Between Set and Bool: Categories of Aristotelian diagrams

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Abstract.

Aristotelian diagrams, such as the square of opposition, have long been used as pedagogical tools in logic and philosophy [3]. With the conception of logical geometry in the early 21st century, they became objects of study in their own right [4]. Formally, a (classical) Aristotelian diagram can be viewed as a couple $(\mathcal{F}, B)$, where $\mathcal{F}$ is a subset of a Boolean algebra B. When visualizing these diagrams, the Aristotelian relations (contradiction, (sub)contrariety and subalternation) that hold between the elements of $\mathcal{F}$ are drawn. Therefore, Aristotelian diagrams exhibit two different levels of structure. The first level is defined solely by the Aristotelian relations, while the second level also cares about the other identities that hold between the elements of $\mathcal{F}$ in B.

To establish solid mathematical foundations for the study of Aristotelian diagrams, recent research in logical geometry has incorporated category theory, by creating categories in which the objects are precisely the Aristotelian diagrams [1]. Among them, the category $\mathbb{D}^{Inc}_{\mathcal{OR}\times\mathcal{IR}}$ looked the most promising to describe the first strucutral level of these diagrams.

In a forthcoming book, we deepen this category-theoretical approach to logical geometry [2]. In this book, we show that $\mathbb{D}^{Inc}_{\mathcal{OR}\times\mathcal{IR}}$ is bicomplete by providing constructions for all its (small) limits and colimits. Additionally, we introduce the category $\mathbb{D}_{\mathcal{B}}$, which captures the second structural level of Aristotelian diagrams, and prove its bicompleteness by providing constructions for all its (small) limits and colimits. The (co)limits in these categories relate closely to those in Set and Bool, and examples of each of them can be found in previous research in logical geometry (which did not yet use the language of category theory). Furthermore, we establish three adjunctions: between Set and $\mathbb{D}^{Inc}_{\mathcal{OR}\times\mathcal{IR}}$, between $\mathbb{D}^{Inc}_{\mathcal{OR}\times\mathcal{IR}}$ and $\mathbb{D}_{\mathcal{B}}$, and between $\mathbb{D}_{\mathcal{B}}$ and Bool. Their composition is proven to be precisely the free/forgetful adjuntion between Set and Bool. These results, though proven, remain unpublished as the book is still in progress. We hope to present them for the first time at CT2025.

References

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