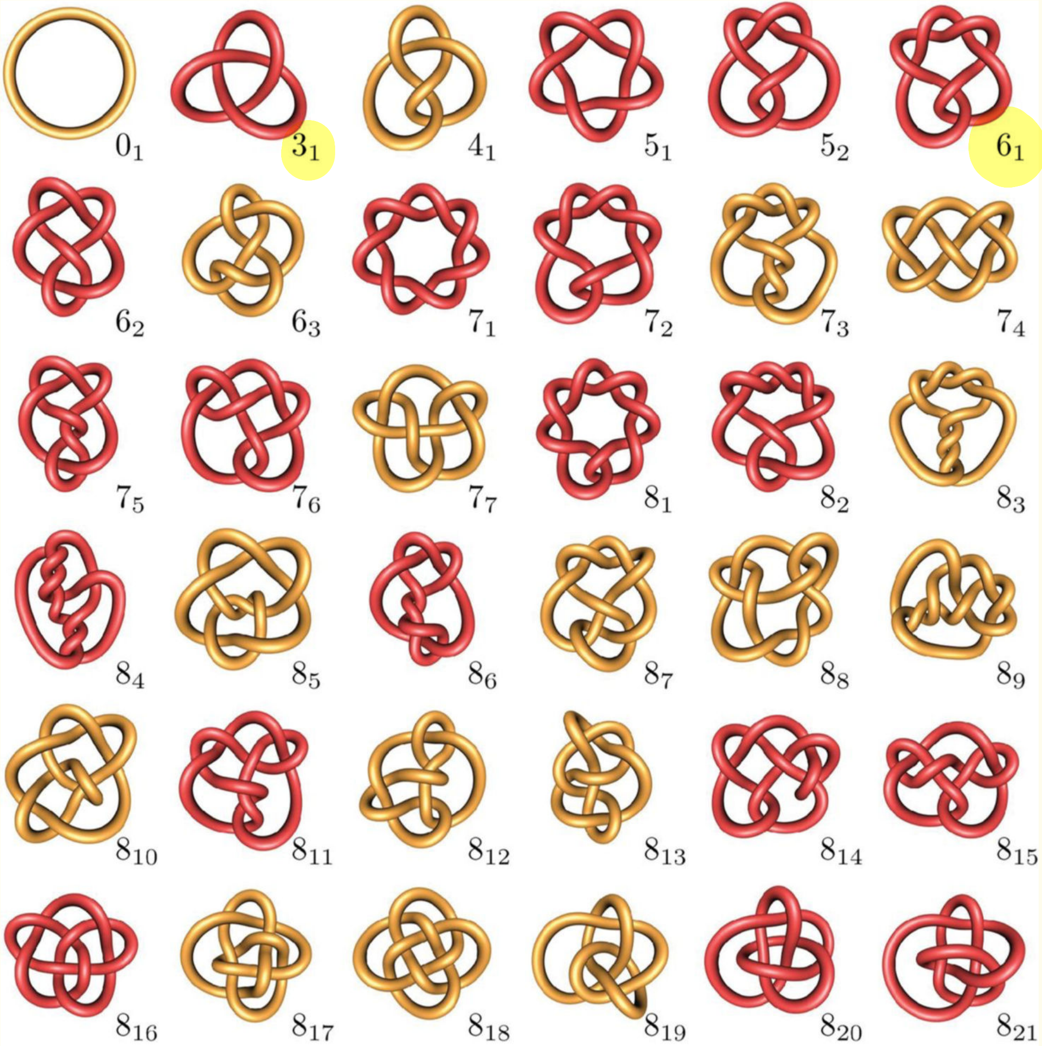
2 :

3 : trefoil

4 : figure 8

⋮

picture better then name !



