Structures for the category of torsion theories

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Abstract.

Torsion theories were introduced in [3] to generalize the concept of torsion and torsion-free abelian groups to the context of abelian categories. Since then, a rich theory has been developed, even in more general categorical contexts. Despite this, little is known in the literature about the category of torsion theories and appropriate functors between them. We contribute to fill this gap by studying various structures for the 2-category of torsion theories.

We construct the cofree torsion theory on a pointed category. Explicitly, this is given by the category of short exact sequences in the starting pointed category. We then extend this construction to prove a comonadicity result for torsion theories. More precisely, the 2-category of torsion theories is comonadic over the 2-category of pointed categories with functors that preserve the zero object and short exact sequences. We can thus apply such result to study (2-dimensional) colimits in the 2-category of torsion theories.

We then prove that the 2-category of torsion theories has all products, calculated in the 2-category of pointed categories. This makes us wonder whether torsion theories are also monadic over the 2-category of pointed categories. While such result does not seem to hold in full generality, we prove a partial monadicity result. More precisely, we show that those torsion theories $(\mathbb{C}, \mathcal{T}, \mathcal{F})$ for which the canonical functor $\mathbb{C} \to \mathcal{T} \times \mathcal{F}$ is an equivalence of categories are monadic over the 2-category of pointed categories with functors that preserve the zero object. The involved 2-monad sends a pointed category $\mathbb{C}$ to the product $\mathbb{C} \times \mathbb{C}$. So, interestingly, this 2-monad is a 2-dimensional analogue of the monad on **Set** whose algebras are *rectangular bands* in semigroup theory [4], i.e. idempotent semigroups satisfying $xyz = xz$. We thus see the special torsion theories described above as internal rectangular bands in the 2-category of pointed categories, and we call them *rectangular torsion theories.* We prove that rectangular torsion theories are precisely the ones equivalent to products in the 2-category of torsion theories of form $(\mathcal{T}, \mathcal{T}, \mathbf{0}) \times (\mathcal{F}, \mathbf{0}, \mathcal{F})$, where $\mathbf{0}$ is the full subcategory of the base category consisting of all zero objects.

References

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