Call *doctrines* by your name

I. Di Liberti

Ivan Di Liberti (`diliberti.math@gmail.com`)
University of Gothenburg

Abstract.

What's a *doctrine* to you? Is it a **lax-idempotent pseudomonad**, is it some form of Lawverian **doctrine**, or a type of **classifying topoi** for a specific fragment of predicate logic?

We show that:

(1) a class of geometric morphisms $\mathcal{H}$ can be understood as specifying a fragment of geometric logic. A topos $\mathcal{E}$ is Kan injective with respect to $\mathcal{H}$ when its models have the desired property specified by $\mathcal{H}$.

(2) every such class of morphisms induces a lax-idempotent relative pseeudomonad $\mathcal{H} \mapsto \mathsf{T}^{\mathcal{H}}$ on the 2-category Lex. Every algebra for such a monad has a *classifying topos*, $\mathcal{C}l : \mathsf{Alg}(\mathsf{T}^{\mathcal{H}})^{\mathrm{op}} \to \mathsf{RInj}(\mathcal{H})$. Let's see two extreme cases below, but the real fun is *everything in between*.

- when $\mathcal{H}$ is all geometric morphisms, $\mathsf{T}^{\mathcal{H}}$ is the identity and $\mathcal{C}l$ is computing the classifying topos of a lex theory, i.e. its presheaf topos.
- when $\mathcal{H}$ is empty, $\mathsf{T}^{\mathcal{H}}$ is the presheaf construction and $\mathcal{C}l$ is *tautological* duality between Logoi and Topoi.

(3) Every lax-idempotent pseudomonad T on Lex induces a lax-idempotent pseudomonad $\mathsf{T}^{\mathrm{fbr}}$ on a reasonable 2-category of Lawverian doctrines, so that we have a representation functor,

$$\mathsf{Sub} : \mathsf{Alg}(\mathsf{T}) \to \mathsf{Alg}(\mathsf{T}^{\mathrm{fbr}}).$$

When T is the identity, so is $\mathsf{T}^{\mathrm{fbr}}$. When T is the presheaf construction, $\mathsf{T}^{\mathrm{fbr}}$ is the free locale completion of a dependent meetlattice.

These constructions allow to give definitions (!) of what *is* a (fragment of geometric) logic, and to move between these notions *safely*, recovering several constructions from the classical literature of categorical logic (classifying topos, syntactic category, doctrine of subobjects...).

This talk presents two ongoing collaborations. One with **Lingyuan Ye** (on (1) and (2)), the other with **Joshua Wrigley** and **Jacopo Emmenegger** on (3).