Knots

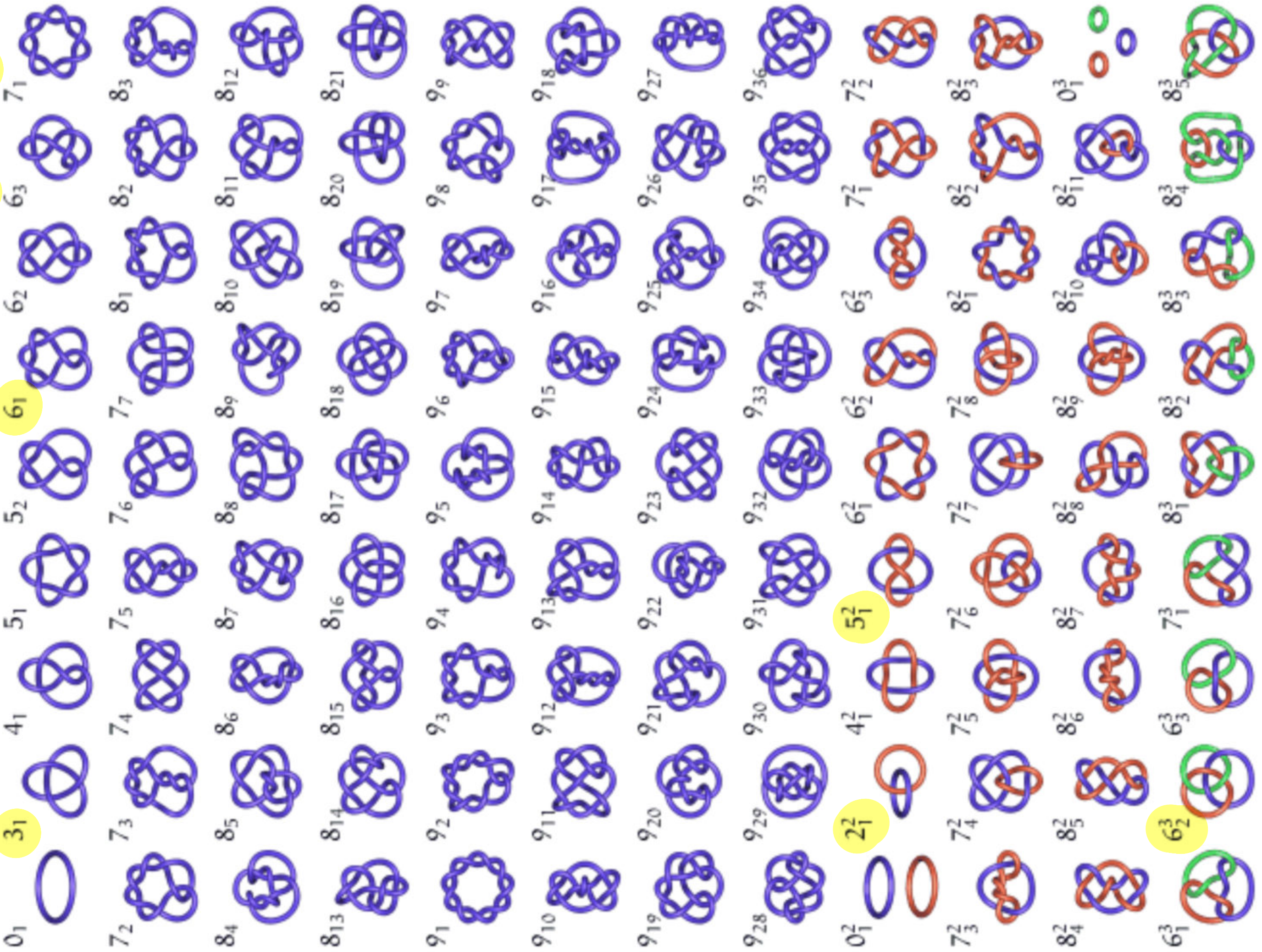
&

Links:

Hopf

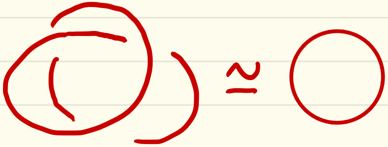
Wh

Bor



Mathematical definition

A knot is a smooth embedding $S^1 \xhookrightarrow{K} \mathbb{R}^3$



There is a space of knots $\mathcal{K}$.

Knot tables list elements in

$\pi_0(\mathcal{K})$ = isotopy classes of knots.

A knot invariant is a math. object associated to a knot that does not change under isotopy.

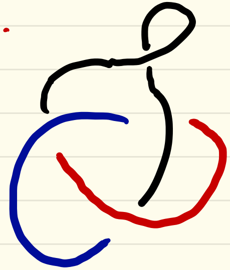
- knot complement $\mathbb{R}^3 \setminus K(S^1)$, up to homeom.,
- knot group $:= \pi_1(\mathbb{R}^3 \setminus K)$. It comes with a Wirtinger presentation for each diagram for K

Theorem :

[Gordon-Lücke]: Knots are classified by their group!

[Haken-Hemion]: Knot groups are classifiable.

$$\left\{ \begin{array}{l} \text{3-colorings of a} \\ \text{diagram for } K \end{array} \right\} \longleftrightarrow \mathrm{Hom}(\pi_1(\mathbb{R}^3 \setminus K), S_3)$$



To get numerical invariants, we can fix G and let

$$\mathbb{N} \ni I_G(K) := \#\, \mathrm{Hom}(\pi_1(\mathbb{R}^3 \setminus K), G) \,/\, \text{conjugation by } G$$

$$= \#\, \text{flat } G\text{-connections on } \mathbb{R}^3 \setminus K \,/\, \text{isomorphism}$$

$$= \#\, \text{classical solutions of Chern-Simons functional} \,/\, \text{Gauge equiv}$$

Quantization

More interesting are the quantum CS-invariants, where we perform a Feynman integral over all connections: $CS_{G,V}(K) =$ "$\int_{\mathcal{A}/\mathcal{G}} Tr_V(hol_K(A))\, e^{\frac{iCS(A)}{\hbar}} dA$"

$V \in Rep(G)$

Physicists [Axelrod-Singer, Altschuler-Freidel, ...] know how to make this integral precise by a perturbative expansion (around the trivial connection).

Math. contributions by [Bott-Taubes, D. Thurston, Poirier, ...]

Configuration space integrals

$$CS_{G,V}(K) = \sum_{n\geq 0} \hbar^n \cdot \left(\sum_{\substack{\text{Feynman} \\ \text{diagrams } \Gamma \\ \text{of degree } n}} |\mathrm{Aut}\,\Gamma|^{-1} \cdot I_\Gamma(K) \cdot W_\Gamma(G,V) \right)$$

$\rotatebox{90}{$\in$}$

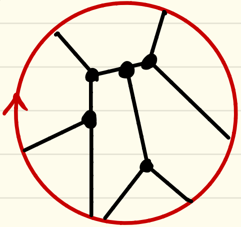
$\mathbb{C}[[\hbar]]$

analytic part

algebraic part

e.g.

