A new tool for detecting cofibrant generation

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Abstract.

A class $\mathcal{M}$ of monomorphisms is *cofibrantly generated* if it is the closure of a (small) set of morphisms under pushouts, transfinite compositions, and retracts. If $\mathcal{M}$ is cofibrantly generated then, under certain technical conditions, Quillen's Small Object Argument ensures that $\mathcal{M}$ forms the left part of a Weak Factorization System.

In locally finitely presentable categories, Borceux and Rosický [1] isolate a condition ("effective unions of pure subobjects") that is sufficient, but not necessary, for the class of pure monomorphisms to be cofibrantly generated. We noticed that if their condition is only required to hold "almost everywhere", then it exactly characterizes when pure monos are cofibrantly generated. The notion of "almost everywhere" can be made precise in several ways, most concisely via a certain interpretation of Shelah's *Stationary Logic*. Moreover, the characterization works for any class $\mathcal{M}$ of monomorphisms that is closed under pushouts, transfinite compositions, and retracts; i.e., such a class is cofibrantly generated if and only if one has "almost everywhere" effective unions of $\mathcal{M}$-subobjects.

As an application, we prove:

Theorem: Suppose $\mathcal{M}$ is a monoid, and that the class of pure monomorphisms is cofibrantly generated in the category of $\mathcal{M}$-acts. Then $\mathcal{M}$ has only a set of locally cyclic acts. (an $\mathcal{M}$-act A is locally cyclic if for every $a, b \in A$, the subact $\langle a, b \rangle$ is contained in a cyclic subact of A.)

References

[1] F. Borceux, J. Rosický, *Purity in algebra*, Algebra Universalis 56 (2007), no. 1, 17—35.