Category Theory within a 2-Category: Internal Enrichment, Presheaf Objects and Convolution Products

S. d'Espalungue

Sophie d'Espalungue (sophie@espalungue.xyz)

Abstract.

Category theory offers convenient tools (such as Kan extensions and Day convolution) for the description of mathematical structures. However, the 2-category of categories is not general enough to encapsulate certain structures. For instance, it is often needed to place within a $\mathcal{V}$-enriched setting for a closed monoidal category $\mathcal{V}$. More generally, it has been noticed that most of the tools of category theory (equivalences, adjunctions, Kan extensions...) can be generalized to the framework of a 2-category. While $\mathcal{V}$-enriched ends and coends provide explicit formulas for $\mathcal{V}$-natural transformations, Kan extensions, and Day convolution product within the 2-category of $\mathcal{V}$-enriched categories, more general 2-categories lack such formulas. Based on the first chapter of [1], this work is motivated by the example given by the 2-category of symmetric sequences of categories $\mathrm{CAT}^{\mathfrak{S}}$, with the goal of defining the presheaf operad of an operad - notably, the presheaf operad of the face poset of the associahedra.

The purpose is to extend categorical constructions to the framework of a 2-category with enough structure, pushing the internalization process further while aligning with the principles of formal category theory. We proceed to define internal enrichment, from which we obtain an internal definition of ends and coends, notably providing explicit formulas for Kan extensions - hence Day convolution. For this purpose, we define suitable notions on a cartesian closed 2-category $(\Lambda, \times_\Lambda, [\_,\_]_\Lambda)$ so that it sufficiently ressembles CAT. Those notions include an oppositization 2-functor $(\_)^{op} : \Lambda^{op_2} \to \Lambda$, an internally complete and cocomplete closed monoidal object $(\mathcal{S}, \times_\mathcal{S}, [\_,\_]_\mathcal{S})$ in Λ - e.g. SET, $\mathcal{V}$, $\mathrm{SET}^{\mathfrak{S}}$ within CAT, resp. $\mathrm{CAT}_\mathcal{V}$, resp. $\mathrm{CAT}^{\mathfrak{S}}$. We first define relative notions of ends and coends, relatively to a morphism $F : \mathcal{C}^{op} \times_\Lambda \mathcal{C} \to \mathcal{S}$ in Λ. We introduce the notion of an $\mathcal{S}$-enrichment in Λ, which involves a coherent data of morphisms $\mathcal{C}(\_,\_) : \mathcal{C}^{op} \times_\Lambda \mathcal{C} \to S$ in Λ for each object $\mathcal{C}$ of Λ. This notion of coherence is expressed by using a 2-functor (see I.4.3 of [1]), and yields the following isomorphism in $\mathcal{S}$

$$[\mathcal{C}, \mathcal{D}]_\Lambda(F, G) \simeq \int^{\mathcal{C}(\_,\_)} \mathcal{D}(F\_, G\_)$$

for objects $\mathcal{C}, \mathcal{D}$ and $F, G : \mathcal{C} \to \mathcal{D}$. We define ends and coends of morphisms $\mathcal{C}^{op} \times_\Lambda \mathcal{C} \to \mathcal{S}$ as those relative to $\mathcal{C}(\_,\_)$. As a result, we obtain a pointwise expression of the internal left Kan extension

$$\mathrm{Lan}^\mathcal{S} : [\mathcal{C}_2, \mathcal{C}_1]_\Lambda^{op} \to [[\mathcal{C}_2, \mathcal{S}]_\Lambda, [\mathcal{C}_1, \mathcal{S}]_\Lambda]_\Lambda$$

whose value $\mathrm{Lan}^\mathcal{S}_\mu$ on $\mu : \mathcal{C}_2 \to \mathcal{C}_1$ is the internal left adjoint of $[\mu, \mathcal{S}]_\Lambda$, given in $F : \mathcal{C}_2 \to S$ by $\mathrm{Lan}^\mathcal{S}_\mu F \simeq \int_{x:*_\Lambda \to \mathcal{C}_2} \mathcal{C}_1(\mu x, \_) \times_\mathcal{S} Fx$. In addition, we obtain an internal version of Yoneda embedding, which satisfies the corresponding lemma and exhibits the presheaf object $[\mathcal{C}^{op}, \mathcal{S}]_\Lambda$ of any object $\mathcal{C}$ as the free cocomplete completion of $\mathcal{C}$ in Λ.

We eventually consider the case where Λ is further equipped with an additional monoidal structure $(\Lambda, \otimes_\Lambda)$ which is compatible with its former closed monoidal structure. Extending Day's convolution product to this framework, we obtain a monoidal structure on the Yoneda embedding which exhibits the presheaf object of an internal monoid $(\mathcal{C}, \otimes_\mathcal{C})$ as the free cocomplete completion of $\mathcal{C}$ in the 2-category of monoids internal to $(\Lambda, \otimes_\Lambda)$ - provided that $\mathcal{S}$ has an additional internal monoidal structure as well.

References

[1] S. d'Espalungue d'Arros, *Operads in 2-categories and models of structure interchange*, PhD Thesis (2024). https://theses.hal.science/tel-04617115.