$n=6$



$$\Gamma = (\underbrace{U \sqcup T}_{\text{vertices}},\ U \subseteq S^1,\ \underbrace{E}_{\text{edges}}) \quad \text{up to isom.}$$

$$n := \tfrac{1}{2}\cdot v(\Gamma) = -\chi(\Gamma \cup S^1)$$

$$I_\Gamma(K) := \int_{C_\Gamma(K)} \Psi^*(\Omega)$$

$\rotatebox{90}{$\in$}$

$\mathbb{Q}$

integral over configuration space

$$C_\Gamma(K) := \left\{ c : v(\Gamma) \hookrightarrow \mathbb{R}^3 \quad c/_U \cong K(U) \subseteq K(S^1) \right\}$$

$$\begin{tikzcd} C_\Gamma(K) \arrow[d, "\Psi"'] \\ \times_{e\in E} S^2_e \end{tikzcd}$$

$$\Psi(c) := \prod_{e\in E} \overrightarrow{c(\partial_0 e)\, c(\partial_1 e)}$$

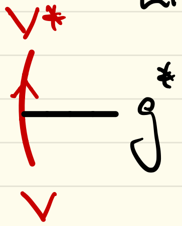
dimension $= \#U + 3\cdot\#T = 2\cdot\#E$

$$CS_{G,V}(K)_n = \sum_{\substack{\text{Feynman} \\ \text{diagrams } \Gamma \text{ of deg } n}} |\mathrm{Aut}\,\Gamma|^{-1} \cdot I_\Gamma(K) \cdot W_\Gamma(G,V] \in \mathbb{C}$$

$W_\Gamma(G,V)$ is the usual Feynman weight, given by contracting the following tensors:

$$\mathrm{Lie}(G) =: \mathfrak{g} \xrightarrow{\rho} \mathrm{End}(V)$$

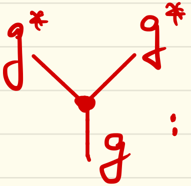
$$\Updownarrow$$



: $\mathfrak{g}^* \otimes V \otimes V^*$

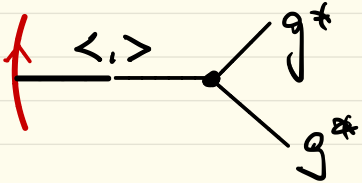
$$\mathfrak{g} \otimes \mathfrak{g} \xrightarrow{[\,,\,]} \mathfrak{g}$$

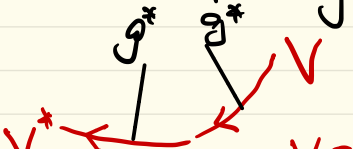
$$\Updownarrow$$



: $\mathfrak{g}^* \otimes \mathfrak{g}^* \otimes \mathfrak{g}$

Uses also a non-deg. $\mathfrak{g}$-invariant pairing on $\mathfrak{g}$ "level":





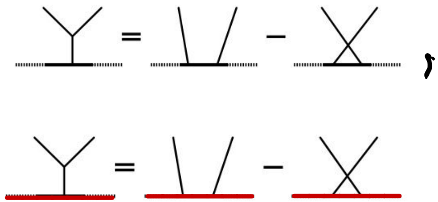
$(,) : V \otimes V^* \longrightarrow \mathbb{C}$

Miracle 1 : $CS_{G,V}(K)_n$ is an isotopy invariant modulo the "anomaly"

Miracle 2 : $Z_n(K) := \sum_{\Gamma \text{ of deg } n} |\mathrm{Aut}\,\Gamma|^{-1} \cdot I_\Gamma(K) \cdot [\Gamma]$

is also an isotopy invariant when considered in

$\mathcal{A}_n^{\mathbb{Q}} :=$ $\mathbb{Q}$- linear combinations of Feynman diagrams Γ of degree n / Jacobi - relations



$[a,b] = a\cdot b - b\cdot a$

Goal for next 2 lectures:

Explain a geometric meaning of this version of the **"Kontsevich integral"** $Z(K)$

