Action representability of internal 2-groupoids

Nadja Egner

Nadja Egner (nadja.egner@uclouvain.be)
Université catholique de Louvain

Marino Gran (marino.gran@uclouvain.be)
Université catholique de Louvain

Abstract. Based on [1], we prove that, in any regular Mal'tsev category $\mathscr{C}$ with coequalizers, the category 2-Grpd($\mathscr{C}$) of internal 2-groupoids is a Birkhoff subcategory of the category $\mathrm{Grpd}^2(\mathscr{C})$ of internal double groupoids, i.e., it is a full reflective subcategory that is also closed under subobjects and quotients. It follows that 2-Grpd($\mathscr{C}$) is a semi-abelian category if $\mathscr{C}$ is so, and we identify sufficient conditions on $\mathscr{C}$ for 2-Grpd($\mathscr{C}$) to be action representable if $\mathscr{C}$ is so. This result applies, for example, to the categories Grp of groups, Lie_R of Lie algebras over a commutative ring R, and $\mathrm{Hopf}_{\mathrm{coc},K}$ of cocommutative Hopf algebras over a field K.

If $\mathscr{C}$ is a Mal'tsev category, i.e., a finitely complete category in which any internal reflexive relation is an equivalence relation, then the inclusion functor from the category Grpd($\mathscr{C}$) of internal groupoids to the category Cat($\mathscr{C}$) of internal categories is an isomorphism of categories, and the inclusion functor from Cat($\mathscr{C}$) to the category RG($\mathscr{C}$) of internal reflexive graphs is full. Examples of Mal'tsev categories include any Mal'tsev variety, i.e., a variety of universal algebras whose theory admits a ternary term p that satisfies the identities $p(x, x, y) = y = p(y, y, x)$, such as the varieties of groups, of rings, of Lie algebras, and of Heyting algebras.

It is known that, if $\mathbb{V}$ is a Mal'tsev variety, then 2-Grpd($\mathbb{V}$) is a subvariety of the Mal'tsev variety $\mathrm{Grpd}^2(\mathbb{V})$. Generalizing this result, we show that 2-Grpd($\mathscr{C}$) is a Birkhoff subcategory of $\mathrm{Grpd}^2(\mathscr{C})$ whenever $\mathscr{C}$ is a regular Mal'tsev category with coequalizers, relying on the result of [2] that Grpd($\mathscr{C}$) is a regular Mal'tsev category whenever $\mathscr{C}$ is so. An explicit description of the reflector $\mathrm{Grpd}^2(\mathscr{C}) \to$ 2-Grpd($\mathscr{C}$) is provided, which was not known in the varietal case, that reveals an interesting link with commutator theory of equivalence relations.

A category $\mathscr{C}$ with finite limits and finite colimits is called action representable if the functors $\mathrm{Act}(-, X)$ that map an object B to the set of its actions on X are representable, i.e., there exists an object $[X]$, called the actor of X, such that $\mathrm{Act}(B, X)$ and $\mathrm{Hom}(B, [X])$ are naturally isomorphic. Examples of action representable categories are given by Grp and Lie_R. The actor of a group G is given by its automorphism group $\mathrm{Aut}(G)$, while the actor of a Lie algebra L is given by its Lie algebra of derivations $\mathrm{Der}(L)$. If $\mathscr{C}$ is a semi-abelian category, the functor $\mathrm{Act}(-, X)$ is isomorphic to the functor $\mathrm{SplExt}(-, X)$ that maps an object B to the set of isomorphism classes of split extensions of B by X. It is shown in [3] that Grpd($\mathscr{C}$) is semi-abelian, action representable, algebraically coherent and with normalizers if and only if $\mathscr{C}$ is so. Using the fact that 2-Grpd($\mathscr{C}$) is coreflective and reflective in $\mathrm{Grpd}^2(\mathscr{C})$ if $\mathscr{C}$ is a semi-abelian category, we apply this result to show that 2-Grpd($\mathscr{C}$) shares the same properties if $\mathscr{C}$ does.

References

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[3] M. Gran and J.R.A. Gray, *Action representability of the category of internal groupoids*, Theory Appl. Categ. 37 (2021), no. 1, 1-13.