Open power-objects in categories of algebras

Quentin Aristote

Quentin Aristote (quentin.aristote@irif.fr)
Université Paris Cité, CNRS, Inria, IRIF, F-75013, Paris, France

Abstract. Categories of algebras are seldom toposes, because they usually do not have power-objects. But can they still admit weaker forms of these power-objects? We will see that this is the case for the algebras of monads whose endofunctors and multiplications are *nearly cartesian*.

In a regular category, define a square to be a *near pullback* [1] when its universal morphism into the corresponding pullback is a regular epimorphism. By [1, Theorem 6] (aggregating older results from Carboni *et al.* and Schubert), a functor $\mathsf{F} : \mathsf{C} \to \mathsf{D}$ between regular categories has a monotone extension $\mathbf{Rel}(\mathsf{F}) : \mathbf{Rel}(\mathsf{C}) \to \mathbf{Rel}(\mathsf{D})$ if and only if F is *nearly cartesian*, i.e. preserves near pullbacks; and a natural transformation $\alpha : \mathsf{F} \Rightarrow \mathsf{G}$ between such functors extends to a natural transformation $\mathbf{Rel}(\mathsf{F}) \Rightarrow \mathbf{Rel}(\mathsf{G})$ if and only if α is *nearly cartesian*, i.e. has near pullbacks for its naturality squares.

In particular, as noticed in [2], a monad T on $\mathbf{Set}$ whose endofunctor and multiplication are nearly cartesian admits a weak distributive law over $\mathbf{P}$ and $\mathbf{P}$ then has a weak lifting $\overline{\mathbf{P}}$ in $\mathbf{EM}(\mathsf{T})$. This was later generalized to the power-object monad of any elementary topos in [1]. Examples of such weakly lifted powerset monads include the Vietoris monad in compact Hausdorff spaces, the convex subset monad in barycentric algebras, the monad of subsets closed under non-empty joins in complete join semi-lattices, and the powerset monad in (commutative) monoids or actions thereof.

In this talk, based on [3], we describe the Kleisli categories of these weakly lifted powerset monads as subcategories of relations. Call a morphism of algebras $f : (R, r) \to (X, x)$ *open* (the name comes from [4, Definition 3.1.1]) when the square $f \circ r = x \circ \mathsf{T} f$ – that makes f into a morphism of algebras – is a near pullback.

Theorem 1. *In $\mathbf{EM}(\mathsf{T})$, Kleisli-morphisms $(X, x) \to \overline{\mathbf{P}}(Y, y)$ are in one-to-one correspondance with $\mathbf{EM}(\mathsf{T})$-relations $(X, x) \xleftarrow{f} (R, r) \xrightarrow{g} (Y, y)$ such that f is open.*

The main use of this is to characterize monotone extensions to $\mathbf{Kl}(\overline{\mathbf{P}})$ in the spirit of [1, Theorem 6], and thus the existence of monotone weak distributive laws over $\overline{\mathbf{P}}$ in categories of algebras, thereby iterating this construction of weak liftings of powerset monads.

In this talk we will also focus on what this means in terms of elementary-topos-like properties: it entails the existence of a classifier of *open* subobjects, and more generally of an *open* power-object, classifying those relations that are in $\mathbf{Kl}(\overline{\mathbf{P}})$. We will give numerous examples.

References

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