SRNI Winter School

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SRNÍ WINTER SCHOOL
2018

SINGULAR VOLUME PROBLEMS
& CONTACT QUANTUM MECHANICS

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w/- Rod GOVER & Gabe HERCZEG

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CLASSICAL MECHANICS

Phase space

q^a generalized positions $\in \mathbb{R}^n$

p_a generalized velocities $\in \mathbb{R}^n$

Time

$t \in \mathbb{R}$

Symplectic Geometry

$\mathbb{R}^{2n} \ni (q^a, p_a)$ coordinates for S

Symplectic manifold (S, ω)

$$\omega \in \Omega^2 S \;, \quad d\omega = 0 \;, \quad \omega^{-1} \text{ exists}$$

Dynamics = flows of Hamiltonian vector field

$$\frac{\partial}{\partial t} = \omega^{-1}(dH, d\cdot) = \{H, \cdot\}_{PB}$$

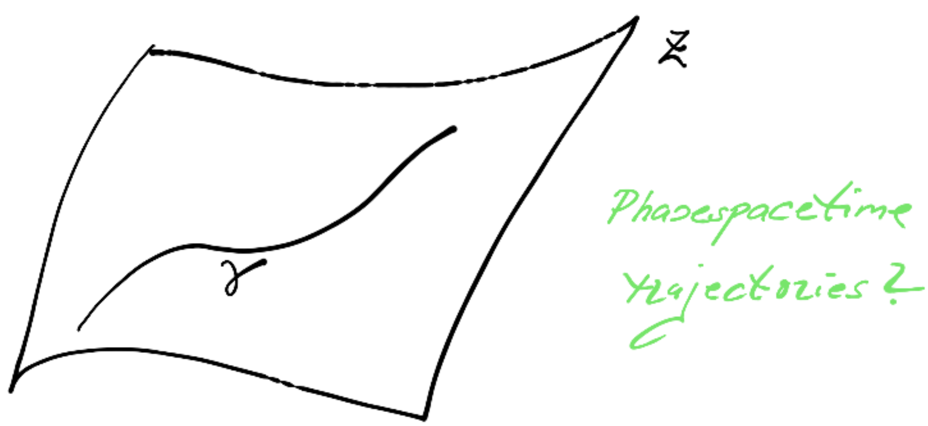
GENERAL COVARIANCE ???

"PHASE SPACE TIME"

Unify phase space & time

$$(q^a, p_a, t) \in Z$$

Dynamics ?



PRINCIPLE OF LEAST ACTION

L5

$$S[\gamma;\alpha] = \int_\gamma \alpha$$

$\alpha \in \Omega^1 Z$ can be integrated over a curve γ.

Examples:

Hamiltonian dynamics

$$Z = \{\vec{q}, \vec{p}, t\}, \quad \alpha = \vec{p}\cdot d\vec{q} - H(\vec{p}, \vec{q}, t)\,dt$$

Relativistic particle/geodesics

$$Z = T_1^* M \ni \{x^\mu, \hat{p}_\mu\}, \quad \alpha = \hat{p}_\mu\, dx^\mu$$

VARIATIONS

L6

$$\delta S = \frac{dS[\gamma_t]}{dt}\Big|_{t=0} = \int \varphi(\dot{\gamma}, \delta\gamma)$$

$$\varphi := d\alpha$$

Equation of motion

$$\varphi(\dot{\gamma}, \cdot) = 0 \quad \Leftrightarrow \quad (\partial_i \alpha_j - \partial_j \alpha_i)\, \dot{z}^i = 0$$

"Levy form"

Tangent vector to trajectory

Example:

$$\alpha = \vartheta - H(\vec{p},\vec{q},t)\,dt\,, \qquad \vartheta = \vec{p}\cdot d\vec{q}$$ symplectic current/potential

$$\Rightarrow \varphi = \vec{E}\wedge\vec{F} \qquad \begin{cases} \vec{E} = d\vec{p} - \dfrac{\partial H}{\partial \vec{q}}\,dt \\ \vec{F} = d\vec{q} + \dfrac{\partial H}{\partial \vec{p}}\,dt \end{cases}$$

tangent vector

$$\dot{\gamma} \propto \frac{\partial}{\partial t} + \frac{\partial H}{\partial \vec{q}}\frac{\partial}{\partial \vec{p}} - \frac{\partial H}{\partial \vec{p}}\frac{\partial}{\partial \vec{q}}$$

$$\mathcal{L}_{\dot{\gamma}} \longrightarrow \text{Hamiltonian dynamics}$$

(Strict) Contact Manifold Z,

Levy form $(2n+1)\times(2n+1)$ antisymmetric maximal rank matrix

avoid degenerate directions

When $Vol_\alpha := \alpha\wedge\varphi^{\wedge 2n} \neq 0$

maximal non-integrability

say α is a contact form

$$\Gamma(TZ) \supset \ker\alpha =: \Delta$$

hyperplane distribution

Then (Z,Δ) is a contact manifold

REEB DYNAMICS

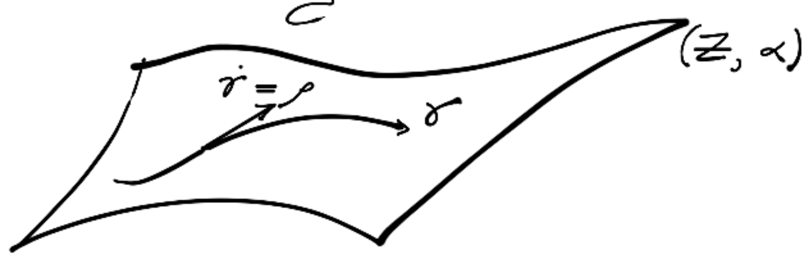
The unique vector ρ such that

$$\begin{cases} \varphi(\rho, \cdot) = 0 \\ \alpha(\rho) = 1 \end{cases}$$

normalization

is called the **Reeb vector**

Dynamics = Reeb flows



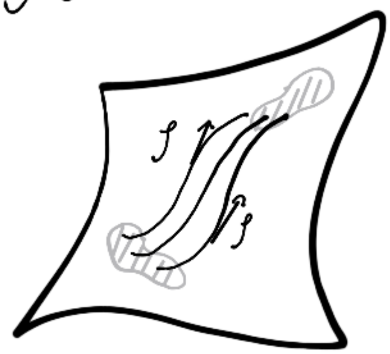
CONTACT LIOUVILLE THEOREM

$$\mathcal{L}_\xi \alpha = (d\,\iota_\xi + \iota_\xi d)\alpha = d\,1 + \varphi(\xi,\cdot) = 0$$

Cartan magic

$$\mathcal{L}_\xi \varphi = (d\,\iota_\xi + \iota_\xi d)\,d\alpha = d\varphi(\xi,\cdot) = 0$$

$$\Longrightarrow \quad \boxed{\mathcal{L}_\xi \, \mathrm{Vol}_\alpha = 0}$$



c.f. Uncertainty principle

(Z, α)

DARBOUX THEOREM

There exist local coordinates $(\vec{x}, \vec{\pi}, \tau)$ where

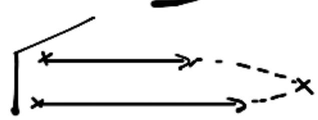
$$\alpha = \vec{\pi} \cdot d\vec{x} - 1\, d\tau$$

trivial Hamiltonian

Proof via Moser trick $\alpha_t = t\alpha_0 + (1-t)\alpha$, contact,

existence of isotopy $\varphi_t^* \alpha_0 = \alpha_t$?