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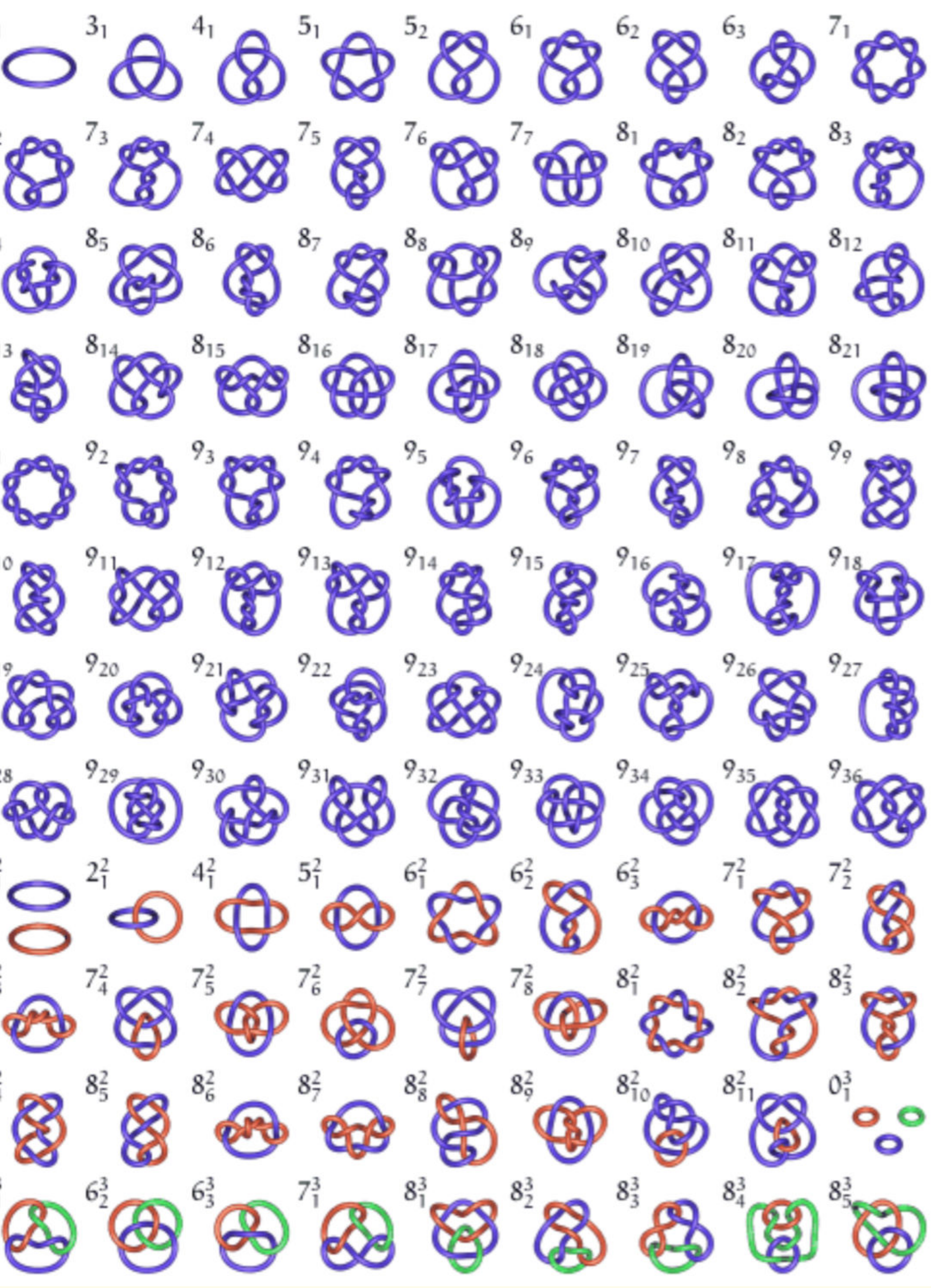
Lecture 2 :
The grope filtration on the space of knots


3_1
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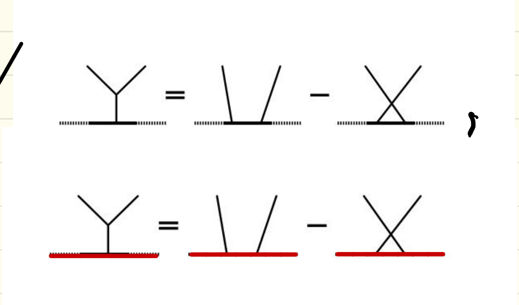
8^2_1
8^2_2
8^2_3
8^2_5
8^2_6
8^2_7
8^2_8
8^2_9
8^2_{10}
8^2_{11}
0^3_1
6^3_2
6^3_3
7^3_1
8^3_1
8^3_2
8^3_3
8^3_4
8^3_5

$\pi_0(\mathcal{K})$ = Isotopy classes of knots ≅ previous table

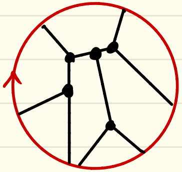
$$\begin{tikzcd} \pi_0(\mathcal{K}) \arrow[d, "Z_n"', "\mathbb{Q}"] \\ \mathcal{A}_n \arrow[d, "W(G,V)"] \\ \mathbb{Q} \end{tikzcd}$$

$\mathcal{A}_n := \mathbb{Q}$- linear combinations of Feynman diagrams Γ of degree n /

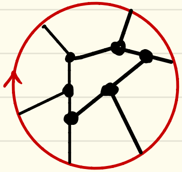


$W(G,V)$ if $\mathfrak{g}, V / \mathbb{Q}$.

$$\mathbb{Q} \ni \mathrm{CS}_{G,V}(K)_6 \longleftarrow Z_6(K) = \frac{3}{4} \cdot$$



$$- \frac{2}{5} \cdot$$



$$\in \mathcal{A}_6^{\mathbb{Q}}$$

Question 1: What do these invariants measure geometrically?

Question 2: How non-trivial are these CS-invariants?

Question 3: Are there integer versions in $\mathcal{A}_n^{\mathbb{Z}} := \mathbb{Z}$-linear combinations....?

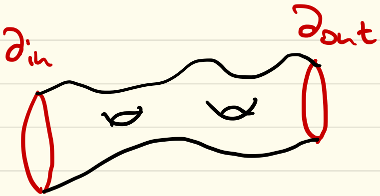
Q3 is open, one approach in Lecture 3. Today Q1, Q2.

