Monilmorphisms and relative extensivity

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Abstract.

We introduce a new class of morphisms in a pointed category with kernels, which contains all monomorphisms and all null morphisms. We call a morphism $f\colon X \to Y$ in such a category a *monilmorphism* when for any two morphisms $g_1, g_2\colon W \to X$ we have:

$$[f \circ g_1 = f \circ g_2 \ \wedge\ \mathsf{ker}(f \circ g_1) = \mathsf{ker}(g_1) \ \wedge\ \mathsf{ker}(f \circ g_2) = \mathsf{ker}(g_2)] \implies [g_1 = g_2].$$

It is easy to see that in the category of sets and partial functions, monilmorphisms are precisely the injective partial functions. More generally, in any restriction category [1] with a restriction zero, the monilmorphisms are the same as restricted monics if and only if all restriction idempotents are monilmorphisms. On the other hand, in the category **Rel** of sets and relations as morphisms, a morphism is a monilmorphism if and only if it is a monomorphism or a zero morphism.

In the first part of the talk, we establish general properties of monilmorphisms and characterize them in various types of categories. In particular, we show that while every monilmorphism has the property that subobjects of its codomain can have at most one section, this property characterises monilmorphisms in any semi-abelian [4] category. In a number of particular semi-abelian categories (e.g. $\mathbf{Set}_*^{\mathrm{op}}$ and $\mathbf{Vect}_K$) monilmorphisms again reduce to just monos and zero morphisms. In any 0-regular variety, though, every morphism from an algebra with distributive subalgebra lattice is a monilmorphism. For example, in the category of groups, any homomorphism from a locally cyclic group is a monilmorphism.

In the second part of the talk, we discuss an application of monilmorphisms in the study of relative extensivity. In [3] it is shown that every monoidal sum structure $\oplus$ on a category can be described in terms of a certain kind of binary relation $\sqsubset$ on morphisms. Every morphism f is $\sqsubset$-preserving in the sense that $f_1 \sqsubset f_2 \Rightarrow ff_1 \sqsubset ff_2$. But, such a morphism might not be "totally honest" to its domain about all structure pertaining to $\oplus$ in its codomain. The morphisms that are totally honest in this sense are those where $ff_1 \sqsubset ff_2 \Rightarrow f_1 \sqsubset f_2$ — the $\sqsubset$-reflecting morphisms. Now, one can define extensivity relative to a morphism exactly as in [2] but with coproduct replaced by $\oplus$. One might require that all morphisms are extensive w.r.t. $\oplus$, but this is often too strong a condition on $\oplus$. Less demanding is that the extensivity requirement holds only for the $\sqsubset$-reflecting morphisms. This is the case, for example, in various categories of sets equipped with relational structure, where $\oplus$ is the stacking operation (every element of A is related to every element of B in $A \oplus B$). As we will show, monilmorphisms allow us to build examples of categories equipped with a sum structure $\oplus$ where $\sqsubset$-reflecting morphisms are exactly the morphisms that are extensive with respect to $\oplus$.

References

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