Initial position & velocity conserved for short times



$\boxed{\text{CLASSICAL MECHANICS} = \text{CONTACT TOPOLOGY}}$

QUANTUM MECHANICS

Copenhagen:

- Measure $\psi_i \in \mathcal{H}$
- Evolve $\frac{i}{\hbar}\dot{\psi} = \hat{H}\psi, \quad \psi(0) = \psi_i$
- Predict probability of final state $\psi_f \in \mathcal{H}$ at later time T

$$\boxed{P_{fi} = \frac{|\langle \psi_f, \psi(T) \rangle|^2}{|\psi_f|^2 \, |\psi(T)|^2}}$$

WIGNER FUNCTIONS

Probabilities computed from

$$|\psi\rangle\langle\psi| \in \mathrm{End}(\mathcal{H})$$

$\Rightarrow$ Gelfand, Naimark, Segal construction

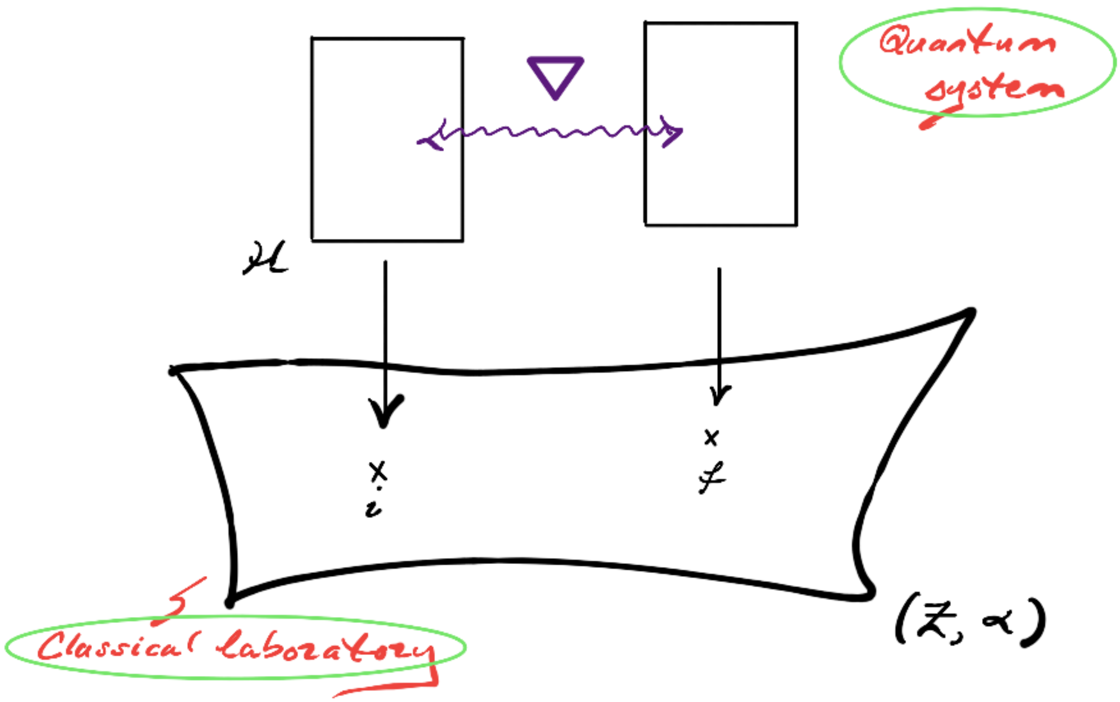
Wigner's quasiprobability distribution:

Let $L^2(\mathbb{R}) = \mathcal{H} \ni \psi(x)$

$$\boxed{W_\psi(x,p) = \int dy\, e^{ipy}\, \psi\left(x+\tfrac{y}{2}\right) \psi^*\left(x-\tfrac{y}{2}\right)}$$

Moyal: $*$ product $\Rightarrow$ phase space quantum mechanics!

QUANTUM CONTACT GEOMETRY FANTASY



BRST QUANTIZATION

- *Not* quantum!

- Constrained dynamics ⇝ Cohomology

- Discovered via well definedness of quantum path integrals

- Faddeev - Popov determinant

Grassmann symmetry

REEB DYNAMICS CONSTRAINT ANALYSIS

"Pathologies" of action principle schematically

$$S(p, x, \lambda) = \int \Big(p(x)\,\dot{x} - \lambda\, C(p,x) - H(p,x) \Big)$$

Make system 1st order

Momenta depend on positions

λ non-dynamical, imposes $C(p,x) = 0$

$\Downarrow$

2nd class constraints

$\Downarrow$

1st class constraints

GAUGE INVARIANCE!

Want to study $S = \int_\gamma \alpha$

Choose coordinates z^i for Z

$$S(z^i(\tau)) = \int \alpha_i(z)\, \dot{z}^i \, d\tau$$

parameterized path

— Momenta $p_i = \alpha_i(z) \Rightarrow 2^{nd}$ class constraints

— $\int_\gamma \alpha$ is path parameterization independent

$\Rightarrow$ gauge invariance $\Rightarrow 1^{st}$ class constraint

More carefully...

Equivalent action

Lagrange multipliers for constraints

$$S = \int\Big(p_i \dot{z}^i - \lambda^i \underbrace{\big[p_i - \alpha_i(z)\big]}_{\text{Constraints}}\Big)$$

standard symplectic current/potential

$$C_i := p_i - \alpha_i(z)$$

Poisson brackets

$$\{z^i, p_j\}_{PB} := \big[d(p\,dz)\big]^{-1}(dz^i, dp_j) = \delta^i_j$$

Poisson algebra of constraints

$$\{C_i, C_j\}_{PB} = g_{ij}$$

The Levy form!

CONSTRAINT COUNTING

$$\left.\begin{array}{lll} 2n+1 & \text{constraints} & C_i \\ 2n & \text{rank} & \varphi_{ij} \end{array}\right\} \Rightarrow \begin{array}{ll} 2n & 2^{nd}\ \text{class} \\ 1 & 1^{st}\ \text{class} \end{array}$$

Short term plan...

