Fundamental Properties of Monads in Double Categories

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Abstract.

It is well established that monoidal categories provide an appropriate context in which internal monoids can be considered. On a higher level, the notion of (pseudo) double category, originally introduced by Ehresmann, can be viewed as a generalization of monoidal categories in which there is accordingly a natural notion of a monad [6, 3].

In this work, we establish various fundamental properties of the category $\mathsf{Mnd}(\mathbb{D})$ of monads and monad maps in a fibrant double category $\mathbb{D}$. For many particular choices of $\mathbb{D}$, the category of monads is a familiar category of interest. For example, if $\mathbb{D}$ is the double category $\mathbb{S}\mathsf{pan}(\mathcal{C})$ of spans in a category $\mathcal{C}$, then $\mathsf{Mnd}(\mathbb{D}) = \mathsf{Cat}(\mathcal{C})$, the category of internal categories in $\mathcal{C}$. Similarly, if $\mathbb{D}$ is the double category $\mathcal{V}$-$\mathbb{M}\mathsf{at}$ of matrices over a monoidal category $\mathcal{V}$, then $\mathsf{Mnd}(\mathbb{D}) = \mathcal{V}$-$\mathsf{Cat}$, the category of enriched categories in $\mathcal{V}$.

On one hand, we consider the existence of limits in $\mathsf{Mnd}(\mathbb{D})$ and their relationship to limits of the same type in categories related to $\mathbb{D}$, such as the category of objects, of arrows or the category of endomorphisms $\mathsf{End}(\mathbb{D})$. In this regard, we rely on a notion of *parallel (co)limits* in a double category, explicitly introduced in [1] – which is not a double-categorical (co)limit, but of a simpler flavor which is of practical use in the fibrant setting.

On the other hand, we investigate the existence of free monads and the monadicity of $\mathsf{Mnd}(\mathbb{D})$ over $\mathsf{End}(\mathbb{D})$. The results here can be seen as a generalization of those in [2], which deals with the case of $\mathcal{W}$-Cat where $\mathcal{W}$ is a bicategory. In the end of this line of ideas, we also approach the question of local presentability of $\mathsf{Mnd}(\mathbb{D})$, in an attempt to generalize known results from the monoidal case [5] and to recover cases of interest like $\mathcal{V}$-Cat [4].

References

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