Properties of coalgebraic models of a Lawvere theory

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Abstract.

Objects in the categories of bialgebras, Hopf algebras or Hopf braces have in common an underlying coalgebra structure. When this structure is cocommutative, they can be interpreted as categories of product-preserving functors from a Lawvere theory to the category Coalg of (cocommutative) coalgebras, analogously to how an algebraic theory defines a functor from a Lawvere theory to Set. The idea of passing from models in Set to models in Coalg was considered in [1] where a "linearization process" was used to study some non associative algebraic structures. This process associates a classical term in the algebraic theory with a linearized one, which is a homomorphism of coalgebras, thereby describing, for instance, how group identities transfer to Hopf algebra identities.

Coalgebraic models of a Lawvere theory seem to have some properties that are similar to those of algebraic varieties. We mention, among others, the construction of colimits in the category $\mathsf{Hopf}_{\mathsf{coc}}$ of Hopf algebras presented in [2], that of a free functor from Coalg to $\mathsf{Hopf}_{\mathsf{coc}}$ [3] and recent similar results on the category HBR of Hopf braces —which serve as coalgebraic models of the Lawvere theory of skew braces— obtained in [4]. We describe a general way to construct colimits and free functors between categories of coalgebraic models of Lawvere theories.

In light of the semi-abelian nature of $\mathsf{Hopf}_{\mathsf{coc}}$ [5] and HBR [6], it is natural to investigate protomodularity, regularity or semi-abelianness in these categories more broadly. In fact, it turns out that, under suitable hypotesis, if a category of coalgebraic models admits a forgetful functor to $\mathsf{Hopf}_{\mathsf{coc}}$, it is semi-abelian. This result allows us to construct new examples of semi-abelian categories. In particular, considering the Lawvere theories of radical rings and of digroups, we take their models in coalgebras defining the categories HRadRng of *Hopf radical rings* and HDGrp of *Hopf digroups*. Hence, the chain of functors $\mathsf{RadRng} \hookrightarrow \mathsf{SKB} \hookrightarrow \mathsf{DiGrp} \to \mathsf{Grp}$, becomes a chain of arrows between categories of coalgebraic models

$$\mathsf{HRadRng} \hookrightarrow \mathsf{HSKB} \hookrightarrow \mathsf{HDiGrp} \longrightarrow \mathsf{Hopf}_{\mathsf{coc}}$$

where all the involved categories turn out to be semiabelian. Additionally, all functors in this chain admit left adjoints and the first two inclusions determine two Birkhoff subcategories of HDiGrp.

References

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