2^{nd} class constraints

DIRAC BRACKET...

1^{st} class constraints

BRST...

DIRAC BRACKET

Consider system with momenta & positions (p_A, q^A)

$$\{p_A, q^A\}_{PB} = \delta_A^B$$

subject to (2nd class) constraints

$$\{C_a(p,q), C_b(p,q)\}_{PB} = \Phi_{ab}$$

w.l.o.g. constant invertible

Dirac says dynamics on constraint surface governed by Dirac bracket

$$\boxed{\{\cdot,\cdot\}_{DB} = \{\cdot,\cdot\}_{PB} - \{\cdot, C_a\}_{PB}\, \Phi^{-1}_{ab}\, \{C_b, \cdot\}_{PB}}$$

b/c $\{C_a, \cdot\}_{DB} = 0$

DIRTY RUSSIAN TRICK...

Batalin, Fradkin, Fradkina, Tyutin

Dirac bracket symplectic current

$$\boxed{\theta_{Dirac} = \theta - \frac{1}{2} C_a \, \Phi^{-1}_{ab} \, dC_b}$$

$$\Rightarrow \int \theta_{Dirac} = \int \Big(\theta - \frac{1}{2} s_a \, \Phi^{-1}_{ab} \, ds_b - \lambda^a [s_a - C_a]\Big)$$

"Fibre coordinates" cf. Fedosov...

Lagrange multipliers

Extended constraints are first class!

$$\boxed{C^{ext}_a(s,z,p) = s_a - C_a(z,p)} \quad b/c \quad \{s_a, s_b\}_{PB} = -\Phi_{ab}$$

EXTENDED REEB DYNAMICS

Introduce $2n$ fibre coordinates s^a

$$\{ s^a , s^b \}_{PB} = J^{ab}$$

$2n \times 2n$, antisymmetric constant, max. rank

Soldering / "square root" of Levy

$$\varphi = J_{ab}\, e^a \wedge e^b$$

α, e^a basis for T^*Z

$e^a(\rho) = 0$

Extended 1st class constraints

$$\mathcal{C}_i^{ext} = p_i - \alpha_i(z) - e_i{}^b(z) J_{ab} s^a + \cdots$$

Integrate out

Corrections such that $\{\mathcal{C}_i^{ext}, \mathcal{C}_j^{ext}\}_{PB} = 0$

Claim

$$\boxed{S^{ext} = \int_\Gamma \left(\frac{1}{2} s^a J_{ab}\, ds^b + \Omega\right)}$$

where $\Omega = \alpha + e^b J_{ab} s^a + \omega(z,s)$

obeys $d\Omega + \{\Omega \wedge \Omega\}_{PB} = 0$, describes Reeb dynamics!

Gauge invariance

Zero curvature condition $d\Omega + \{\Omega \wedge \Omega\}_{PB} = 0$

$\Rightarrow$ $2n+1$ local invariances of S^{ext}

$$\boxed{\begin{aligned} \delta z^i &= \varepsilon^i(\tau) \\ \delta s^a &= \varepsilon^i(\tau) J^{ab} \frac{\partial \Omega_i}{\partial s^b} \end{aligned}}$$

Reason: First class constraints C_i^{ext} generate gauge symmetries via

$$\delta \cdot = \varepsilon^i(\tau) \{ C_i^{ext} , \cdot \}$$

Equations of motion

$\dot{\Gamma} = (\dot{z}^i, \dot{s}^a)$ tangent vector to path in extended space

$$\begin{cases} \dot{z}^i(\partial_i \Omega_j - \partial_j \Omega_i) - \dot{s}^a \dfrac{\partial \Omega_j}{\partial s^a} = 0 \\[2ex] J_{ab}\dot{s}^b + \dot{z}^i \dfrac{\partial \Omega_i}{\partial s^a} = 0 \end{cases}$$

But $s^a \sim 0 \implies d\alpha(\dot{z}, \cdot) = \varphi(\dot{\gamma}, \cdot) = 0 = e^a(\dot{\gamma})$

gauge equivalent

Reeb dynamics as claimed

BRST QUANTIZATION (Batalin, Fradkin Vilkovisky)

Lie algebra cohomology:

$$\mathfrak{g} = \mathrm{span}\{g_a\} \qquad [g_a, g_b] = f_{ab}^{c}\, g_c$$

and some representation V,

Consider $\Lambda_V \mathfrak{g} = \mathrm{span}_V\{c^a\}$ where $c^a \wedge c^b =: c^a c^b = -c^b c^a$

Nilpotent Chevalley-Eilenberg differential:

$$\boxed{Q = c^a g_a - \frac{1}{2} c^a c^b f_{ab}^{c} \frac{\partial}{\partial c^c}} \qquad \frac{\ker Q}{\operatorname{im} Q} = H^*(\mathfrak{g}, V)$$

Example (SINGLETON) (M_{d+2}, g) where $g_{MN} = \nabla_M X_N$ is a Fefferman–Graham ambient metric determined by a conformal geometry (M_d, c)

Constrained dynamics (Marnelius, Bars ...)

$$S = \int_\gamma \Big[P_M dY^M - \underbrace{e P_M g^{MN} P_N - \lambda P_M X^M - \mu X^M g_{MN} X^N}_{\text{Triplet of constraints obey } \mathfrak{g} = s\ell_2 \text{ w.r.t. } \{\cdot,\cdot\}_{PB}} \Big]$$

(P, Y) coordinates on T^*M_{d+2}

Tautological 1-form

BRST quantization: $V = C^\infty M$, $\mathfrak{g} = sp(2) = \{\Delta^g, \nabla_X + \frac{d+2}{2}, X^2\}$

$$H^*_0(\mathfrak{g}, V) = \text{Spectrum of Yamabe operator on } (M_d, c).$$

grading

BRST QUANTIZATION OF CLASSICAL MECHANICS

First class extended constraints (abelian)

$$C_i^{ext} = p_i - \alpha_i - c_i{}^b J_{ab} s^a - \omega_i(z,s)$$

BRST symplectic potential

$$\Theta = p_i dz^i + \tfrac{1}{2} s^a J_{ab} ds^b + b_i dc^i$$

Grassmann ghost canonical (b,c) pairs

BRST charge

$$\boxed{Q = c^i C_i^{ext}}$$

PATH INTEGRAL QUANTIZATION

Quantum correlators

$$Z = \int [Dz\, Dp\, Db\, Dc]\, e^{\frac{i}{\hbar} S_{qu}}$$

Functional integration

Quantum action

$$\boxed{S_{qu} = \int \left[\Theta - \{Q, \Psi\}_{PB} \right]}$$

Gauge fixing fermion

CANONICAL QUANTIZATION 29

— Direct connection to geometry

BRST Hilbert space

Choice of polarization:

$$x^i,\ p_i = \frac{\hbar}{i}\frac{\partial}{\partial z^i};\ c^i,\ b_i = \frac{\hbar}{i}\frac{\partial}{\partial c^i};\ \underbrace{S^A,\ \frac{\hbar}{i}\frac{\partial}{\partial S^A}}_{\hat{S}^a};$$

⇝ Watch choices carefully!!!