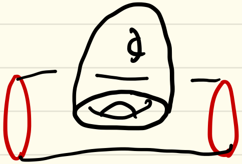
Thm.: [Conant-T., Bar-Natan-Garoufalidis-D. Thurston]

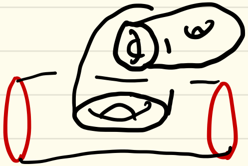
If K and K' cobound an embedded capped grope of degree n in S^3 then

$$Z_{<n}(K) = Z_{<n}(K')$$

Abstract gropes are the following types of 2-complexes:

$Y=$ 



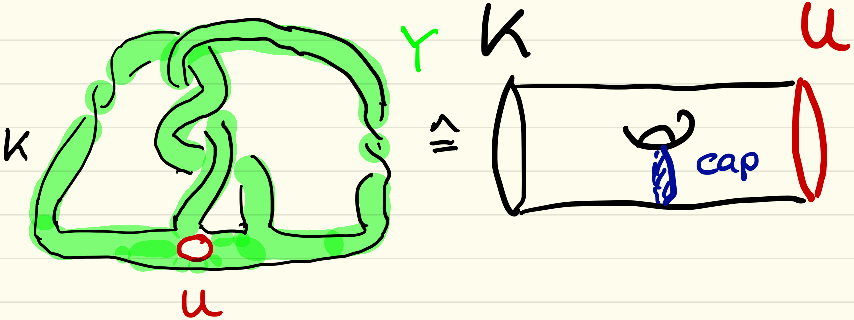


degree = 2 deg = 3 deg = 4 ......

If $Y \to X$, it measures $[\partial_{in} Y]^{-1} \cdot [\partial_{out} Y] \in \pi_1(X) / \pi_1(X)_{(n)}$ where the lower central series is $[G, G_{(n-1)}] =: G_{(n)}$.

Def.: A (capped) grope cobordism between $K, K': S^1 \hookrightarrow \mathbb{R}^3$ is a (capped) embedded grope $Y \hookrightarrow \mathbb{R}^3$ with $\partial Y = K \sqcup K'$. Write $K \underset{n}{\sim} K'$.

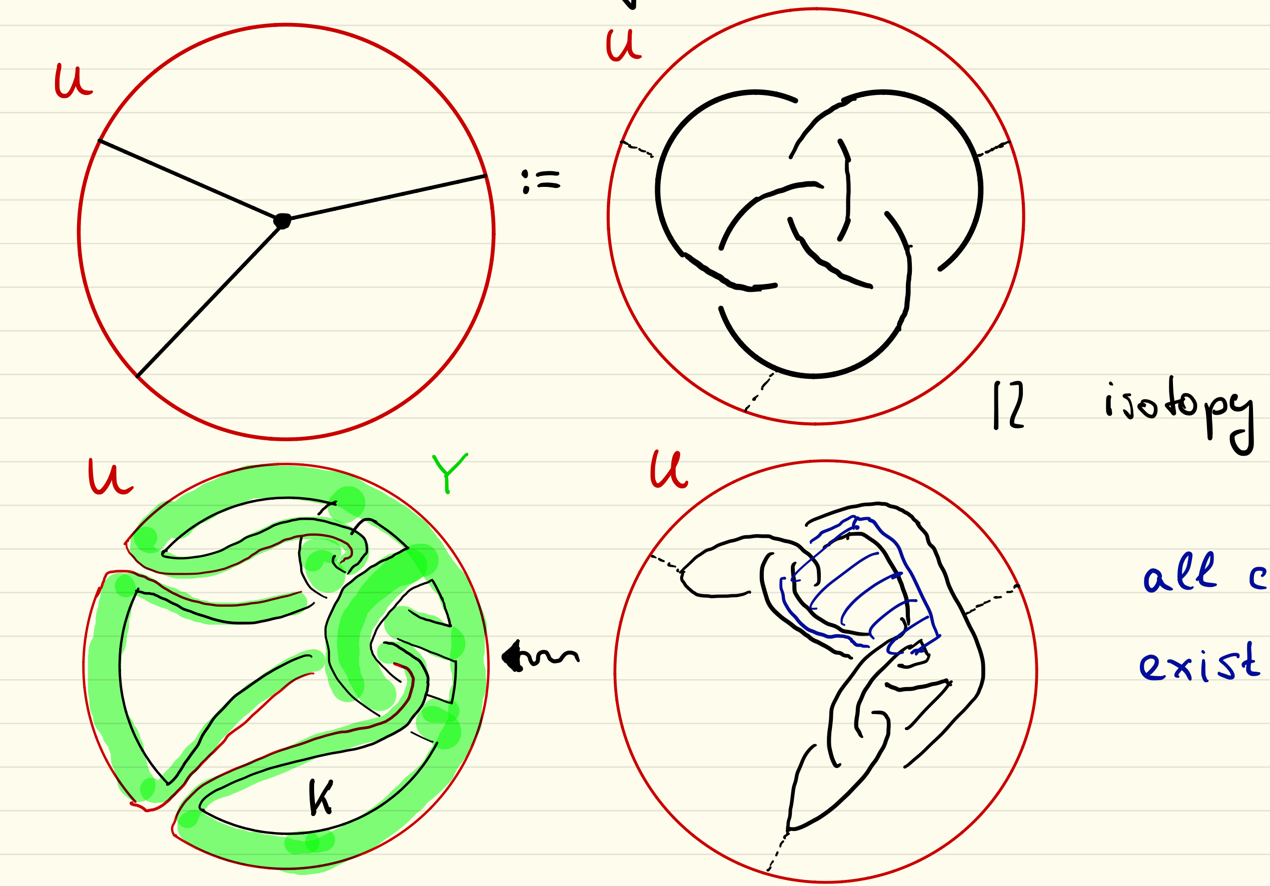
Example: A Seifert surface for K gives a deg 2 grope cobordism from K to the unknot U.



Note that a cap for a band in Y exists $\Longleftrightarrow$ the band is unknotted and untwisted.

