An adjunction between categories of monads

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Abstract

Given a monad T on $\mathscr{A}$ and a functor $G\colon \mathscr{A} \to \mathscr{B}$, if the right Kan extension of GT along G exists, then it has a canonical monad structure; we call this monad the pushforward of T along G and denote it by $G_\# T$. If F is left adjoint to G, then $G_\# T$ is the familiar monad structure on GTF. If T is the identity monad, then $G_\# T$ is the codensity monad of T. Even though it was introduced by Street [2] in the 1970s, the pushforward constructions has not yet been studied in depth. This talk will report on work towards this goal, the details of which can be found in [1].

Under suitable smallness and completeness conditions, $G_\#$ becomes a functor $\mathsf{Mnd}(\mathscr{A}) \to \mathsf{Mnd}(\mathscr{B})$. In fact, when G is full and faithful, $G_\#$ has a partial left adjoint $G^\#$, defined on those monads on $\mathscr{B}$ whose endofunctor restricts to $\mathscr{A}$, i.e. those monads S such that there is an endofunctor S' of $\mathscr{A}$ and an isomorphism $SG \cong GS'$. We denote the corresponding subcategory of $\mathsf{Mnd}(\mathscr{B})$ by $\mathsf{Mnd}(\mathscr{B})^{\mathrm{res}\mathscr{A}}$, giving an adjunction

$$\mathsf{Mnd}(\mathscr{A}) \underset{G^\#}{\overset{G_\#}{\underset{\longleftarrow}{\overset{\longrightarrow}{\top}}}} \mathsf{Mnd}(\mathscr{B})^{\mathrm{res}\mathscr{A}} \subseteq \mathsf{Mnd}(\mathscr{B}),$$

with $G_\#$ full and faithful and $G^\#$ given by restriction.

The conditions which result in this situation are not hard to satisfy. For example, one may take G to be the inclusion $\mathsf{FinSet} \hookrightarrow \mathsf{Set}$, the inclusion $\mathsf{Set}_{\neq\varnothing} \hookrightarrow \mathsf{Set}$ (where the domain is the category of nonempty sets), the discrete-topology functor $\mathsf{Set} \to \mathsf{Top}$, or the forgetful functor $\mathsf{Field} \to \mathsf{CRing}$. The first two will be the guiding examples for this talk.

In the case of $\mathsf{Set}_{\neq\varnothing} \hookrightarrow \mathsf{Set}$, the category on the right-hand side of the adjunction is just $\mathsf{Mnd}(\mathsf{Set})$, and the corresponding reflective subcategory on the left is spanned by those monads all of whose pseudoconstants are induced by actual constants.

In the case of $\mathsf{FinSet} \hookrightarrow \mathsf{Set}$, the right-hand side of the adjunction includes monads such as the identity, the $(-) + E$ monad (for a finite set E), the $M \times (-)$ monad (for a finite monoid M), the powerset monad, the double-powerset monad, the ultrafilter monad, and the filter monad. I will explain what the pushforward of their restriction to FinSet is; for example, starting with the powerset monad, this process produces the theory of continuous lattices. I will also explain why, with two trivial exceptions, pushforwards along $\mathsf{FinSet} \hookrightarrow \mathsf{Set}$ never have rank.

References

[1] A. Doña Mateo, Pushforward monads, preprint arXiv:2406.15256, 2025

[2] R. Street, *The formal theory of monads*, Journal of Pure and Applied Algebra 2.2 (1972)