BRST wave functions

$$\Psi_{BRST} = \Psi(x^i, c^i, S^A)$$

Wavefunctions & Sections 30

Differential forms

$$\Psi(x^i, c^i) \in \Omega^{\cdot} Z$$

commuting coordinates for Z   anti-commuting variables

by identifying $c^i = dz^i$

since

$$c^i c^j = -c^j c^i \iff dz^i \wedge dz^j = -dz^j \wedge dz^i$$

Vector bundle

$$\Psi_{BRST}(x^i, c^i, S^A)$$

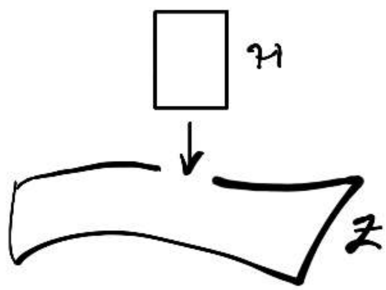
Differential form on Z

Fiber-wise wave function

Choose inner product fibrewise, e.g. $L^2(\mathbb{R}^n)$

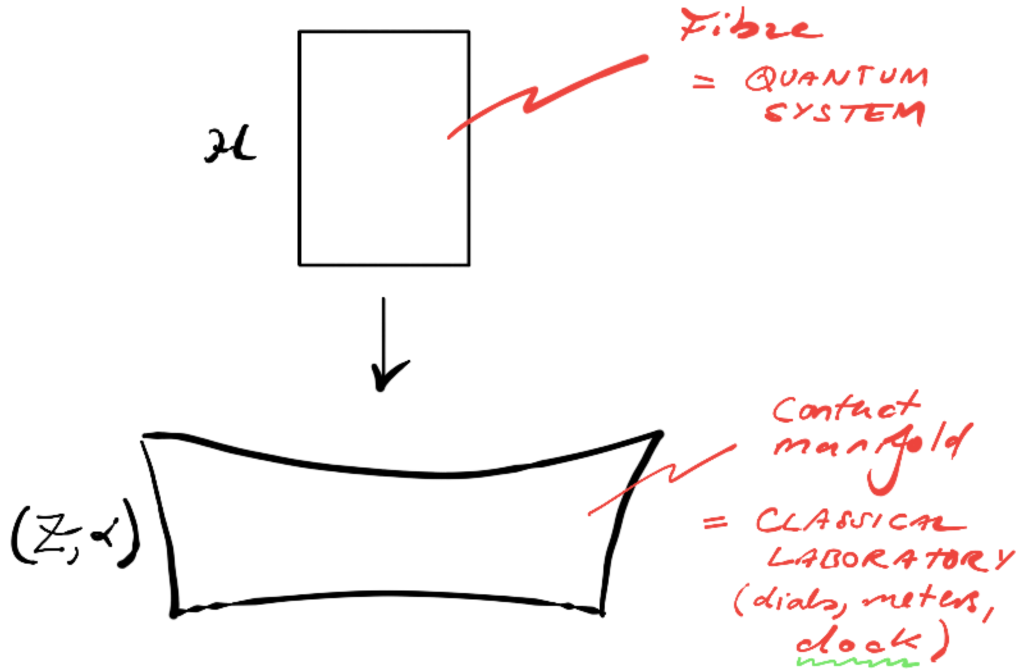
$\Rightarrow$ Ψ_{BRST} is differential form on Z taking values in Hilbert space $\mathcal{H}$.

Zero forms = Sections of vector bundle:



Cf. Krysl

Physical Interpretation



FLAT CONNECTION

– Compare fibres over differing base points!

= Compare quantum system for differing classical times + configurations

Claim:

$$\text{Connexion} = \text{BRST Charge}$$

$\nabla = d + A$
1-form taking values in $\mathrm{End}(\mathcal{H})$

Nilpotent operator on BRST wave-functions

QUANTUM BRST CHARGE

Classical

$$Q_{BRST} = c^i \left(p_i - \Omega_i\right) \quad , \quad \Omega_i = \alpha_i + \cdots$$

Quantize

$$p_i \mapsto \frac{\hbar}{i}\frac{\partial}{\partial z^i} \quad , \quad s^a \mapsto \hat{s}^a = \left(S^A, \frac{\hbar}{i}\frac{\partial}{\partial S^A}\right)$$

$$c_i \longmapsto dz^i$$

$$\Rightarrow \frac{i}{\hbar}\hat{Q}_{BRST} = d - \frac{i}{\hbar}\left(\alpha + e_a \hat{s}^a + \hat{\omega}(z,\hat{s})\right) =: \nabla$$

① Determine $\hat{\omega}$ by flatness $\nabla^2 = 0$

QUANTIZATION = ZERO CURVATURE CONDITION

cf. Fedosov

GAUGE ALGEBRA:

- Solve for $\hat{\omega}$ as expansion in $\left(s^A, \frac{\hbar}{i}\frac{\partial}{\partial s^A}\right) = \hat{s}^a$

  $\rightsquigarrow$ formally always solvable

- Heisenberg algebra

$$\mathfrak{heis} = \{1, \hat{s}^a\}$$

- $\nabla = d - \frac{i}{\hbar}\left(\alpha . 1 + e_a \hat{s}^a + \cdots\right)$

$\Rightarrow$ Connexion is $\mathcal{U}(\mathfrak{heis})$ – valued

Schrödinger Equation

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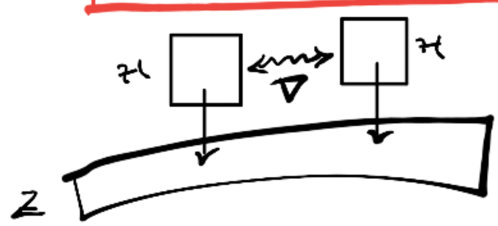
- Physical states = Cohomology of $\hat{Q}_{BRST}$ at fixed zero? ghost number