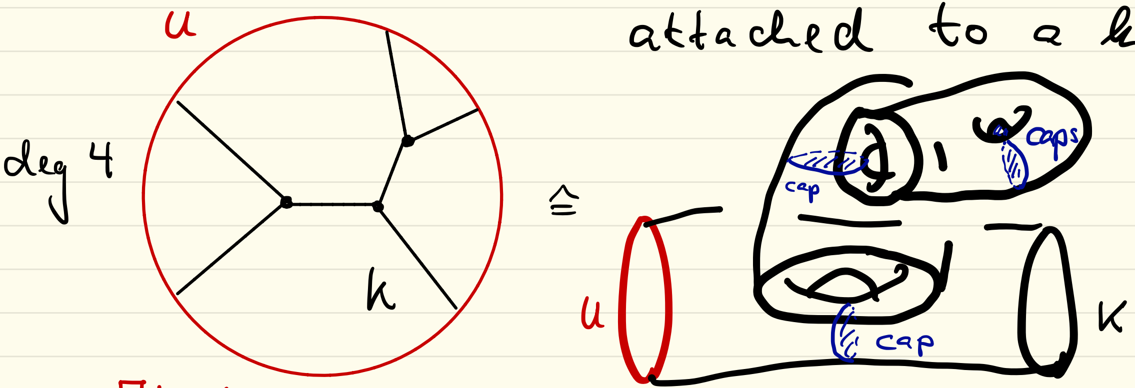
Question: How to construct higher degree grope cobordisms?

A shortcut : Capped gropes in $\mathbb{R}^3$ from trees



all caps exist ¡

⟹ Capped gropes of higher degree are obtained by induction from trees attached to a knot:

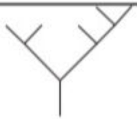
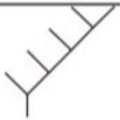
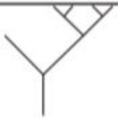


Thm.: $\mathcal{K}/\underset{n}{\sim}$ are finitely generated abelian groups (under #).
[Conant-T.]

Def.: The grope filtration is

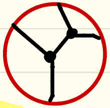
$$\dots \subseteq G_{n+1} \subseteq G_n := \{K \in \mathcal{K} \mid K \underset{n}{\sim} U\} \subseteq \dots \subseteq \mathcal{K}$$

Open problem: $\bigcap_n G_n = \{U\}$ "do Vassiliev inv. detect knots?"

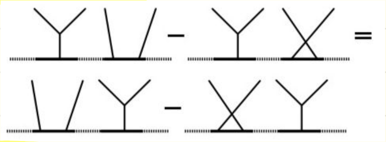
grope degree	type of grope	$\mathcal{K}$/capped gropes	$\mathcal{K}$/grope cobordism
2		$\{0\}$	$\{0\}$
3		$\mathbb{Z}(c_2)$	$\mathbb{Z}/2(\mathrm{Arf})$
4		$\mathbb{Z}(c_3) \oplus \mathbb{Z}(c_2)$	$\mathbb{Z}(c_2)$
4		$\mathbb{Z}(c_3) \oplus \mathbb{Z}(c_2)$	$\mathbb{Z}(c_2)$
5		$\mathbb{Z}(c_4) \oplus \mathbb{Z}(c_4') \oplus \mathbb{Z}(c_3) \oplus \mathbb{Z}(c_2)$	$\mathbb{Z}/2(c_3) \oplus \mathbb{Z}(c_2)$
5		$\mathbb{Z}(c_4) \oplus \mathbb{Z}(c_4') \oplus \mathbb{Z}(c_3) \oplus \mathbb{Z}(c_2)$	$\mathbb{Z}/2(c_3) \oplus \mathbb{Z}(c_2)$
5		?	S-equivalence

$\mathcal{A} := \bigoplus_{n\geq 0} \mathcal{A}_n^{\mathbb{Z}} \xrightarrow{\varepsilon} \mathcal{A}_0 = \mathbb{Z}$ is an augmented commutative ring

Indecomposables are $\mathcal{A}^I := \mathcal{A}/I^2$, where $I := \mathrm{Ker}(\varepsilon)$.

[Conant]: $\mathcal{A}_2^I \cong \mathbb{Z}\cdot$  $\mathcal{A}_3^I \cong \mathbb{Z}\cdot$ 

and in general, $\mathcal{A}_n^I$ is generated by trees and the only relations are IHX, STU^2:



Thm.: $\exists\ R_n : \mathcal{A}_n^I \to \mathbb{G}_n := G_n/\sim_{n+1}$

[Conant-T, Habiro]

using our trees to gropes correspondence.

Thm.: $Z_{\leq n}\big(R_n(\Gamma)\big) = \Gamma \in \mathcal{A}^I_{\leq n}$

[BN-G-DT]

and hence $\mathcal{A}_n^I \otimes \mathbb{Q} \cong \mathbb{G}_n \otimes \mathbb{Q}$!

