Strongly Finitary Metric Monads are Too Strong

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Abstract.
For finitary algebras in the cartesian closed categories of sets, posets, cpos, or ultrametric spaces, varieties correspond precisely to the Eilenberg-Moore categories of strongly finitary monads [1, 2]. Several authors (e.g. Rosický, Lucyshyn-Wright, and Parker) have recently asked whether the same holds for **Met**, the category of (extended) metric spaces. It is symmetric monoidal closed, but not cartesian closed. Finitary algebras in this category play an impotant role in computer science, where they are called *quantitative algebras* [3]. For equational presentations one uses *quantitative equations* $t =_\varepsilon t'$ between terms, where $\varepsilon \geq 0$ is a rational number. An algebra satisfies this equation iff every interpretation of variables of t and t' yields elements of distance at most ε.
The answer to the question mentioned above is negative:

Example. The monad for the variety of algebras on binary operations $+$ and $\star$, presented by the quantitative equation $x + y =_1 x \star y$, is not strongly finitary.

Let us denote by $\mathbf{Mnd_p}$ the category of enriched monads on **Met** that are *prefinitary*: they preserve collectively surjective cocones of directed diagrams. The category $\mathbf{Mnd_p}$ is enriched by taking the distance of parallel monad morphisms as the supremum of the distances of their components.

Theorem. *A monad on* **Met** *corresponds to a variety of quantitative algebras iff it is a weighted colimit of strongly finitary monads in* $\mathbf{Mnd_p}$.

A functor between varieties V and W is *concrete* if it commutes with the forgetful funtors of V and W (to **Met**) on the nose.

Corollary. *The following ordinary categories are dually equivalent:*
(1) The category of varieties of quantitative algebras and concrete functors.
(2) The closure of strongly finitary monads under weighted colimits in $\mathbf{Mnd_p}$.

References

[1] J. Adámek, M. Dostál, and J. Velebil, Quantitative algebras and a classification of metric monads, preprint, arXiv:2210.01565, 2024.

[2] J. Adámek, M. Dostál, and J. Velebil, *Sifted colimits, strongly finitary monads and continuous algebras, Theory and Application of Categories 44 (2025), 84-131.*

[3] R. Mardare, P. Panangaden, and G. Plotkin, *On the axiomatizability of quantitative algebras, Proceeding of Logic in Computer Science 2017, IEEE Computer Science 2017, 1-12.*