- Cohomology at ghost number = form degree zero given by parallel transport

$$\boxed{\nabla \Psi = 0}$$



Summary

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QUANTIZATION

$$\nabla^2 = 0$$

CLASSICAL MECHANICS

$$(Z, \alpha) \longmapsto \nabla = d + \underbrace{\alpha.1 + e_a \hat{s}^a}_{\text{HEIS}} + \cdots$$

strict contact

$$\nabla \Psi = 0$$

QUANTUM DYNAMICS

MEASUREMENT

– Solve contact Schrödinger equation

$$\nabla \Psi_{\varepsilon} = 0$$

physical quantum numbers

where $\{\Psi_{\varepsilon} =: |\varepsilon; z\rangle\}$ are fibrewise complete & orthonormal, for L^2 norm

$$\langle \varepsilon'; z | \varepsilon; z\rangle := \int \Psi^*_{\varepsilon'}(z, S)\, \Psi_{\varepsilon}(z, S)\, d^n S = \delta_{\varepsilon\varepsilon'}$$

same point $z \in Z$

Generalized Wigner Distribution

$$\mathcal{W}_{\varepsilon,\varepsilon'}(z,z') := \langle \varepsilon'; z' | \varepsilon; z \rangle$$

— If Classical laboratory evolves from z to z'

$$|\mathcal{W}_{\varepsilon,\varepsilon'}(z,z')|^2$$

Time interval replaced by pair of phase space time points

then gives probability of measuring outcome ε' if ε was prepared.

HAMILTONIAN MECHANICS

— Contact form

$$\alpha = p_A \, dq^A - H(p, q, t)\, dt$$

$$\varphi = d\alpha = e_A \wedge f^A \Rightarrow \begin{cases} e_A = dp_A + \dfrac{\partial H}{\partial q^A} dt \\ f^A = dq^A - \dfrac{\partial H}{\partial p_A} dt \end{cases}$$

— Flat connexion

$$\nabla = d - \frac{i}{\hbar}\alpha + \frac{i}{\hbar} e_A s^A - f^A \frac{\partial}{\partial s^A} + \frac{i}{\hbar} dt \sum_{\sigma \geq 2} \frac{\partial^\sigma H}{\partial z^{a_1} \ldots \partial z^{a_\sigma}} \hat{s}^{a_1} \ldots \hat{s}^{a_\sigma}$$

fibre coords.

$z = (p, q)$

$\left(s, \frac{\hbar}{i}\frac{\partial}{\partial s}\right)$

CONTACT SCHRÖDINGER EQUATIONS

— $\nabla\Psi = 0 \Rightarrow$ triplet dq^A, dp_A, dt

$$\begin{cases} \left(\dfrac{\partial}{\partial q^A} - \dfrac{\partial}{\partial s^A}\right)\Psi = \dfrac{i}{\hbar} p_A \Psi \\ \left(\dfrac{\partial}{\partial p_A} - \dfrac{i}{\hbar} s^A\right)\Psi = 0 \\ \dfrac{\partial \Psi}{\partial t} = \dfrac{1}{\hbar} \displaystyle\sum_{\sigma \geq 0} \dfrac{\partial^\sigma H}{\partial z^{a_1} \cdots \partial z^{a_\sigma}} \hat{s}^{a_1} \cdots \hat{s}^{a_\sigma} \Psi \end{cases}$$

$$\Longrightarrow \Psi = e^{-\frac{i}{\hbar} p_A s^A} \psi(Q, t) \quad \& \quad i\hbar\dot{\psi} = \left[H\left(Q, \frac{\hbar}{i}\frac{\partial}{\partial Q}\right)\right]_{\text{WEYL}} \psi$$

$Q := q + s$

Weyl-ordered quantization

Wigner Functions

— $H \neq H(t) \Rightarrow \left\{ |\Psi_E(t)\rangle = e^{-\frac{iEt}{\hbar}} \psi_E(Q) \,\middle|\, E \in \mathrm{spec}(\hat{H}) \right\}$

complete & orthonormal along fibres

remember $Q = q + S$

$$\Rightarrow \mathcal{W}_{E,E'}(z,z') \xrightarrow{z \to z'} \delta_{EE'}$$

— Single state limit $E = E'$

$$\boxed{\mathcal{W}_E(z,z') = e^{\frac{i}{\hbar}[E\,\delta t - \delta p_A\, \bar{q}^A]} \int \frac{d^nS\, d^nP}{(2\pi\hbar)^n}\, W_E(S,P)\, e^{\frac{i}{\hbar} P_A \delta q^A}}$$

q, p, t — difference of time/momenta — average position — WIGNER FUNCTION

Open Questions & Future Directions

— New key problems in contact geometry:

* study flat $U(\mathrm{heis})$ connexions
* compute overlaps $\mathcal{W}_{E,E'}(z,z')$

— Choices ? Uniqueness of quantization

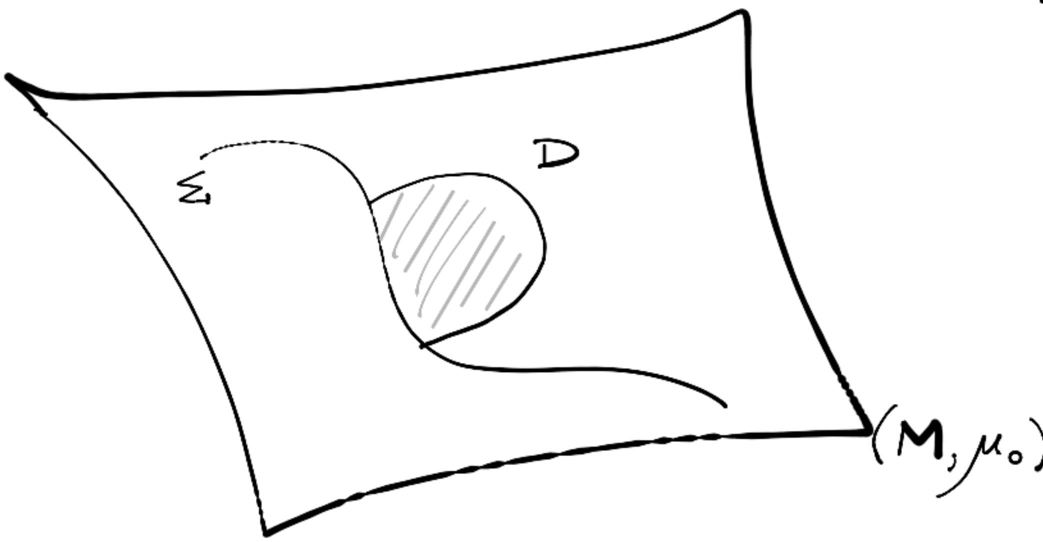
— Contact topology & quantum mechanics?