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Lecture 3 :
Feynman diagrams in Goodwillie-Weiss towers

Chern-Simons invariants

Witten, 1989

Vassiliev, 1990

Bar-Natan, thesis 1991

Axelrod-Singer, 1991

Altschuler-Freidel, 1992

Kontsevich, 1992

Bott-Taubes, 1994

Goussarov, thesis 1994

Habiro, thesis 1999

Conant-T., 2002, Poirer

Goodwillie-Weiss calculus

Goodwillie, 1989

Weiss, 1996

Goodwillie-Weiss, 1999

Sinha, thesis 2004

Volic, thesis 2006

Conant, 2008

Boavida-Weiss, 2013

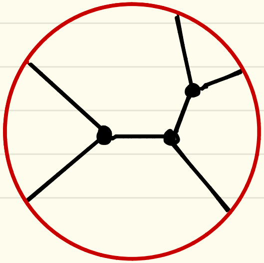
Munson-Volic, book 2015

Budney-Conant-Koytcheff-Sinha, 2017

Chern-Simons

$$\begin{tikzcd}
\mathcal{K} \arrow[r, "Z"] \arrow[d] & \hat{\mathcal{A}}^I \otimes \mathbb{Q} \arrow[d, two heads, "\pi_n"] \\
\mathcal{K}/\sim_{n+1} \arrow[r, "Z_{\le n}"] & \mathcal{A}^I_{\le n} \otimes \mathbb{Q} \\
\mathcal{G}_n \arrow[u, phantom, "\cup"] \arrow[r, "Z_n"] \arrow[dr, red, "?"] & \mathcal{A}^I_n \otimes \mathbb{Q} \arrow[u, phantom, "\cup"] \\
& \mathcal{A}^I_n \arrow[u] \arrow[ul, two heads, "R_n"]
\end{tikzcd}$$

$\widehat{=}$



$\in \mathcal{A}^I_n \cong E^2_{-n-1,n+1}$

Goodwillie - Weiss

$$\begin{tikzcd}
\mathcal{K} \arrow[r, "ev_\infty"] \arrow[d, two heads] & \lim(\cdots \to T_{n+1} \to T_n \to \cdots) \arrow[l, bend left, "h"] \arrow[d, two heads, "\pi_{n+1}"] \\
\mathcal{K}/\sim_{n+1} \arrow[r, two heads, "ev_n"] & \pi_0(T_{n+1}) \\
\mathcal{G}_n \arrow[u, phantom, "\cup"] \arrow[r, two heads] \arrow[d, red, "\nexists\, Z_n\ ?"] & \mathrm{Ker}(\pi_0 T_{n+1} \to \pi_0 T_n) \arrow[u, phantom, "\cup"] \\
& E^\infty_{-n-1,n+1} \arrow[u, phantom, "\cong"] \\
\mathcal{A}^I_n \arrow[r, phantom, "\cong"] & E^2_{-n-1,n+1} \arrow[u, two heads, "\text{mod im } d_r,\ r \ge 2"']
\end{tikzcd}$$

Goodwillie-Weiss calculus

Fix mfd. M, N and an embedding $\partial M \hookrightarrow \partial N$

$F : \mathrm{Open}(M) \to \mathrm{Top}$, $U \mapsto \mathrm{Emb}_\partial(U, N)$ has

Taylor approximation

$$\begin{tikzcd} \cdots \arrow[r] & T_k F \arrow[r] & T_{k-1} F \arrow[r] & \cdots \arrow[r] & T_1 F \arrow[r] \arrow[d, equal] & T_0 F \arrow[d, equal] \\ & & F \arrow[ul] \arrow[u] \arrow[rr] & & \mathrm{Imm}(-,N) & * \end{tikzcd}$$

This is the **Goodwillie-Weiss tower** of F

It converges if $\dim M \le \dim N - 3$ but $\forall \dim, k$

$\pi_0 F(M) \to \pi_0 T_k F(M)$ are **isotopy invariants**.

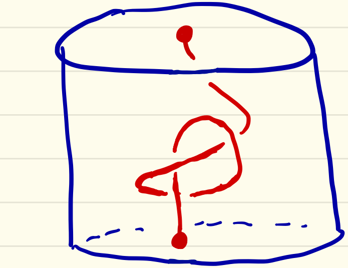
Our main example is $M = \mathbb{D}^m$, $N = \mathbb{D}^n$, unknot on ∂.

For $m=1$ on has **cosimplicial models [Sinha]**:

$$\mathrm{Emb}_\partial(\mathbb{D}^1, N) \xrightarrow{C_k} \mathrm{Strat}(\Delta^k, C_k(N)) \simeq T_k \mathrm{Emb}_\partial(\mathbb{D}^1, N)$$

where $\Delta^k \cong \{-1 \le t_0 \le t_1 \le \dots \le t_k \le 1\} \subseteq C_k(\mathbb{D}^1)$

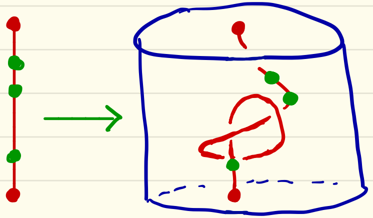
Goodwillie-Weiss tower:



for the space of "long knots"

$$\begin{tikzcd}
 & & \cdots \arrow[d] \\
 & & T_{n+1} := T_{n+1}(\mathcal{K}) \arrow[d] \\
\mathcal{K} \arrow[uurr] \arrow[urr] \arrow[rr, "ev_n"] \arrow[d, equal] & & T_n \arrow[d, phantom, sloped, "\simeq"] \\
\mathrm{Emb}_\partial(D^1, D^3) \arrow[rr, "C_n"] & & \mathrm{Strat}(\Delta^n, C_n(D^3))
\end{tikzcd}$$

Idea: Approximate an interval by a finite number n of points! $n=3$:



$\pi_0(ev_n): \pi_0(\mathcal{K}) \longrightarrow \pi_0(T_n)$ are knot invariants.

$$\begin{tikzcd}
\pi_0(\mathcal{K}) \arrow[r] \arrow[dr] & \pi_0(T_n) \\
 & K/\sim_n \arrow[u, "\widetilde{ev}_n"']
\end{tikzcd}$$

Thm.: They factor through $K/\sim_n$, i.e. through degree n grope cobordism.

