

Premonoidal and Kleisli double categories

B. Femić

Bojana Femić (femicenelsur@gmail.com)

Mathematical Institute of Serbian Academy of Sciences and Arts

Abstract.

Premonoidal categories appeared in Computer Science as a model that encodes the structure of effectful languages. Recently in [1] the notion of premonoidal bicategories was introduced covering semantic models that have more structure and supply more detailed and determined data. In this talk we give a double categorical version of the notion and the accompanying results.

We introduce two kinds of funny products on double categories and the corresponding closed representable multicategories, which we call *funny multicategories*, providing the category Dbl of double categories structures of closed funny monoidal products. We show that a monoid in one of the monoidal structures and a pseudomonoid in the monoidal 2-category $(Dbl_2, \square_2)$ induced by the other funny monoidal product are premonoidal double categories of certain kinds. We also describe binoidal structures induced by pseudodouble funny functors (multimaps for funny multicategories).

We prove that a premonoidal double category $\mathbb{D}$ is purely central if and only if its binoidal structure is given by a pseudodouble quasi-functor of two variables (a multimap for a Gray type of multicategory) if and only if $\mathbb{D}$ admits a monoidal structure. For such double categories we introduce pure center double categories and show that the monoidal structure on $\mathbb{D}$ extends to a one on the corresponding pure center. We differentiate a (general) center double category in which left and right centrality structures are not “sufficiently well” related and that henceforth does not present a monoidal structure.

Provided existence of certain companions, we prove a series of further results by applying companion-lifting of vertical structures into their horizontal counterparts. We firstly introduce a vertical strength t on a vertical double monad T on a monoidal double category, and also a horizontal strength s on a horizontal double monad S on a horizontally monoidal double category. We prove that vertical strengths induce horizontal strengths $\hat{t}$ on the induced horizontal double monads $\hat{T}$. We show that vertical strengths induce actions of the induced horizontally monoidal double category $(\mathbb{D}, \otimes, \hat{\alpha}, \hat{\lambda}, \hat{\rho})$ on the horizontal Kleisli double category $\mathbb{Kl}(\hat{T})$. Moreover, we prove that there is a 1-1 correspondence between horizontal strengths and extensions of the canonical action of $\mathbb{D}$ on itself (a canonical action consists of a horizontal action on the Kleisli double category and a horizontal icon). Finally, we show that for a bistrong vertical double monad T the Kleisli double category $\mathbb{Kl}(\hat{T})$ is premonoidal. By the latter three results we lifted to double categories the corresponding bicategorical results from [1, 2]. We exhibit the advantage of working with vertical structures in double categories: companion-lifts of 2-cells appearing in the computations are globular 2-cells of certain type that satisfy any equation that can be comprised by their compositions. Consequently, for bicategories that are underlying bicategories of double categories with suitable companion-lifts much simpler proofs can be performed by applying companion-lifting of the vertical structures, as is exemplified in this work.

References

- [1] H. Paquet, P. Saville, *Effectful semantics in 2-dimensional categories: premonoidal and Freyd bicategories*, Electronic Proceedings in Theoretical Computer Science **397**/3 (2023), 190–209.
- [2] H. Paquet, P. Saville, *Effectful semantics in 2-dimensional categories: strong, commutative, and concurrent pseudomonads*, LICS '24: Proceedings of the 39th Annual ACM/IEEE Symposium on Logic in Computer Science Article **61**, 1 – 15.