SINGULAR VOLUME PROBLEMS w/ Rod Gover

Problem: Given a manifold M and measure μ_0 that is singular along a hypersurface

$$\Sigma \hookrightarrow M$$

analyze volumes of (possibly higher codimensional) regions intersecting Σ



$$\mathrm{Vol}(D) = \text{“}\int_D \mu_0\text{”} \quad \text{— infinite}$$

MOTIVATIONS

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— Quantum field theories

⇝ formally ill-defined / infinite

— Simplify study of

$$\Sigma \hookrightarrow (M, \mu_0, \text{structure})$$

— Holography

Quantum Field Theory & AdS/CFT

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— infinities of QFT

↳ regulate

↳ renormalize

↳ renormalization point

⇒ physical predictions / couplings

scale dependent

⇒ anomaly in scale invariance

Could scale dependence be described by higher dimensional geometry ?

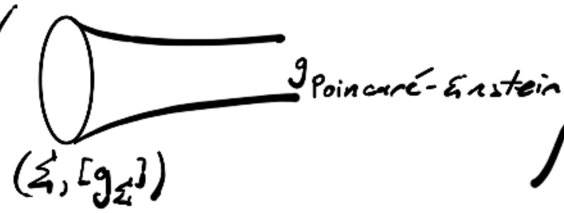
AdS / CFT correspondence

Asymptotically anti de Sitter gravitational bulk theory $\cong$ Boundary conformal field theory

— Fefferman-Graham precursor

WEYL ANOMALY & Q-CURVATURE

- Weyl anomaly ⇝ couple QFT to metric & study anomalous response to $g_{\mu\nu} \mapsto \Omega^2 g_{\mu\nu}$
- Henningson–Skenderis:

"Volume ()"

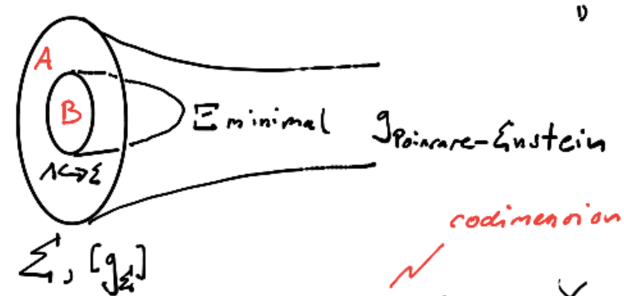
= Divergent + log-divergent × Weyl anomaly
+ Finite

RENORMALIZED VOLUME

BRANSON Q-CURVATURE

SUBMANIFOLD OBSERVABLES

— Graham–Witten

" Area 

"

codimension!

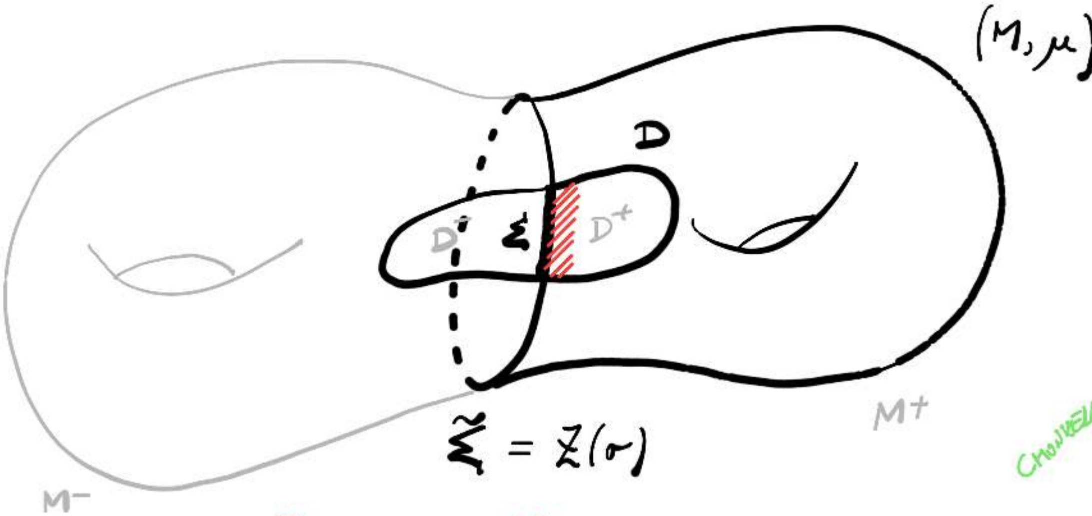
$=$ divergent $+$ log-divergent $\times$ Willmore $+$ finite

Entanglement entropy b/w A & B

Volume enclosed = information of B

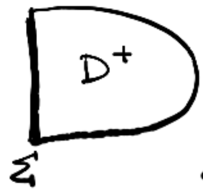
← Ryu, Takanayagi, Susskind, ...

GEOMETRIC DATA



* $\tilde{\Sigma}$ separating
* σ smooth, defining $\Rightarrow d\sigma|_{\tilde{\Sigma}} \neq 0$
* cylinder topology

REGULATORS

$\mathrm{Vol}_{\mu_0}\Bigg($  $\Bigg)$ ill-defined

$\Rightarrow$ regulated volume:

$\boxed{\mathrm{Vol}_{\varepsilon} := \Bigg(}$  $\boxed{\Bigg) = \int_{D^+_\varepsilon} \mu_0}$

Independence from 1-parameter family of regulating hypersurfaces Σ_ε?

DEFINING FUNCTIONS

— Singularity

non-singular measure

$$\mu_0 = \frac{\mu}{\sigma^k} \qquad (\mathbb{Z} \ni k \geq 1)$$

where $\tilde{\Sigma}_1 = Z(\sigma)$, $d\sigma\big|_{\Sigma_1} \neq 0$, $\sigma \in C^\infty M$

— Regulating hypersurfaces

$$\tilde{\Sigma}_\varepsilon = Z\left(\frac{\sigma}{\tau} - \varepsilon\right), \quad 0 < \tau \in C^\infty M$$

* Regulator $= \tau$

* $[\mu; \sigma] \underset{\Omega \in C^\infty M}{=} [\Omega^k \mu, \Omega\tau]$

MAIN THEOREM

$$\mathrm{Vol}_\varepsilon\left(D_j[\mu_j\,\sigma_j\,\tau]\right) := \int_{D_{\frac{\sigma}{\tau}>\varepsilon}} \frac{\mu}{\sigma^k}$$

$$= {\sum_{\ell=k-1}^{1}}' \frac{1}{\varepsilon^\ell}\int_{\Sigma} v_\ell(\sigma,\tau) + \log\varepsilon \int_{\Sigma} a(\sigma) + \mathrm{Vol}_{\mathrm{Ren}}(\sigma,\tau) + \varepsilon R(\varepsilon)$$

Local Divergences

LOCAL Regulator-independent Anomaly

Non-local Renormalized volume

Local anomaly operator

$$\& \quad \mathrm{Vol}_{\mathrm{Ren}}(e^{\omega}\tau) - \mathrm{Vol}_{\mathrm{Ren}}(\tau) = \int_{\Sigma} a(\sigma,\omega)$$