[Budney-Conant-Koytcheff-Sinha 2017]

Refinement :

[Kosanovic-Shi-T.]

$$\begin{tikzcd}
\left\{\begin{array}{l}\text{grope cobordisms} \\ \text{from } U \text{ to } K \text{ of} \\ \text{degree } n\end{array}\right\} \arrow[r, "\exists"] \arrow[d, "\partial_1"] & F_n := \text{fibre}(T_{n+1} \to T_n) \arrow[d] \\
\mathcal{K} \arrow[r, "ev_{n+1}"] \arrow[d] & T_{n+1} \arrow[d] \\
\mathcal{K}/\sim_n \arrow[r, two heads, "\widetilde{ev}_n"] & \pi_0 T_n
\end{tikzcd}$$

Goodwillie-Weiss

$\Downarrow$

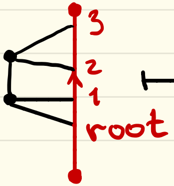
$$F_n \simeq \Omega^{n+1}\left(\begin{array}{c}\text{total fibre} \\ \text{of } (n+1)\text{-cube} \\ S \mapsto C_S(D^3)\end{array}\right) \qquad \bigvee^{k} S^2 \to C_{k+1}(D^3) \to C_k(D^3)$$

Hilton

$$\Longrightarrow \pi_0 F_n \cong \pi_{n+1}(\downarrow) \xleftarrow[\cong]{\text{Whitehead products}} \mathcal{L}_n \cong \mathbb{Z}^{(n-1)!} \text{ where}$$

$$\mathcal{L}_n := \frac{\langle \text{deg } n \text{ trees as before} \rangle}{\text{Jacobi relations}} \subseteq \begin{array}{l}\text{Free Lie alg.} \\ \text{on } x_1, \dots, x_n\end{array}$$

Spanned by basic words, containing all n letters.



$\longmapsto [x_1, [x_2, x_3]]$

The knot invariant $\widetilde{ev}_n : \mathcal{K}/\sim_n \twoheadrightarrow \pi_0(T_n)$ takes values in an unknown abelian group!

Thm. The spectral sequence for GW-tower $T_n \to T_{n-1} \to \dots \to T_0$ converges to $\pi_i(T_n)$

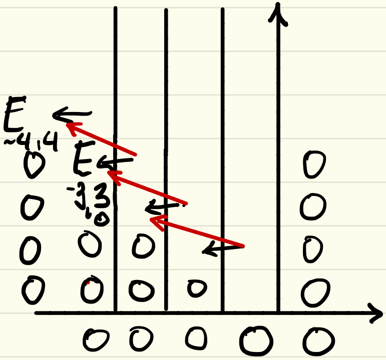
[Conant, Scannell-Sinha]

with $E^1_{-n,n} \cong \mathcal{L}_{n-1} \cong \mathbb{Z}^{(n-2)!}$

and

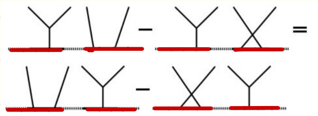
$$\begin{tikzcd}
E^2_{-n,n} = E^1_{-n,n}/\mathrm{im}\, d_1 \arrow[d] & \cong \mathcal{L}_{n-1}/STU^2 \cong \mathcal{A}^I_{n-1} \arrow[ddd, two heads, "\cong"] \\
E^3_{-n,n} = E^2_{-n,n}/\mathrm{im}\, d_2 \arrow[d] & \\
\vdots \arrow[d] & \\
E^{n+1}_{-n,n} = E^\infty_{-n,n} & = \mathrm{Ker}\left(\pi_0(T_n) \to \pi_0(T_{n-1})\right)
\end{tikzcd}$$

Main Conjecture:



Summary :

$$\begin{tikzcd}
 & \pi_0 \mathcal{K} \arrow[d] \arrow[drr, "ev_n"] & & & \\
\text{capped grope cobordism} & \mathcal{K}/\sim_n \arrow[rr, "\widetilde{ev}_n"] & & \pi_0 T_n & \text{iso. by Main Conj.} \\
 & \mathbb{G}_{n-1} = G_{n-1}/\sim_n \arrow[u, phantom, sloped, "\subseteq"] \arrow[rr, "(*)"'] & & E^\infty_{-n,n} \arrow[u, phantom, sloped, "\subseteq"] & \cong \operatorname{Ker}(\pi_0 T_n \to \pi_0 T_{n-1}) \\
\text{rational isom.} & A_{n-1} \arrow[u, two heads, "R_{n-1}", "I"'] \arrow[rr, phantom, "\cong"] & & E^2_{-n,n} \arrow[u, two heads, "\text{mod im } d_r,\ r \geq 2"'] & \\
\text{mod STU}^2: & \mathcal{L}_{n-1} \arrow[u, two heads] \arrow[rr, phantom, "\cong"] & & E^1_{-n,n} \arrow[u, two heads, "\text{mod im } d_1"'] & \cong \pi_0 F_{n-1} \arrow[uu, two heads]
\end{tikzcd}$$



Conj. [Danica, Peter, Yuqing,]: The central square commutes, in part. $\widetilde{ev}_n$ are all onto. and CS & GW are compatible!

True for $n \leq 3$.