Schemes relative to Actegories

Subhajit Das

Abhishek Banerjee (abhi@iisc.ac.in)
Indian Institute of Science

Subhajit Das (subhajitdas@iisc.ac.in)
Indian Institute of Science

Surjeet Kour (surjeetkour@iitd.ac.in)
Indian Institute of Technology Delhi

Abstract.

In [3], Toën and Vaquié give a category-theoretic notion of a scheme over a closed symmetric monoidal category $\mathcal{C}$ which generalizes the usual algebraic geometry over $(Ab, \otimes_{\mathbb{Z}}, \mathbb{Z})$. They consider the category $Comm(\mathcal{C})$ of internal commutative monoids in $\mathcal{C}$ and define a Grothendieck topology J on $Aff_{\mathcal{C}} := Comm(\mathcal{C})^{op}$. To do this, they use actions of objects $c \in Comm(\mathcal{C})$ on objects of $\mathcal{C}$. By choosing $\mathcal{C}$ appropriately, one opens up a number of new geometries by taking $\mathcal{C}$ for instance to be the category of sets, or commutative monoids, or the category of symmetric spectra.

In this talk, we bring together their novel idea and the notion of vertical categorification of monoid actions, popularly known as "Actegories" ([2]), to present a notion of a scheme relative to a $\mathcal{C}$-actegory $\mathcal{M}$ ([1, § 4]). This is motivated by the Microcosm Principle, due to Baez and Dolan, which suggests that Actegories are the right setups for internalizing monoid actions, just as (braided/symmetric) monoidal categories are the right setups to internalize (commutative) monoids.

We start with a datum $(\mathcal{C}, \mathcal{M})$ consisting of a closed symmetric monoidal category $\mathcal{C} = (\mathcal{C}, \otimes, 1)$ and a left $\mathcal{C}$-actegory $\mathcal{M} = (\mathcal{M}, \boxtimes : \mathcal{C} \times \mathcal{M} \longrightarrow \mathcal{M})$ such that the $\mathcal{C}$-action bifunctor $\boxtimes$ is cocontinuous in both variables. We then use actions of objects $c \in Comm(\mathcal{C})$ on objects of $\mathcal{M}$ to define a Grothendieck topology $J_{\mathcal{M}}$ on $Aff_{\mathcal{C}} = Comm(\mathcal{C})^{op}$. In general, $J_{\mathcal{M}}$ may not be subcanonical (unlike the case of [3, Corollary 2.11] where $(\mathcal{M}, \boxtimes) = (\mathcal{C}, \otimes)$). For technical purposes, we only work with those $\mathcal{M}$ for which $J_{\mathcal{M}}$ is subcanonical (and call $\mathcal{M}$ a "subcanonical actegory"). We then define an "$\mathcal{M}$-scheme" as an object of the Grothendieck topos $Sh(Aff_{\mathcal{C}})_{\mathcal{M}}$ (of sheaves on the site $(Aff_{\mathcal{C}}, J_{\mathcal{M}})$) which can be "nicely covered" by representable sheaves. We see that the full subcategory $Sch_{\mathcal{M}}$ of $Sh(Aff_{\mathcal{C}})_{\mathcal{M}}$, consisting of the $\mathcal{M}$-schemes, is closed under pullbacks, coproducts and certain quotients.

We will end with a theorem which investigates the behaviour of our notion of a scheme under base changes along an adjunction in the 2-category $MonCat_{lax}$ and a lax linear functor between actegories. We also use this to generate a plethora of examples of subcanonical actegories.

References

[1] A. Banerjee, S. Das and S. Kour, Preprint, *Categorification of modules and construction of schemes*, arXiv:2412.08952, 2024.

[2] J. Janelidze and G.M. Kelly, *A note on actions of a monoidal category*, Theory Appl. Categ. 9 (2001/02), 61–91

[3] B. Toën and M. Vaquié, *Au-dessous de* $\operatorname{Spec}\mathbb{Z}$, J. K-Theory 3 (2009), no. 3, 437–500.