CONSTRUCTIVE PROOF

— Key trick

$$\boxed{\mathrm{Vol}_\varepsilon = \int_D \frac{\mu}{\sigma\kappa}\,\theta\!\left(\frac{\sigma}{\tau} - \varepsilon\right)}$$

Heaviside

— Cylinder topology $\Rightarrow$ distributional identities

— want Laurent & $\frac{d\log\varepsilon}{d\varepsilon} = \frac{1}{\varepsilon}$

$$\Rightarrow \quad \boxed{\frac{d}{d\varepsilon}\,\theta\!\left(\frac{\sigma}{\tau} - \varepsilon\right) = \delta\!\left(\frac{\sigma}{\tau} - \varepsilon\right)}$$

Dirac Delta

Hence

$$-\varepsilon^k \frac{dVol_\varepsilon}{d\varepsilon} = \int_D \frac{\mu}{\tau^{k-1}} \delta(\sigma - \tau\varepsilon) = \text{TAYLOR}(\varepsilon)$$

$\sigma \to \tau\varepsilon$

$\delta(\alpha(x)\, x) = \frac{\delta(x)}{\alpha(x)}$

$\alpha > 0$

Local

$\rightsquigarrow \frac{d\,1}{d\,\varepsilon} = 0 \Rightarrow Vol_{ren}$ non-local

Next: Taylor expand...

$\Longrightarrow$ differentiated δ-functions

DIVERGENCES

$$\mathrm{Vol}_\varepsilon = \frac{V_{k-1}}{\varepsilon^{k-1}} + \cdots + \frac{V_1}{\varepsilon} + \cancel{f} \log\varepsilon + \cdots$$

— local integrals

$$\boxed{V_\ell = \frac{(-)^{k-\ell-1}}{(k-\ell-1)!\,\ell} \int_D \frac{\mu}{\tau^\ell}\, \delta^{(k-\ell-1)}(\sigma)}$$

Regulator dependent

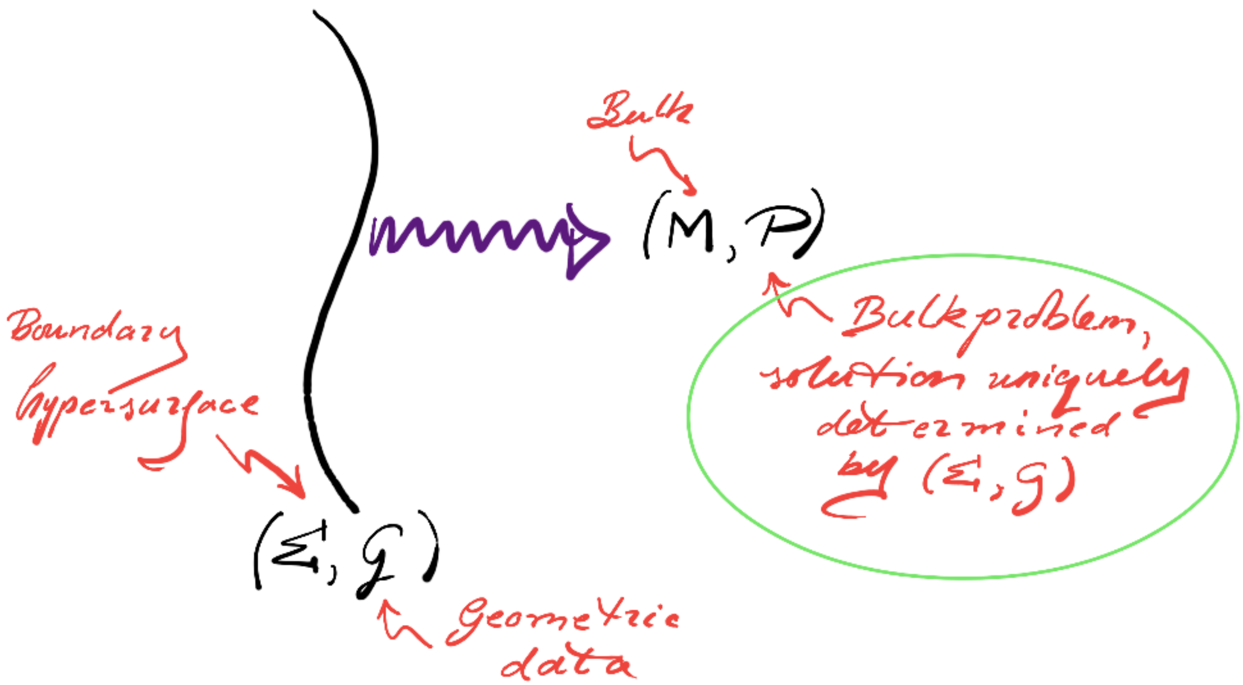
jets $\Rightarrow$ local

ANOMALY

$$\boxed{\mathcal{A} = \frac{(-)^{k-1}}{(k-1)!}\int_D \mu\, \delta^{(k-1)}(\sigma)}$$

- local
- regulator-independent!
- computable
- variational calculus  via $\sigma \mapsto \sigma_t$
- not geometric ? ⟵ Need structure!

HOLOGRAPHY



Bulk
(M, P)
Bulk problem, solution uniquely determined by (Σ, g)
Boundary hypersurface
(Σ, g)
Geometric data

Key examples

Σ	M
CONFORMAL GEOMETRY	POINCARE EINSTEIN
CONFORMAL HYPERSURFACE	SINGULAR YAMABE

Alternatively, given only

$$\Sigma \hookrightarrow (M,[g])$$

compute A as functional of this data!

CONFORMAL GEOMETRY

Data: Conformal structure $(M,[g])$

Defining density

$$\sigma = [g;\sigma] = [\Omega^2 g; \Omega\sigma] \in \mathcal{E}M[1]$$

$$0 \leq \sigma \in C^\infty M$$

$$\mathcal{Z}(\sigma) = \Sigma \hookrightarrow M, \quad d\sigma\big|_\Sigma \neq 0$$

$\Longleftrightarrow$ Σ, is a conformal infinity

for $g_o = \dfrac{g}{\sigma^2}$

$\Rightarrow$ $k = d$ singular measure $\det^{\frac{1}{2}} g_o = \dfrac{\det^{\frac{1}{2}} g}{\sigma^d}$

S-CURVATURE

— Given $\sigma \in \mathcal{E}M[1]$

$$S_\sigma = \Big[g\,;\ |\nabla\sigma|_g^2 - \frac{2\sigma}{d}\big(\Delta^g\sigma + J^g\sigma\big)\Big] \in \mathcal{E}M[0] = C^\infty M$$

