Extending an arithmetic universe by an object

S. Desrochers

Samuel Desrochers (sdesr091@uottawa.ca)
University of Ottawa

Abstract.

In a 1999 paper [4], Vickers speculated that arithmetic universes could be used to provide a "base-free" way to study Grothendieck toposes. As he explains in [5], one usually works with an elementary topos $\mathcal{S}$ as a base, and a Grothendieck topos with respect to $\mathcal{S}$ is an elementary topos together with a bounded geometric morphism to $\mathcal{S}$. In this way, the infinitary operations of the Grothendieck topos are parametrized by objects of $\mathcal{S}$. A problem with this approach is that the most natural notion of morphism between elementary toposes is not geometric morphisms, so the category of elementary toposes and geometric morphisms is not as well behaved.

On the other hand, morphisms of arithmetic universes (that is, list-arithmetic pretoposes [2]) are much closer to geometric morphisms. Instead of having all power objects, these categories have parametrized list objects $L(X)$ for every object X; thus, the only infinities available are free algebra constructions.

Vickers' idea led to a research program exploring whether it is possible to develop the theory of Grothendieck toposes using AUs, which has seen some success [3, 5]. For instance, Vickers recently [5] developed an analogue for classifying toposes; specifically, he gave a syntactic construction of the arithmetic universe $\mathbf{AU}\langle\mathbb{T}\rangle$ freely generated by a context $\mathbb{T}$. This extends to a full and faithful 2-functor from contexts to arithmetic universes.

In this talk, I'll examine the special case of extending an arithmetic universe $\mathcal{A}$ by a single indeterminate object X to get $\mathcal{A}[X]$. I'll show that the same description as for Grothendieck toposes holds [1, B3.2.9]: $\mathcal{A}[X]$ is equivalent to the category $\mathrm{CoPsh}(\mathbf{Fin}_{\mathcal{A}})$ of internal copresheaves on the internal category $\mathbf{Fin}_{\mathcal{A}}$ of finite sets in $\mathcal{A}$. Looking forward, one could consider the extension by an arbitrary collection of objects; together with localization, this would give a concrete description of classifying toposes over an arbitrary arithmetic universe, analogous to the case of a base elementary topos.

This work is the result of PhD research under the supervision of Simon Henry and Philip J. Scott, and is based on a conjecture of Simon Henry.

References

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