Then $$\boxed{\int_\Sigma \mu_{[g_\Sigma]}\,\bullet \;=\; \int_M \mu_{[g]}\, S_\sigma^{1/2}\,\delta(\sigma)\,\bullet_{\mathrm{ext}}}$$

— $\delta(\sigma) = [g\,;\ \delta(\sigma)]$ distribution-valued weight -1 density

— $S_\sigma = I_\sigma^2$ squared scale factor

$\Rightarrow$ $I_\sigma^2 = 1 + \sigma^d B$ for Singular Yamabe

LAPLACE ROBIN OPERATOR

$$L_\sigma : \Gamma(\mathcal{E}M[w]) \longrightarrow \Gamma(\mathcal{E}M[w-1])$$

where

$$L_\sigma \stackrel{g}{=} (d+2w-2)(\nabla_n + w\rho_\sigma) - \sigma(\Delta^g + wJ^g)$$

$$\& \quad \begin{cases} n = \nabla\sigma \\ \rho_\sigma = -\frac{1}{d}(\Delta^g\sigma + J^g\sigma) \end{cases}$$

— Formally self-adjoint

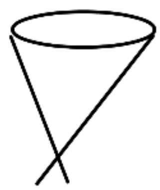
— $L_\sigma = I_\sigma \cdot D$ Thomas D-operator

$\Rightarrow$ also tractor-coupled!

SOLUTION GENERATING ALGEBRA

— View flat model as moduli space of massless excitations in $d+1,1$ spacetime

$\boxed{\mathbb{R}^{d+1,1}}$



— $so(d+2,2) = conf(\mathbb{R}^{d+1,1})$ symmetry of massless Lorentzian wave equation

— Čap-Gover-Petersen:

Ambient tensors ⇝ Tractors

Find subalgebras of $conf(\mathbb{R}^{d+1,1})$ valid in curved setting

Indeed $\forall\ (c, \sigma)$

$$\{\sigma,\ d+2\underline{w},\ \underbrace{S_\sigma^{-1} L_\sigma}\}\ \underbrace{\text{obey}}\ sl_2$$

weight operator

assume $S_c^{\ 0_0} \neq 0$

⇝ useful for many applications...

EVALUATING DIVERGENCES & ANOMALY

— integrals over $\delta^{(\ell)}(\sigma)$

Key Lemma I :

$$\boxed{\left(\mathcal{S}_\sigma^{-1} L_\sigma\right)^{\ell} \delta(\sigma) = (d-\ell-1)\cdots(d-2)\, \delta^{(\ell)}(\sigma)}$$

Formally self-adjoint

DIVERGENCES

$$\boxed{\frac{1}{\varepsilon^{\ell}} \int_D \frac{\delta^{(d-\ell-1)}(\sigma)}{\pi^{\ell}} \overset{\ell \neq d-1}{\propto} \int_{\Sigma} \mathcal{S}_\sigma^{-\frac{1}{2}} \left(L_\sigma \mathcal{S}_\sigma^{-1}\right)^{d-\ell-1} \frac{1}{\pi^{\ell}}}$$

regulator $0 < \pi \in \Gamma(\mathcal{E}M[1])$

— leading divergence $v_{d-1} \propto \operatorname{Area}(\Sigma)$

. Sing. Yam. π-scale

— subleading divergence $v_{d-2} \propto \int_\Sigma H$ evenness!

ANOMALY

— Lemma I fails

$$L_\sigma\, \delta^{(d-2)}(\sigma) = 0$$

but want to compute $\int \delta^{(d-1)}(\sigma)$ 

— $\int \delta^{(d-1)}(\sigma) = \int \dfrac{\delta^{(d-1)}(\sigma)}{\tau^0} \rightsquigarrow \log\tau$?

— log density of weight w:

$$\boxed{\boldsymbol{\lambda} = [g;\lambda] = [\Omega^2 g\,;\, \lambda + w\log\Omega]}$$

— $\underline{w}\log\tau = \dfrac{\underline{w}\,\tau}{\tau} = 1$ b/w τ weight 1

LEMMA II

$$\boxed{\int_M \delta^{(d-1)}(\sigma) = -\int_M \delta^{(d-2)}(\sigma)\, L_\sigma \log \pi}$$

— assumes $S_\sigma = 1$ for simplicity only

— integrated identity!

— analogue for $\partial M \neq \phi$ known

ANOMALY

$$\boxed{\mathcal{A} = \frac{(-)^{d-1}}{(d-1)!\,(d-2)!} \int_{\Sigma} Q_\sigma}$$

Extrinsic Q-curvature, σ-Singular Yamabe

$$\boxed{Q_\sigma \stackrel{\Sigma}{=} L_\sigma^{d-1} \log \tau}$$

— changing scale $\tau \mapsto e^{\omega}\tau$

$$Q_\sigma \longmapsto Q_\sigma + P_\sigma e^{\omega}$$

$\boxed{P_\sigma \stackrel{\Sigma}{=} L_\sigma^{d-1}}$ $\longrightarrow$ total derivative $\Longrightarrow$ $\int Q$ invariant

Q-CURVATURE

— $Q_\sigma \log \tau|_\Sigma$ extrinsically coupled generalization of Branson Q-curvature

e.g. $d = 2+1$,

$$Q^{\text{extrinsic}} = J_\Sigma - \frac{1}{2} \mathring{II}_{ab} \mathring{II}^{ab}$$

Willmore energy

$d = 3+1$

$$Q^{\text{extrinsic}} = \nabla^a_\Sigma \nabla^b_\Sigma \mathring{II}_{ab} + 2 \mathring{II}^{ab} F_{ab}$$

Fialkow

EXTRINSICALLY COUPLED LAPLACIAN POWERS

— $\mathbb{P}_\sigma\big|_\Sigma$ extrinsically coupled analog of GJMS operators

e.g. $$P_2^{extrinsic} = \Delta_\Sigma + \frac{1-n}{2}\left[J_\Sigma - \frac{\mathring{II}_{ab}\mathring{II}^{ab}}{2(n-1)}\right]$$

$$: \Gamma\mathcal{E}\Sigma[\tfrac{1-n}{2}] \to \Gamma\mathcal{E}\Sigma[\tfrac{-1-n}{2}]$$

$$P_3^{extrinsic} = -8\,\mathring{II}^{ab}\nabla_a^\Sigma\nabla_b^\Sigma + \text{MORE}$$

$$: \Gamma\mathcal{E}\Sigma[\tfrac{3-n}{2}] \to \Gamma\mathcal{E}\Sigma[-\tfrac{n}{2}]$$

— tractor coupled

GENERALIZATIONS

— higher codimension ✓

— boundary transgressions ✓

— corners ?